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R-Matrix Approach to the Krall–Sheffer Problem

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The complete set of commuting invariants for integrable systems arising in the framework of the Krall–Sheffer problem is derived using the classical R-matrix approach, based on the loop algebra $\widetilde{sl}(2)_R$. The separating coordinates are also deduced from this framework.

1 Introduction

Krall and Sheffer studied the problem of finding all polynomial eigenfunctions of second order linear differential operators in two variables having polynomial coefficients of degree equal to the order of derivative under certain further restrictions relating to its symmetrizability and the orthogonality of its eigenfunctions (for details see [2]). They classified all possible normal forms of the operators satisfying the required properties. It was shown in [3] that all the operators in the Krall–Sheffer list are reducible by gauge transformations to the form of a Laplace–Beltrami operator on a space of constant curvature plus some potential, the magnetic field being absent. Moreover, they all are related to two-dimensional superintegrable systems on spaces of constant curvature [2].

In this paper we show how to construct a complete set of commuting invariants to the integrable systems arising in the Krall–Sheffer framework using the classical R-matrix approach, based on the loop algebra $\widetilde{sl}(2)_R$. We give both the quantum and classical formulations in terms of Lax matrices depending on a loop parameter. The main construction is based on the well-known procedure of symmetry reduction from a free system in a higher dimension space (in particular, quadrics in $\mathbb{R}^6$ or $\mathbb{C}^6$). Classically this corresponds to reduction of geodesic flow, while quantum mechanically it involves reduction of the Laplacian. The reduction process leaves a residue of the original system, providing a complete set of commuting integrals.

2 General construction scheme

We begin with a phase space $\mathbb{M}$ of $\dim \mathbb{M} = 12$, with canonical variables $(x_i, y_i)_{i=1,\ldots,6}$ which form the components of a pair (X, Y) of (either real or complex) column vector.

From these we form a Lax matrix $N(\lambda)$, depending on a spectral parameter $\lambda \in \mathbb{C}$ as follows:

$$N(\lambda) := \frac{1}{2}\left(Y^T, -X^T J\right)(\lambda - A)^{-1}(X, JY) = \sum_{i=1}^{n}\sum_{a=1}^{m_i} \frac{N_i^a}{(\lambda - \alpha_i)^a},$$

where A, J are fixed 6×6 matrices with A having either $n = 1, 2$ or 3 distinct eigenvalues $\{\alpha_i\}_{i=1,\dots,n}$ and minimal polynomial

$$\prod_{i=1}^{n} (\lambda - \alpha_i)^{m_i}$$

and J is a symmetric real matrix with antidiagonal blocks of the form

$$\begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & \dots & 1 & 0 \\ \dots & & & \\ 1 & \dots & & \end{pmatrix}$$

for each Jordan block of A.

The dynamics is generated by Hamiltonians chosen from the algebra of spectral invariants of $N(\lambda)$. Classically, these Poisson commute and hence generate isospectral flows satisfying a Lax equation:

$$\frac{dN}{dt} = [B, N] \, .$$

It is easily verified that $N(\lambda)$ satisfies the standard rational R-matrix Poisson bracket relations:

$$\{N(\lambda) \otimes N(\mu)\} = [r(\lambda), N(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes N(\mu)],$$

where both sides are viewed, for fixed $\lambda \neq \mu$ as elements of $\operatorname{End}\left(\mathbb{C}^6 \otimes \mathbb{C}^6\right)$ and

$$r(\lambda) = \frac{P_{1,2}}{(\lambda - \mu)}, \qquad P_{1,2}(u \otimes v) = v \otimes u.$$

In the cases considered below, we only study Hamiltonians that are $O(6, J)$ invariant and restrict to the quadric defined by

$$X^T J X = 1.$$

Quotienting by the stabilizer $G_A \subset O(6, J)$ of A we reduce to a 2-dimensional configuration space, however the reduced system is no longer free.

In this case the algebra of spectral invariants is generated by the coefficients of:

$$-\frac{1}{2} \operatorname{Tr} N(\lambda)^2 = \sum_{i=1}^{n} \sum_{d=1}^{2m_i} \frac{H_i}{(\lambda - \alpha_i)^d}$$

with $2m_i \leq n_i$. The numerators H_i of this partial fraction expansion all Poisson commute and generate the algebra of spectral invariants. They are not all independent, however, since:

$$\sum_{i=1}^{n} H_{id} = 0$$

and H_{id} with $m_i < d \leq 2m_i$ are Casimir invariants.

The connection between configuration space coordinates in 6-dimensional space and the separating coordinates λ_1, λ_2 in the reduced 2-dimensional space is given by

$$X^T J (\lambda - A)^{-1} X = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{a(\lambda)},$$

where $a(\lambda)$ is the minimal polynomial of the matrix A.

The quantum version of this approach is simply obtained through canonical quantization with conjugate (momentum) variables y_j replaced by the partial derivatives $i\,\partial/\partial x_j$. The relation between the quantum integrals and the ones in the corresponding Krall–Sheffer cases is obtained applying a suitable gauge transformation.

3 Case 1. Sphere. Neuman–Rosochatius system

In the case of a sphere in $\mathbb{R}^6$, the matrices A and J are just:

$$A = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{pmatrix}, \qquad J = id$$

with $\alpha \neq \beta \neq \gamma$. The symmetry algebra g_A corresponding to the stabilizer $G_A \subset O(6, \mathbb{R})$ is a maximal torus with generators

$$\{x_1y_2 - x_2y_1,\ x_3y_4 - x_4y_3,\ x_5y_6 - x_6y_5\}$$

and the Lax matrix has the form:

$$N(\lambda) = \frac{N_1}{(\lambda - \alpha)} + \frac{N_2}{(\lambda - \beta)} + \frac{N_3}{(\lambda - \gamma)} = \begin{pmatrix} h(\lambda) & f(\lambda) \\ e(\lambda) & -h(\lambda) \end{pmatrix},$$

where the N_i are elements of $sl(2)$

$$N_1 = \frac{1}{2}\begin{pmatrix} x_1y_1 + x_2y_2 & y_1^2 + y_2^2 \\ -x_1^2 - x_2^2 & -x_1y_1 - x_2y_2 \end{pmatrix},$$
$$N_2 = \frac{1}{2}\begin{pmatrix} x_3y_3 + x_4y_4 & y_3^2 + y_4^2 \\ -x_3^2 - x_4^2 & -x_3y_3 - x_4y_4 \end{pmatrix},$$
$$N_3 = \frac{1}{2}\begin{pmatrix} x_5y_5 + x_6y_6 & y_5^2 + y_6^2 \\ -x_5^2 - x_6^2 & -x_5y_5 - x_6y_6 \end{pmatrix}.$$

The invariants are the coefficients of:

$$-\frac{1}{2}\operatorname{Tr} N(\lambda)^2 = \frac{H_1}{(\lambda - \alpha)} + \frac{H_2}{(\lambda - \beta)} + \frac{H_3}{(\lambda - \gamma)} + \frac{\mu_1^2}{(\lambda - \alpha)^2} + \frac{\mu_2^2}{(\lambda - \beta)^2} + \frac{\mu_3^2}{(\lambda - \gamma)^2}.$$

Here μ_1, μ_2 and μ_3 are constants defining the restriction to level sets of invariants of motion under the reduction procedure (the components of the moment map generating the torus action), namely:

$$\mu_1 = x_1y_2 - x_2y_1, \qquad \mu_2 = x_3y_4 - x_4y_3, \qquad \mu_3 = x_5y_6 - x_6y_5.$$

Integrals H_1, H_2 and H_3 are not all independent, since their sum is equal to zero. The Hamiltonian of the problem is given by the linear combination:

$$H = \alpha H_1 + \beta H_2 + \gamma H_3.$$

The constraint to a sphere $\mathbb{S}^5 \subset \mathbb{R}^6$ is given by $X^T J X = 1$:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 1.$$

The reduced ambient coordinates are given by the radial distance in three planes (X_1, X_2), (X_3, X_4) and (X_5, X_6):

$$s_1^2 = x_1^2 + x_2^2, \qquad s_2^2 = x_3^2 + x_4^2, \qquad s_3^2 = x_5^2 + x_6^2.$$

The reduction of the constraint gives

$$s_1^2 + s_2^2 + s_3^2 = 1.$$

The reduced Hamiltonian is:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{\mu_1^2}{s_1^2} + \frac{\mu_2^2}{s_2^2} + \frac{\mu_3^2}{s_3^2}.$$

which is the kinetic energy on the sphere in $\mathbb{R}^3$ plus Rosochatius potential. Here (p_1, p_2, p_3) are canonical conjugate to (s_1, s_2, s_3).

The reduced separating coordinates (λ_1, λ_2) in this case are sphero-conical coordinates related to (s_1, s_2, s_3) by:

$$s_1^2 = \frac{(\alpha - \lambda_1)(\alpha - \lambda_2)}{(\alpha - \beta)(\alpha - \gamma)}, \qquad s_2^2 = \frac{(\beta - \lambda_1)(\beta - \lambda_2)}{(\beta - \alpha)(\beta - \gamma)}, \qquad s_3^2 = \frac{(\gamma - \lambda_1)(\gamma - \lambda_2)}{(\gamma - \alpha)(\gamma - \beta)}.$$

In terms of the reduced ambient space coordinates the integrals H_1, H_2 and H_3 are:

$$\begin{aligned}
H_1 &= -\frac{1}{2}\frac{L_{13}^2 + \mu_3^2 s_1^2/s_3^2 + \mu_1^2 s_3^2/s_1^2}{\alpha - \gamma} - \frac{1}{2}\frac{L_{12}^2 + \mu_1^2 s_2^2/s_1^2 + \mu_2^2 s_1^2/s_2^2}{\alpha - \beta},\\
H_2 &= -\frac{1}{2}\frac{L_{23}^2 + \mu_3^2 s_2^2/s_3^2 + \mu_2^2 s_3^2/s_2^2}{\beta - \gamma} + \frac{1}{2}\frac{L_{12}^2 + \mu_1^2 s_2^2/s_1^2 + \mu_2^2 s_1^2/s_2^2}{\alpha - \beta},\\
H_3 &= \frac{1}{2}\frac{L_{23}^2 + \mu_3^2 s_2^2/s_3^2 + \mu_2^2 s_3^2/s_2^2}{\beta - \gamma} + \frac{1}{2}\frac{L_{13}^2 + \mu_3^2 s_1^2/s_3^2 + \mu_1^2 s_3^2/s_1^2}{\alpha - \gamma},
\end{aligned}$$

where $L_{ij} = s_1 p_2 - s_2 p_1$. The quantum versions of these integrals are denoted by $(\hat{H}_1, \hat{H}_2, \hat{H}_3)$ and are obtained by replacing the matrix elements of $N(\lambda)$ by the corresponding differential operators. This leads to replacing L_{ij} by their quantum version:

$$\hat{L}_{ij} = \sqrt{-1}(s_i \partial/\partial s_j - s_j \partial/\partial s_i).$$

Note that whereas the Hamiltonian H is independent of the parameters (α, β, γ), which only serve to determine the separating coordinate system, the invariants H_1, H_2 individually do depend on those. Therefore, different choices for these parameters give distinct integrals that commute with H, but do not commute with each other. This provides an explanation for the superintegrability of this system.

To relate the invariants to the ones obtained in [2] for the corresponding Krall–Sheffer case we apply the gauge transformation consisting of conjugation by the function:

$$\Phi = x^{d_1} y^{d_2} (1 - x - y)^{d_3},$$

where

$$d_1 = \frac{1}{2}(d_{00} + 1/2), \qquad d_2 = \frac{1}{2}(e_{00} + 1/2), \qquad d_3 = \frac{1}{2}(1/2 - d_{00} - e_{00} - B)$$

and d_{00}, e_{00}, B are the parameters appearing in Krall–Sheffer setting (see [2]).

The following are the relations between the integrals constructed in these two approaches:

$$\tilde{H}_1 = 4\frac{\alpha_1 - \gamma_1}{\beta_1 - \gamma_1}\hat{I}_x + 4\hat{I}_y - 4\hat{L} - c_0, \qquad \tilde{H}_2 = 4\frac{\gamma_1 - \beta_1}{\gamma_1 - \alpha_1}\hat{I}_y + 4\hat{I}_x - 4\hat{L} - c_1,$$

where $\tilde{H}_i = \Phi \hat{H} \Phi^{-1}$ and $\hat{L}$ is the Krall–Sheffer operator corresponding to case I, c_0 and c_1 depend on α, β, γ, d_{00}, e_{00}, B.

4 Case 2. Hyperboloid

For the case of a hyperboloid embedded in $\mathbb{R}^6$, matrices (A, J) may be taken as

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that J has an antidiagonal block corresponding to each Jordan block of A and a diagonal block corresponding to the diagonal part of A.

The symmetry algebra g_A again has three generators

$$\{x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4,\ x_2y_1 - x_4y_3,\ x_5y_6 - x_6y_5\}$$

but the Lax matrix now has a second order pole at $\lambda = \alpha$:

$$N(\lambda) = \frac{N_1}{(\lambda - \alpha)} + \frac{N_2}{(\lambda - \alpha)^2} + \frac{N_3}{(\lambda - \beta)},$$

where

$$N_1 = \frac{1}{2}\begin{pmatrix} x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 & 2y_1y_4 + 2y_2y_3 \\ -2x_1x_4 - 2x_2x_3 & -x_1y_1 - x_2y_2 1 x_3y_3 - x_4y_4 \end{pmatrix},$$

$$N_2 = \frac{1}{2}\begin{pmatrix} -x_4y_3 + x_2y_1 & -2y_3y_1 \\ 2x_2x_4 & -x_2y_1 + x_4y_3 \end{pmatrix},$$

$$N_3 = \frac{1}{2}\begin{pmatrix} x_5y_5 + x_6y_6 & y_5^2 + y_6^2 \\ -x_5^2 - x_6^2 & -x_5y_5 - x_6y_6 \end{pmatrix}.$$

Here (N_1, N_2) should be viewed as an element of the jet extension $sl(2)^{(1)*}$ while $N_3 \in sl(2)$. The invariants again give us only two independent H_1 and H_2

$$-\frac{1}{2}\operatorname{Tr} N(\lambda)^2 = \frac{H_1}{(\lambda - \alpha)} + \frac{H_2}{(\lambda - \alpha)^2} - \frac{\mu_1\mu_2}{(\lambda - \alpha)^3} + \frac{\mu_2^2}{2(\lambda - \alpha)^4} + \frac{H_3}{(\lambda - \beta)} - \frac{\mu_3^2}{2(\lambda - \beta)^2},$$

where

$$H_1 + H_3 = 0.$$

The Hamiltonian is now defined by:

$$H = (\alpha - \beta)H_1 + H_2 - \frac{1}{2}\mu_3^2.$$

The reduced ambient space coordinates (s_1, s_2, s_3) are now defined by:

$$s_1^2 = \frac{(x_1x_4 + x_2x_3)^2}{2x_2x_4}, \qquad s_2^2 = 2x_2x_4, \qquad s_3^2 = x_5^2 + x_6^2.$$

The constraint to the quadric $X^TJX = 1$ reduces to define a hyperboloid in $\mathbb{R}^3$

$$2s_1s_2 + s_3^2 = 1.$$

In these coordinates the integrals H_1 and H_2 are

$$H_1 = \frac{(s_1p_3 - s_3p_2)(s_3p_1 - s_2p_3) - \mu_3^2 s_1 s_2/s_3^2 + \mu_1\mu_2 s_3^2/s_2^2 - \mu_2^2 s_1 s_3^2/s_2^2}{\alpha - \beta}$$
$$- \frac{(s_3p_1 - s_2p_3)^2 + \mu_3^2 s_2^2/s_3^2 - \mu_2^2 s_3^2/s_2^2}{2(\alpha-\beta)^2},$$
$$H_2 = \frac{1}{2}(s_1p_1 - s_2p_2)^2 - 2\frac{\mu_2^2 s_1^2}{s_2^2} + 2\frac{\mu_1\mu_2 s_1}{s_2} + \frac{(s_3p_1 - s_2p_3)^2 + \mu_3^2 s_2^2/s_3^2 - \mu_2^2 s_3^2/s_2^2}{2(\alpha-\beta)}.$$

The quantized operators $\hat{H}_1$, $\hat{H}_2$, $\hat{H}_3$ are obtained as before by replacing all conjugate variables by corresponding differential operators. And again, whereas Hamiltonian H does depend on the parameters (α, β) the integrals H_1, H_2 do, thereby again providing an explanation for the superintegrability in this case.

5 Case 3. Pseudoeuclidean plane

Matrix A in this case has only one degenerate eigenvalue:

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 0 & \alpha \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

J is antidiagonal.

The symmetry algebra g_A is generated by

$$\{-x_1y_4 - x_2y_5 - x_3y_6 + x_4y_1 + x_5y_2 + x_6y_3,\ x_6y_1 - x_3y_4,\ -x_2y_4 - x_3y_5 + x_5y_1 + x_6y_2\}$$

and the Lax matrix is of the form:

$$N(\lambda) = \frac{N_1}{(\lambda-\alpha)} + \frac{N_2}{(\lambda-\alpha)^2} + \frac{N_3}{(\lambda-\alpha)^3},$$

where

$$N_1 = \frac{1}{2}\begin{pmatrix} \begin{matrix} x_1y_1 + x_2y_2 + x_3y_3 \\ + x_4y_4 + x_5y_5 + x_6y_6 \end{matrix} & 2y_1y_3 + y_2^2 + 2y_4y_6 + y_5^2 \\ -2x_1x_3 - x_2^2 - 2x_4x_6 - x_5^2 & \begin{matrix} -x_1y_1 - x_2y_2 - x_3y_3 \\ - x_4y_4 - x_5y_5 - x_6y_6 \end{matrix} \end{pmatrix},$$
$$N_2 = \frac{1}{2}\begin{pmatrix} -x_3y_2 - x_2y_1 - x_6y_5 - x_5y_4 & -2y_2y_1 - 2y_4y_5 \\ 2x_2x_3 + 2x_5x_6 & x_3y_2 + x_2y_1 + x_6y_5 + x_5y_4 \end{pmatrix},$$
$$N_3 = \frac{1}{2}\begin{pmatrix} x_3y_1 + x_6y_4 & y_1^2 + y_4^2 \\ -x_3^2 - x_5^2 & -x_3y_1 - x_6y_4 \end{pmatrix}.$$

The trace formula again gives only two independent integrals H_1 and H_2

$$-\frac{1}{2}\operatorname{Tr} N(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)^2} + \frac{H_2}{(\lambda-\alpha)^3} - \frac{2\mu_1\mu_2 - \mu_3^2}{2(\lambda-\alpha)^4} + \frac{\mu_2\mu_3}{2(\lambda-\alpha)^5} - \frac{\mu_2^2}{2(\lambda-\alpha)^6}.$$

The Hamiltonian of the problem is:

$$H = -2p_1p_3 - p_2^2 + 2\gamma_1\gamma_3 + \gamma_2^2,$$
$$\gamma_1 = \frac{\mu_1}{s_1} - \frac{\mu_2 s_2}{s_1^2} - \frac{\mu_3 s_2^2}{s_1^2} - \frac{\mu_3 s_3}{s_1^2}, \qquad \gamma_2 = \frac{\mu_2}{s_1} - \frac{\mu_3 s_2}{s_1^2}, \qquad \gamma_3 = \frac{\mu_3}{s_1}.$$

In this case the parameter α may be absorbed in the definition of λ and therefore no parameter dependence appears in the integrals H_1 and H_2:

$$H_1 = (p_2s_3 - s_2p_1)(s_1p_1 - s_3p_3) - 2s_2s_3\left(p_2^2 + 2p_1p_3\right) - \frac{\mu_1\mu_2}{s_1^2} - \frac{3\mu_3\mu_2 s_3}{s_1^3}$$
$$- \frac{\mu_3\mu_2 s_1}{s_2^2} - \frac{4s_3\mu_3\mu_1(1-2s_1s_2)}{s_1^4} - \frac{\mu_3 s_2^2}{s_1^2} - \frac{(\mu_2^2 + \mu_3\mu_1)s_2}{s_1^3},$$
$$H_2 = \left(p_2^2 + 2p_1p_3\right)\left(s_2^2 + 2s_1s_3\right) + \frac{2\mu_3^2 s_1}{s_2^2} + \frac{4\mu_3^2 s_3^2}{s_1^2}$$
$$+ \frac{4\mu_3\mu_2 s_2}{s_1^3} - \frac{\mu_2^2 - 2\mu_3\mu_1}{s_1^2} + \frac{\mu_3^2\left(1-2s_2^2\right)}{s_1^4}.$$

Reduced coordinates in $\mathbb{R}^3$

$$s_1^2 = -\frac{(x_1x_3 + x_4x_6)^2}{x_3^2 + x_6^2}, \qquad s_2^2 = x_2^2 + x_5^2, \qquad s_3^2 = -\left(x_3^2 + x_6^2\right).$$

The constraint to the quadric $X^TJX = 1$ reduces to $2s_1s_3 + s_2^2 = 1$.

6 Conclusions

The approach based on Lax matrices satisfying the rational R-matrix structure gives a systematic way to derive the Hamiltonians and commuting invariants for these three cases corresponding to Krall–Sheffer operators on quadrics. This also provides a prescription for the separating coordinates, both in the classical and quantum cases. The presence of the additional parameters (α, β, γ) in the Case I, and (α, β) in the case II provides an explanation for their superintegrability.

A similar analysis may be made for the cases of Euclidean space arised in the Krall–Sheffer problem, they may be obtained as limiting cases of the above, providing an R-matrix approach to the remaining Krall–Sheffer operators. The details for all these cases will be provided elsewhere.

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