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On Galilei Invariance of Continuity Equation

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Classes of the nonlinear Schrödinger-type equations compatible with the Galilei relativity principle are obtained. Solutions of these equations satisfy the continuity equation.

The continuity equation is one of the most fundamental equations of quantum mechanics

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0. \tag{1}$$

Depending on definition of ρ (density) and $\vec{j} = (j^1, \ldots, j^n)$ (current), we can construct essentially different quantum mechanics with different equations of motion, which are distinct from classical linear Schrödinger, Klein–Gordon–Fock, and Dirac equations.

At the beginning we study a symmetry of the continuity equation considering $(\rho, \vec{j})$ as dependent variables related by (1).

Theorem 1 [1]. *The invariance algebra of equation (1) is an infinite-dimensional algebra with basis operators*

$$X = \xi^\mu(x) \frac{\partial}{\partial x_\mu} + \Big(a^{\mu\nu}(x) j^\nu + b^\mu(x) \Big) \frac{\partial}{\partial j^\mu} \ , \tag{2}$$

where $j^0 \equiv \rho$; $\xi^\mu(x)$ *are arbitrary smooth functions;* $x = (x_0 = t, x_1, x_2, \ldots, x_n) \in \mathbf{R}^{n+1}$; $a^{\mu\nu}(x) = \dfrac{\partial \xi^\mu}{\partial x_\nu} - \delta_{\mu\nu} \left(\dfrac{\partial \xi^i}{\partial x_i} + C \right)$; $C = \mathrm{const}$, $\delta_{\mu\nu}$ *is the Kronecker delta;* $\mu, \nu, i = 0, 1, \ldots, n$, $\Big(b^0(x), b^1(x), \ldots, b^n(x) \Big)$ *is an arbitrary solution of equation (1).*

Here and below we imply summation over repeated indices.

An infinite-dimensional algebra with basis operators (2) contains as subalgebras the generalized Galilei algebra

$$AG_2(1,n) = \langle P_\mu, J_{ab}, G_a, D^{(1)}, A \rangle \tag{3}$$

and the conformal algebra

$$AP_2(1,n) = AC(1,n) = \langle P_\mu, J_{ab}, J_{0a}, D^{(2)}, K_\mu \rangle. \tag{4}$$

We use the following designations in (3) and (4)

$$P_\mu = \partial_\mu, \qquad J_{ab} = x_a \partial_b - x_b \partial_a + j^a \partial_{j^b} - j^b \partial_{j^a} \quad (a < b),$$

$$G_a = x_0 \partial_a + \rho \partial_{j^a}, \qquad J_{0a} = x_a \partial_0 + x_0 \partial_a + j^a \partial_\rho + \rho \partial_{j^a},$$

$$D^{(1)} = 2x_0 \partial_0 + x_a \partial_a - n\rho \partial_\rho - (n+1) j^a \partial_{j^a}, \qquad D^{(2)} = x_\mu \partial_\mu - n\rho\partial_\rho - n j^a \partial_{j^a},$$

$$A = x_0^2 \partial_0 + x_0 x_a \partial_a - n x_0 \rho \partial_\rho + (x_a \rho - (n+1) x_0 j^a) \partial_{j^a},$$

$$K_\mu = 2x_\mu D^{(2)} - x_\nu x^\nu g_{\mu i} \partial_i - 2x^\nu S_{\mu\nu}, \quad S_{\mu\nu} = g_{\mu i} j^\nu \partial_{j^i} - g_{\nu i} j^\mu \partial_{j^i},$$

$$g_{\mu\nu} = \begin{cases} 1, & \mu = \nu = 0 \\ -1, & \mu = \nu \neq 0 \\ 0, & \mu \neq \nu, \end{cases} \qquad \mu, \nu, i = 0, 1 \ldots, n; \quad a, b = 1, 2, \ldots, n.$$

Thus, the continuity equation satisfies the Galilei relativity principle as well as the Lorentz–Poincare–Einstein relativity principle and, depending on the definition of ρ and $\vec{j}$, we will come to different quantum mechanics.

Let us consider the scalar complex-valued wave functions and define ρ and $\vec{j}$ in the following way

$$\rho = f(uu^*), \qquad j^k = -\frac{1}{2} i g(uu^*) \left(\frac{\partial u}{\partial x_k} u^* - u \frac{\partial u^*}{\partial x_k} \right) + \frac{\partial \varphi(uu^*)}{\partial x_k}, \quad k = 1, 2, \ldots, n, \tag{5}$$

where f, g, φ are arbitrary smooth functions, $f \neq \text{const}$, $g \neq 0$. Without loss of generality, we assume that $f \equiv uu^*$.

Let us describe all functions $g(uu^*)$, $\varphi(uu^*)$ for continuity equation (1), (5) to be compatible with the Galilei relativity principle, defined by the following transformations:

$$t \to t' = t, \qquad x_a \to x'_a = x_a + v_a t.$$

Here we do not fix transformation rules for the wave function u.

If ρ and $\vec{j}$ are defined according to formula (5), then the continuity equation (1) is Galilei-invariant iff

$$\rho = uu^*, \qquad j^k = -\frac{1}{2} i \left(\frac{\partial u}{\partial x_k} u^* - u \frac{\partial u^*}{\partial x_k} \right) + \frac{\partial \varphi(uu^*)}{\partial x_k}, \qquad k = 1, 2, \ldots, n. \tag{6}$$

The corresponding generators of Galilei transformations have the form

$$G_a = x_0 \partial_a + i x_a \left(u \partial_u - u^* \partial_{u^*} \right), \quad a = 1, 2, \ldots, n.$$

If in (6)

$$\varphi = \lambda uu^*, \qquad \lambda = \text{const}, \tag{7}$$

then the continuity equation (1), (6), (7) coincides with the Fokker–Planck equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} + \lambda \Delta \rho = 0, \tag{8}$$

where

$$\rho = uu^*, \qquad j^k = -\frac{1}{2} i \left(\frac{\partial u}{\partial x_k} u^* - u \frac{\partial u^*}{\partial x_k} \right), \quad k = 1, 2, \ldots, n. \tag{9}$$

The continuity equation (1), (6), (7) was considered in [3, 5].

In [1] we investigated the symmetry properties of the nonlinear Schrödinger equation the following form

$$i u_0 + \frac{1}{2} \Delta u + i \frac{\Delta \varphi(uu^*)}{2uu^*} u = F\left(uu^*, (\vec{\nabla}(uu^*))^2, \Delta(uu^*) \right) u, \tag{10}$$

where F is an arbitrary real smooth function.

For the solutions of equation (10), equation (1), (6) is satisfied and therefore this equation is compatible with the Galilei relativity principle.

In terms of the phase and amplitude $\Big(u = R\exp(i\Theta)\Big)$, equation (10) has the form

$$\begin{aligned} & R_0 + R_k \Theta_k + \frac{1}{2} R \Delta \Theta + \frac{1}{2R} \Delta \varphi = 0, \\ & \Theta_0 + \frac{1}{2} \Theta_k^2 - \frac{1}{2R} \Delta R + F\left(R^2, \left(\vec{\nabla}\left(R^2\right) \right)^2, \Delta R^2 \right) = 0. \end{aligned} \tag{11}$$

Theorem 2 [1]. *The maximal invariance algebras for system (11) if $F = 0$ are the following:*

$$1. \quad \langle P_\mu, J_{ab}, Q, G_a, D\rangle \tag{12}$$

when φ is an arbitrary function;

$$2. \quad \langle P_\mu, J_{ab}, Q, G_a, D, I, A\rangle \tag{13}$$

when $\varphi = \lambda R^2$, $\lambda =$ const.

In (12) and (13) we use the following designations:

$$\begin{gathered}
P_\mu = \partial_\mu, \qquad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a}, \qquad a < b, \\
G_a = x_0\partial_{x_a} + ix_a\partial_\Theta, \quad Q = \partial_\Theta, \quad D = 2x_0\partial_{x_0} + x_a\partial_{x_a}, \quad I = R\partial_R, \\
A = x_0^2\partial_{x_0} + x_0x_a\partial_{x_a} - \frac{n}{2}x_0R\partial_R + \frac{1}{2}x_a^2\partial_\Theta, \\
\mu = 0, 1, \ldots, n; \qquad a, b = 1, 2, \ldots, n.
\end{gathered} \tag{14}$$

Algebra (13) coincides with the invariance algebra of the linear Schrödinger equation.

Corolarry. *System (11), (7) is invariant with respect to algebra (13) if*

$$F = R^{-1}\Delta R\, N\left(\frac{R\Delta R}{(\vec\nabla R)^2}\right),$$

where N is an arbitrary real smooth function.

Let us consider a more general system than (10)

$$iu_0 + \frac{1}{2}\Delta u = (F_1 + iF_2)u, \tag{15}$$

where F_1, F_2 are arbitrary real smooth functions,

$$F_m = F_m\left(uu^*, (\vec\nabla(uu^*))^2, \Delta(uu^*)\right)u, \qquad m = 1, 2. \tag{16}$$

The structure of functions F_1, F_2 may be described in form (16) by virtue of conditions for system (15) to be Galilei-invariant.

In terms of the phase and amplitude, equation (15) has the form

$$\begin{gathered}
R_0 + R_k\Theta_k + \frac{1}{2}R\Delta\Theta - RF_2 = 0, \\
\Theta_0 + \frac{1}{2}\Theta_k^2 - \frac{1}{2R}\Delta R + F_1 = 0,
\end{gathered} \tag{17}$$

where $F_m = F_m\left(R^2, \left(\vec\nabla\left(R^2\right)\right)^2, \Delta R^2\right), \quad m = 1, 2.$

Theorem 3. *System (17) is invariant with respect to algebra (13) if it has the form*

$$\begin{gathered}
R_0 + R_k\Theta_k + \frac{1}{2}R\Delta\Theta - \Delta R\, M\left(\frac{R\Delta R}{(\vec\nabla R)^2}\right) = 0, \\
\Theta_0 + \frac{1}{2}\Theta_k^2 - \frac{1}{2R}\Delta R + \frac{\Delta R}{R}N\left(\frac{R\Delta R}{(\vec\nabla R)^2}\right) = 0,
\end{gathered} \tag{18}$$

where N, M are arbitrary real smooth functions.

System (18) written in terms of the wave function has the form

$$iu_0 + \frac{1}{2}\Delta u = \frac{\Delta|u|}{|u|}\left(N\left(\frac{|u|\Delta|u|}{(\vec{\nabla}|u|)^2}\right) + iM\left(\frac{|u|\Delta|u|}{(\vec{\nabla}|u|)^2}\right)\right)u. \qquad (19)$$

Thus, equation (19) admits an invariance algebra which coincides with the invariance algebra of the linear Schrödinger equation with arbitrary functions M, N.

With certain particular M and N the symmetry of system (18) can be essentially extended. If in (18) $N = \frac{1}{2}$, then the second equation of the system (equation for the phase) will be the Hamilton–Jacobi equation [4].

Let us consider some forms of the continuity equation (1) for equation (19).

Case 1. If $M = 0$, then for solutions of equation (18) equation (1) holds true, where the density and current can be defined in the classical way (9).

Case 2. If $M\Delta R = -\lambda\Big(\Delta R + \frac{(\vec{\nabla}R)^2}{R}\Big)$, then for solutions of equation (19), the continuity equation (1), (6), (7) (or the Fokker–Planck equation (8), (9)) is valid.

Case 3. If M is arbitrary then for solutions of equation (19), the continuity equation is valid, where the density and current can be defined by the conditions

$$\rho = uu^*, \qquad \vec{\nabla}\cdot\vec{j} = \frac{\partial}{\partial x_k}\left(-\frac{1}{2}i\left(\frac{\partial u}{\partial x_k}u^* - u\frac{\partial u^*}{\partial x_k}\right)\right) - 2|u|\Delta|u|\, M\left(\frac{|u|\Delta|u|}{(\vec{\nabla}|u|)^2}\right).$$

Thus, we constructed wide classes of the nonlinear Schrödinger-type equations which are invariant with respect to algebra (13) (maximal invariance algebra of the linear Schrödinger equation) and for whose solutions the continuity equation (1) is valid.

The necessary and sufficient condition for the Lorentz invariance of the continuity equation for the electromagnetic field, where energy density and Poiting vectors depend on the vector fields $\vec{E}$, $\vec{H}$ has been obtained in [6].

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