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The Asymptotic Solutions of the Systems of Nonlinear Differential Equations

Mykola SHKIL † *and Genadiy ZAVIZION* ‡

† *National Pedagogical University Ukraine, 9 Pyrogova Str., Kyiv, Ukraine*

‡ *Kirovograd State Pedagogical University, Ukraine*
E-mail: *zavizion@kspu.kr.ua*

The method of asymptotic integration of a singular perturbed nonlinear system of differential equations is offered.

In works [1, 2] the systems of singular perturbed linear differential equations were studied. The construction of asymptotic solution of nonlinear systems of differential equations were studied in works of W. Wasow, R. Langer, M. Iwano, A. Vasilieva, S. Lomov. In this work the method of asymptotic integration of the singular perturbed nonlinear system of differential equations is suggested.

Let us study the system of equations

$$\varepsilon\frac{dx}{dt} = A(t,\varepsilon)x + f(t,\varepsilon,x), \qquad x(0,\varepsilon) = x_0, \tag{1}$$

where ε $(0 < \varepsilon \le \varepsilon_0)$ is a small parameter, $f(t,\varepsilon,x)$, $x(t,\varepsilon)$, x_0 is n-dimensional vectors. We suppose to carry out such conditions:

1) vector $f(t,\varepsilon,x)$ has the decomposition into uniform convergent series

$$f(t,\varepsilon,x) = \sum_{|r|=2}^{\infty} a_r(t,\varepsilon)x^r, \tag{2}$$

where $a_r(t,\varepsilon)$ is n-dimensional vectors, $x^r = x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}$, $|r| = \sum\limits_{i=1}^{n} r_i$; x_i $(i = 1,\ldots,n)$, components of vector $x(t,\varepsilon)$;

2) the matrix $A(t,\varepsilon)$ and vectors $a_r(t,\varepsilon)$ have the decomposition using degrees of small parameters

$$A(t,\varepsilon) = \sum_{s=0}^{\infty}\varepsilon^s A_s(t), \qquad a_r(t) = \sum_{s=0}^{\infty}\varepsilon^s a_{rs}(t);$$

3) matrix $A_s(t)$ and vectors $a_{rs}(t)$ $(s = 0,1,\ldots)$ are infinite differentiable on the segment $[0;L]$;

4) solutions of characteristic equation

$$\det\|A_0(t) - \lambda(t)E\| = 0 \tag{3}$$

are simple on the segment $[0;L]$, where E is the identity matrix that has order n.

Let us use substitution into system (1)

$$x(t,\varepsilon) = U_m(t,\varepsilon)y(t,\varepsilon), \tag{4}$$

where $y(t,\varepsilon)$ is an n-dimensional vector, $U_m(t,\varepsilon)$ is an $n\times n$ matrix

$$U_m(t,\varepsilon) = \sum_{s=0}^{m}\varepsilon^s U_s(t)$$

the result

$$\varepsilon U_m(t,\varepsilon)y' = (A(t,\varepsilon)U_m(t,\varepsilon) - \varepsilon U_m^{'}(t,\varepsilon))y(t,\varepsilon) + f(t,\varepsilon,U_m(t,\varepsilon)y), \tag{5}$$

here $()'$ means the derivative with respect to t.

We will construct matrix $U_s(t)$, $(s = 0,\dots,m)$ in the way that the matrix equation takes place:

$$A(t,\varepsilon)U_m(t,\varepsilon) - \varepsilon U_m^{'}(t,\varepsilon) = U_m(t,\varepsilon)(\Lambda_m(t,\varepsilon) + \varepsilon_{m+1}C_m(t,\varepsilon)), \tag{6}$$

where $\Lambda_m(t,\varepsilon)$ is a diagonal matrix in form

$$\Lambda_m(t,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \Lambda_s(t),$$

$C_m(t,\varepsilon)$ is $n\times n$ matrix. Matrices $U_s(t)$, $\Lambda_s(t)$ $(s = 0,\dots,m)$ are obtained by using methods [1, 2] from equation (6). So, from system (1) we obtain system

$$\varepsilon y'(t,\varepsilon)y' = (\Lambda_m(t,\varepsilon)\varepsilon^{m+1}C_m(t,\varepsilon))y + U_m^{'}(t,\varepsilon)f(t,\varepsilon,U_m(t,\varepsilon)y), \tag{7}$$

Let us substitute $y(t,\varepsilon) = z + q(t,\varepsilon,z)$ to (7), where $q(t,\varepsilon,z)$ has the development

$$q(t,\varepsilon,z) = \sum_{|r|=2}^{\infty} q_r(t,\varepsilon)z^r.$$

So system (7) has form

$$\begin{aligned}\varepsilon z'(t,\varepsilon) = {} & (E + Q_z(t,\varepsilon,z))^{-1}\Big(- \varepsilon q'(t,\varepsilon,z) \\ & + (\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}C_m(t,\varepsilon))z + (\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}C_m(t,\varepsilon))q(t,\varepsilon,z)\Big) \\ & + U_m^{-1}(t,\varepsilon)f(t,\varepsilon,U_m(t,\varepsilon)(z + q(t,\varepsilon,z)),\end{aligned} \tag{8}$$

where $Q_z(t,\varepsilon,z)$ is matrix that consist of partial derivative components of vector $q(t,\varepsilon,z)$.

Let us choose vector $q(t,\varepsilon,z)$ in a way that

$$\begin{aligned}&(E + Q_z(t,\varepsilon,z))^{-1}\Big(- \varepsilon q'(t,\varepsilon,z) + \Lambda_m(t,\varepsilon)(z + q(t,\varepsilon,z))\Big) \\ &\quad + U_m^{-1}(t,\varepsilon)f(t,\varepsilon,U_m(t,\varepsilon)(z + q(t,\varepsilon,z))) = (\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}C_m(t,\varepsilon))z\end{aligned} \tag{9}$$

takes place. After multiplying (9) by matrix $E + Q_z(t,\varepsilon,z)$ and grouping similar terms with $\Lambda_m(t,\varepsilon)z$ we will obtain

$$\begin{aligned}\varepsilon q'(t,\varepsilon,z) = {} & (\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}C_m(t,\varepsilon))q(t,\varepsilon,z) \\ & + U_m^{-1}(t,\varepsilon)f(t,\varepsilon,U_m(t,\varepsilon)(z + q(t,\varepsilon,z)) - Q_z(t,\varepsilon,z)(\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}C_m(t,\varepsilon))z.\end{aligned} \tag{10}$$

Let us present matrix $Q_z(t,\varepsilon,z)$ in the form

$$Q_z(t,\varepsilon,z) = \sum_{|r|=2}^{\infty} z^r q_r(t,\varepsilon)r_z,$$

where $r_z = \left(r_1 z_1^{-1}, \dots, r_n z_n^{-1}\right)$. So, we have

$$Q_z(t,\varepsilon,z)\Lambda_m(t,\varepsilon)z = \sum_{|r|=2}^{\infty}\sum_{j=1}^{n} z^r q_r(t,\varepsilon) r_j \lambda_{mj}(t,\varepsilon),$$

$$Q_z(t,\varepsilon,z)C_m(t,\varepsilon)z = \sum_{|r|=2}^{\infty} \overline{g}_r(t,\varepsilon,z),$$

$$\overline{g}_r(t,\varepsilon,z) = z^r q_r(t,\varepsilon) r_z C_m(t,\varepsilon)z,$$

$$g(t,\varepsilon,z) = \left(U_m(t,\varepsilon)\left(z + \sum_{|s|=2}^{\infty} q_s(t,\varepsilon)z^s\right)\right)^r,$$

where $\lambda_{mj}(t,\varepsilon)$ are elements of the matrix $\lambda_m(t,\varepsilon)$. Decomposing functions $\overline{g}_r(t,\varepsilon,z)$, $g_r(t,\varepsilon,z)$ into power series, substituting to (10) and equating coefficient with similar degrees $z_1^{r_1}\cdots z_n^{r_n}$ we will obtain

$$\begin{aligned} \varepsilon q'(t,\varepsilon) &= \left(\Lambda_m(t,\varepsilon) - \sum_{j=1}^{n} r_j \lambda_{mj}(t,\varepsilon)\cdot E\right) q_r(t,\varepsilon) \\ &\quad + \varepsilon^{m+1} C_m(t,\varepsilon) q_r(t,\varepsilon) + V_r(t,\varepsilon,q_1,\dots,q_{r-1}) + \varepsilon^{m+1}\overline{V}_r(t,\varepsilon,q_1,\dots,q_{r-1}), \\ q_r(0,\varepsilon) &= 0, \end{aligned} \tag{11}$$

where $V_r(t,\varepsilon,q_1,\dots,q_{r-1})$, $\overline{V}_r(t,\varepsilon,q_1,\dots,q_{r-1})$ are expressed in terms of partial derivatives respectively to function $g_r(t,\varepsilon,z)$, $\overline{g}_r(t,\varepsilon,z)$.

System of equation has form

$$\begin{aligned} \varepsilon z'(t,\varepsilon) &= (\Lambda_m(t,\varepsilon) + \varepsilon^{m+1} C_m(t,\varepsilon))z, \\ z(0,\varepsilon) &= U_m^{-1}(0,\varepsilon)x_0 \end{aligned} \tag{12}$$

approximate [1] m (11) we will write in the form

$$q_{rm}(t,\varepsilon) = -\exp\left(\frac{1}{\varepsilon}\int_0^t \Lambda_m(t,\varepsilon)dt\right)\overline{q}_{rm}(0,\varepsilon) + \overline{q}_{rm}(t,\varepsilon),$$

where

$$\overline{q}_{rm}(t,\varepsilon) = \sum_{s=\beta}^{m} \varepsilon^s q_{rs}(t)$$

the particular solution (11); $\beta = 0$ or $\beta = -1$.

Approximate m (12) we will write in the form

$$z_m(t,\varepsilon) = \exp\left(\frac{1}{\varepsilon}\int_0^t \Lambda_m(t,\varepsilon)dt\right) U_m^{-1}(0,\varepsilon)x_0.$$

So, approximate m (1) takes the form

$$\begin{aligned} x_m(t,\varepsilon) &= U_m(t,\varepsilon)\exp\left(\frac{1}{\varepsilon}\int_0^t \Lambda_m(t,\varepsilon)dt\right) U_m^{-1}(0,\varepsilon)x_0 \\ &\quad + U_m(t,\varepsilon)\sum_{|r|=2}^{\infty}\left(\overline{q}_{rm}(t,\varepsilon) - \exp\left(\frac{1}{\varepsilon}\int_0^t \Lambda_m(t,\varepsilon)dt\right)\overline{q}_{rm}(0,\varepsilon)\right) \\ &\quad \times \left(\exp(\frac{1}{\varepsilon}\int_0^t \Lambda_m(t,\varepsilon)dt\right) U_m^{-1}(0,\varepsilon)x_0)^r. \end{aligned} \tag{13}$$

We proved that (13) consists of convergent series, and approximate m has an asymptotic property.

[1] Shkil N., The asymptotic methods in differential equations, Kyiv, 1971 (in Russian).

[2] Shkil N., Starun I. and Yakovets V., The asymptotic integration of the linear systems of differential equations, Kyiv, 1991.