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Differential Invariants and Construction of Conditionally Invariant Equations

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New concept of conditional differential invariant is discussed that would allow description of equations invariant with respect to an operator under a certain condition. Example of conditional invariants of the projective operator is presented.

1 Introduction

Importance of investigation of symmetry properties of differential equations is well-established in mathematical physics. Classical methods for studying symmetry properties and their utilisation for finding solutions of partial differential equations were originated in the papers by S. Lie, and developed by modern authors (see e.g. [1, 2, 3, 4]).

We start our consideration from some symmetry properties and solutions of the nonlinear wave equation

$$\Box u = F(u, u^*) \tag{1}$$

for the complex-valued function $u = u(x_0, x_1, \ldots, x_n)$, $x_0 = t$ is the time variable, $x_1, \ldots, x_n$ are n space variables. F is some function. $\Box u$ is the d'Alembert operator

$$\Box u = -\frac{\partial^2 u}{\partial x_0^2} + \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}. \tag{2}$$

It is well-known that the equation (1) may be reduced to a nonlinear Schrödinger equation with the number of space dimensions smaller by 1, when the nonlinearity F has a special form $F = uf(|u|)$, where $|u| = (uu^*)^{1/2}$, an asterisk designates complex conjugation.

Further we are trying to generalise this relation between the nonlinear wave equation and the nonlinear Schrödinger equation into a relation between differential invariants of the respective invariance algebrae, and introduce new concepts of the reduction of fundamental sets of differential invariants and of conditional differential invariants. Conditional differential invariants may be utilised to describe conditionally invariant equations under certain operators and with the certain conditions, in the same manner as absolute differential invariants of a Lie algebra may be utilised for description of all equations invariant under this algebra.

The concept of non-classical, or conditional symmetry, originated in its various facets in the papers [5, 6, 7, 8, 9, 10] and later by numerous authors was developed into the theory and a number of algorithms for studying symmetry properties of equations of mathematical physics and for construction of their exact solutions. Here we will use the following definition of the conditional symmetry:

Definition 1. The equation $F(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) = 0$ where $\underset{k}{u}$ is the set of all kth-order partial derivatives of the function $u = (u^1, u^2, \ldots, u^m)$, is called conditionally invariant under the operator

$$Q = \xi^i(x, u)\partial_{x_i} + \eta^r(x, u)\partial_{u^r} \tag{3}$$

if there is an additional condition

$$G(x, u, \underset{1}{u}, \ldots, \underset{l_1}{u}) = 0, \tag{4}$$

such that the system of two equations $F = 0$, $G = O$ is invariant under the operator Q.

If (4) has the form $G = Qu$, then the equation $F = 0$ is called Q-conditionally invariant under the operator Q.

2 Differential invariants and description of invariant equations

Differential invariants of Lie algebrae present a powerful tool for studying partial differential equations and construction of their solutions [21, 22, 23].

Now we will present some basic definitions that we will further generalise. For the purpose of these definitions we deal with Lie algebrae consisting of the infinitesimal operators

$$X = \xi^i(x, u)\partial_{x_i} + \eta^r(x, u)\partial_{u^r}. \tag{5}$$

Here $x = (x_1, x_2, \ldots, x_n)$, $u = (u^1, u^2, \ldots, u^m)$.

Definition 2. The function $F = F(x, u, \underset{1}{u}, \ldots, \underset{l}{u})$, is called a differential invariant for the Lie algebra L with basis elements X_i of the form (5) ($L = \langle X_i \rangle$) if it is an invariant of the lth prolongation of this algebra:

$$\underset{l}{X_s} F(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) = \lambda_s(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) F, \tag{6}$$

where the λ_s are some functions; when $\lambda_i = 0$, F is called an absolute invariant; when $\lambda_i \neq 0$, it is a relative invariant.

Further when writing "differential invariant" we would imply "absolute differential invariant".

Definition 3. A maximal set of functionally independent invariants of order $r \leq l$ of the Lie algebra L is called a functional basis of the lth-order differential invariants for the algebra L.

While writing out lists of invariants we shall use the following designations

$$\begin{aligned} &u_a \equiv \frac{\partial u}{\partial x_a}, \qquad u_{ab} \equiv \frac{\partial^2 u}{\partial x_a \partial x_b}, \qquad S_k(u_{ab}) \equiv u_{a_1 a_2} u_{a_2 a_3} \cdots u_{a_{k-1} a_k} u_{a_k a_1}, \\ &S_{jk}(u_{ab}, v_{ab}) \equiv u_{a_1 a_2} \cdots u_{a_{j-1} a_j} v_{a_j a_{j+1}} \cdots v_{a_k a_1}, \\ &R_k(u_a, u_{ab}) \equiv u_{a_1} u_{a_k} u_{a_1 a_2} u_{a_2 a_3} \cdots u_{a_{k-1} a_k}. \end{aligned} \tag{7}$$

In all the lists of invariants j takes the values from 0 to k. We shall not discern the upper and lower indices with respect to summation: for all Latin indices $x_a x_a \equiv x_a x^a \equiv x^a x_a = x_1^2 + x_2^2 + \cdots + x_n^2$.

Fundamental bases of differential invariants for the standard scalar representations of the Poincaré and Galilei algebra of the types (17), (12) were found in [24]. Fundamental bases of differential invariants allow describing all equations invariant under the respective Lie algebrae.

Construction of conditional differential invariants would allow describing all equations, conditionally invariant with respect to certain operators under certain conditions.

Definition 4. $F = F(x, u, \underset{1}{u}, \ldots, \underset{l}{u})$ is called a conditional differential invariant for the operator with X of the form (5) if under the condition

$$G(x, u, \underset{1}{u}, \ldots, \underset{l_1}{u}) = 0, \tag{8}$$

$$\underset{l_{\max}}{X} F(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) = 0, \qquad \underset{l_{\max}}{X} G(x, u, \underset{1}{u}, \ldots, \underset{l_1}{u}) = 0, \tag{9}$$

$\underset{l_{\max}}{X}$ being the $l_{\max}$th prolongation of the operator X. The order of the prolongation $l_{\max} = \max(l, l_1)$.

3 Nonlinear wave equation, nonlinear Schrödinger equation and relation between their symmetries

The Galilei algebra for $n-1$ space dimensions is a subalgebra of the Poincaré algebra for n space dimensions (see e.g. [11]) and references therein), and this fact allows reduction of the nonlinear wave equation (1) to the Schrödinger equation. We will consider the nonlinear wave equations for three space variables, and its symmetry properties in relation to the symmetry properties of the nonlinear Schrödinger equation for two space variables. However, all the results can be easily generalised for arbitrary number of space dimensions.

Reduction of the nonlinear wave equation (1) to the Schrödinger equation can be performed by means of the ansatz

$$u = \exp((-im/2)(x_0 + x_3))\Phi(x_0 - x_3, x_1, x_2). \tag{10}$$

Substitution of the expression (10) into (1) gives the equation $\exp((-im/2)(x_0+x_3))(2im\Phi_\tau + \Phi_{11} + \Phi_{22}) = F(u, u^*)$. Here we adopted the following notations: $\tau = x_0 + x_3$ is the new time variable, $\Phi_\tau = \frac{\partial\Phi}{\partial\tau}$, $\Phi_a = \frac{\partial\Phi}{\partial x_a}$, $\Phi_{ab} = \frac{\partial^2\Phi}{\partial x_a \partial x_b}$.

Further on we adopt the convention that summation is implied over the repeated indices. If not stated otherwise, small Latin indices run from 1 to 2.

If the nonlinearity in the equation (1) has the form $F = uf(|u|)$, then it reduces to the Schrödinger equation

$$2im\Phi_\tau + \Phi_{11} + \Phi_{22} = \Phi f(|\Phi|). \tag{11}$$

Such reduction allowed construction of numerous new solutions for the nonlinear wave equation by means of the solutions of a nonlinear Schrödinger equation [12, 13]. We show that this reduction allowed also to describe additional symmetry properties for the equation (1), related to the symmetry properties of the equation (11).

Lie symmetry of the equation (11) was described in [14, 16]. With an arbitrary function f it is invariant under the Galilei algebra with basis operators

$$\begin{gathered}\partial_\tau = \frac{\partial}{\partial\tau}, \qquad \partial_a = \frac{\partial}{\partial x_a}, \qquad J_{12} = x_1\partial_2 - x_2\partial_1, \\ G_a = t\partial_a + ix_a(\Phi\partial_\Phi - \Phi^*\partial_{\Phi^*}) \quad (a = 1, 2), \qquad J = (\Phi\partial_\Phi - \Phi^*\partial_{\Phi^*}).\end{gathered} \tag{12}$$

When $f = \lambda|u|^2$, where λ is an arbitrary constant, the equation (11) is invariant under the extended Galilei algebra that contains besides the operators (12) also the dilation operator

$$D = 2\tau\partial_\tau + x_a\partial_a - I, \tag{13}$$

where $I = \Phi\partial_\Phi + \Phi^*\partial_{\Phi^*}$, and the projective operator

$$A = \tau^2\partial_\tau + \tau x_a\partial_a + \frac{im}{2}x_a x_a J - \tau I. \tag{14}$$

Lie reductions and families of exact solutions for multidimensional nonlinear Schrödinger equations were found at [15, 16, 17, 18, 19, 20]. Note that the ansatz (10) is the general solution of the equation

$$u_0 + u_3 + imu = 0. \tag{15}$$

We can regard the equation (15) as the additional condition imposed on the nonlinear wave equation with the nonlinearity $F = \lambda u|u|^2$. Solution of the resulting system

$$\Box u = \lambda u|u|^2, \tag{16}$$

with the equation (15) would allow to reduce number of independent variables by one, and obtain the same reduced equation, invariant under the extended Galilei algebra with the projective operator. This allows establishing conditional invariance of the nonlinear wave equation (16) under the projective operator. It is well-known that it is not invariant under this operator in the Lie sense.

The maximal invariance algebra of the equation (1) that may be found according to the Lie algorithm (see e.g. [1, 2, 3, 4]) is defined by the following basis operators:

$$p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{17}$$

where μ, ν take the values $0, 1, \dots, 3$; the summation is implied over the repeated indices (if they are small Greek letters) in the following way: $x_\nu x_\nu \equiv x_\nu x^\nu \equiv x^\nu x_\nu = x_0^2 - x_1^2 - \cdots - x_n^2$, $g_{\mu\nu} = \mathrm{diag}\,(1, -1, \dots, -1)$.

However, summation for all derivatives of the function u is assumed as follows: $u_\nu u_\nu \equiv u_\nu u^\nu \equiv u^\nu u_\nu = -u_0^2 + u_1^2 + \cdots + u_n^2$.

Unlike the standard convention on summation of the repeated upper and lower indices we consider x_ν and x^ν equal with respect to summation not to mix signs of derivatives and numbers of functions.

Theorem 1. *The nonlinear wave equation (16) is conditionally invariant with the condition (15) under the projective operator*

$$\begin{aligned} A_1 = {} & \frac{1}{2}(x_0 - x_3)^2(\partial_0 - \partial_3) + (x_0 - x_3)(x_1\partial_1 + x_2\partial_2) \\ & + \frac{imx^2}{2}(u\partial_u - u^*\partial_{u^*}) + \frac{n-1}{2}(x_0 - x_3)(u\partial_u + u^*\partial_{u^*}). \end{aligned} \tag{18}$$

To prove Theorem 1 it is sufficient to show that the system (16), (15) is invariant under the operator (18) by means of the classical Lie algorithm.

Our further study aims at construction of other Poincaré-invariant equations possessing the same conditional invariance property.

4 Example: construction of conditional differential invariants

Now we adduce fundamental bases of differential invariants that will be utilised for construction of our example of conditional differential invariants.

First we present a functional basis of differential invariants for the Poincaré algebra (17) of the second order for the complex-valued scalar function $u = u(x_0, x_1, \dots, x_3)$. It consists of 24 invariants

$$u^r, \qquad R_k\left(u^r_\mu, u^1_{\mu\nu}\right), \qquad S_{jk}\left(u^r_{\mu\nu}, u^1_{\mu\nu}\right). \tag{19}$$

In (19) everywhere $k = 1, \dots, 4$; $j = 0, \dots, k$. A functional basis of differential invariants for the Galilei algebra (12), mass $m \neq 0$, of the second order for the complex-valued scalar function $\Phi = \Phi(\tau, x_1, \dots, x_2)$ consists of 16 invariants.

For simplification of the expressions for differential invariants we introduced the following notations:

$$\Phi = \exp\phi, \qquad \operatorname{Im}\Phi = \arctan\frac{\operatorname{Re}\phi}{\operatorname{Im}\phi}.$$

The elements of the functional basis may be chosen as follows:

$$\begin{aligned}
&\phi+\phi^*, \quad M_1 = 2im\phi_t + \phi_a\phi_a, \quad M_1^*, \quad M_2 = -m^2\phi_{tt} + 2im\phi_a\phi_{at} + \phi_a\phi_b\phi_{ab}, \quad M_2^*,\\
&S_{jk}(\phi_{ab},\phi_{ab}^*), \quad R_j^1 = R_j(\theta_a,\phi_{ab}), \quad R_j^2 = R_j(\theta_a^*,\phi_{ab}), \quad R_j^3 = R_j(\phi_a+\phi_a^*,\phi_{ab}) \qquad (20)
\end{aligned}$$

Here $\theta_a = im\phi_{at} + \phi_a\phi_{ab}, \phi_{ab}$ are covariant tensors for the Galilei algebra.

A functional basis of differential invariants for the Galilei algebra (12) extended by the dilation operator (13) and the projective operator (14) may be chosen as follows:

$$\begin{aligned}
&N_1 e^{-2(\phi+\phi^*)}, \quad \frac{N_1}{N_1^*}, \quad \frac{N_2}{N_1^2}, \quad \frac{N_2^*}{(N_1^*)^2}, \quad S_{jk}(\rho_{ab},\rho_{ab}^*), \quad R_j(\rho_a,\rho_{ab}),\\
&R_j(\rho_a^*,\rho_{ab}), \quad R_j(\phi_a+\phi_a^*,\rho_{ab})N_1^{-1}, \quad (\phi_{aa}+\phi_{aa}^*)N_1^{-1}, \qquad (21)
\end{aligned}$$

where

$$N_1 = M_1 + \phi_{aa} = 2im\phi_t + \phi_{aa} + \phi_a\phi_a, \qquad N_2 = \frac{1}{n}\phi_{aa}N_1 + \frac{\phi_{aa}^2}{2n} + M_2 \qquad (22)$$

and the covariant tensors have the form

$$\rho_a = \theta_a N_1^{-3/2}, \qquad \rho_{ab} = \left(\phi_{ab} - \frac{\delta_{ab}}{n}\phi_{cc}\right)N_1^{-1}.$$

An algorithm for construction of conditional differential invariants may be derived directly from the Definition 4. Such invariants may be found by means of the solution of the system (9), (8).

We can construct conditional differential invariants of the Poincaré algebra (17) and the projective operator (18) solving the system

$$\underset{2}{A_1}F(\mathrm{Inv}_P) = 0, \qquad u_0 + u_3 + imu = 0,$$

where Inv_P are all differential invariants (19) of the Poincaré algebra (17). Using the fact that the ansatz (10) is the general solution of the additional condition (15), we can directly substitute this ansatz into differential invariants (19). To avoid cumbersome formulae here we did not list expressions for all differential invariants from (19).

The expression $\Box u$ transforms into the following:

$$\Box u = u(2im\phi_\tau + \phi_{aa} + \phi_a\phi_a),$$

where N_1 is an expression entering into expression for differential invariants (20). Further we get

$$\begin{aligned}
u_\mu u_\mu &= u^2(2im\phi_t + \phi_a\phi_a),\\
u_\mu u_\nu u_{\mu\nu} &= u^3(\phi_a\phi_b\phi_{ab} + (\phi_a\phi_a)^2 - m^2(\phi_{tt} + 4\phi_t^2) + \phi_a\phi_b\phi_{ab} + (\phi_a\phi_a)^2\\
&\quad - m^2(\phi_{tt} + 4\phi_t^2) + 2im\phi_a\phi_{at} + 4im\phi_t\phi_a\phi_a), \qquad (23)
\end{aligned}$$

Substituting the ansatz (10) to all elements of the fundamental basis (19) of second-order differential invariants of the Poincaré algebra similarly to (23), we can obtain reduced basis of differential invariants, that may be used for construction of all equations reducible by means of this ansatz. We can give the following representation of the Poincaré invariants using expressions M_k (20) and N_k (21), where in the expressions for M_k, N_k ($k = 1, 2$) time variable is $\tau = x_0 - x_3$:

$$\begin{aligned}
&\Box u = uN_1, \quad u_\mu u_\mu = u^2 M_1, \quad u_\mu u_\nu u_{\mu\nu} = u^3 \left(M_2 + M_1^2\right), \\
&u_{\mu\nu} u_{\mu\nu} = u^2 \left(2M_2 + M_1^2 + \phi_{ab}\phi_{ab}\right), \\
&u_\mu u_\mu^* = \frac{uu^*}{2} \left(M_1 + M_1^* - (\phi_a + \phi_a^*)(\phi_a + \phi_a^*)\right).
\end{aligned} \tag{24}$$

Here a, b take values from 1 to 2.

Whence

$$\begin{aligned}
&M_1 = u_\mu u_\mu u^{-2}, \quad \phi_{aa} = N_1 - M_1 = \frac{u\Box u - u_\mu u_\mu}{u^2}, \\
&M_2 = u_\mu u_\nu u_{\mu\nu} u^{-3} - (u_\mu u_\mu)^2 u^{-4}, \quad N_1 = \frac{\Box u}{u}, \\
&N_2 = \frac{1}{n}\phi_{aa} N_1 + \frac{\phi_{aa}^2}{2n} + M_2 = u_\mu u_\nu u_{\mu\nu} u^{-3} - (u_\mu u_\mu)^2 u^{-4} \\
&\qquad + \frac{1}{n}\frac{\Box u}{u}\frac{u\Box u - u_\mu u_\mu}{u^2} + \frac{1}{2n}\frac{(u\Box u - u_\mu u_\mu)^2}{u^4}, \\
&R_1(\phi_a + \phi_a^*, \rho_{ab}) N_1^{-1} = (\phi_a + \phi_a^*)(\phi_a + \phi_a^*) N_1^{-1} \\
&\qquad = \left(N_1 + N_1^* - \frac{2}{uu^*} u_\mu u_\mu^*\right) N_1^{-1} = \frac{u^*\Box u + u\Box u^* - 2u_\mu u_\mu^*}{u^*\Box u}.
\end{aligned} \tag{25}$$

We construct Poincaré-invariant conditional differential invariants of the projective operator (18) under the condition (15) using differential invariants (20)

$$\begin{aligned}
&I_1 = N_1 e^{-2(\phi + \phi^*)} = \frac{\Box u}{u(uu^*)^2}, \quad I_2 = \frac{N_1}{N_1^*} = \frac{u^*\Box u}{u\Box u^*}, \\
&I_3 = \frac{N_2}{N_1^2} = \left(u u_\mu u_\nu u_{\mu\nu} + \frac{3}{2n} u^2 (\Box u)^2 + \left(\frac{1}{2n} - 1\right)(u_\mu u_\mu)^2 - \frac{2}{n} u\Box u (u_\mu u_\mu)\right)\left(u^2 (\Box u)^2\right)^{-1}, \\
&I_4 = R_1(\phi_a + \phi_a^*, \rho_{ab}) N_1^{-1} = \frac{u^*\Box u + u\Box u^* - 2u_\mu u_\mu^*}{u^*\Box u}.
\end{aligned} \tag{26}$$

Whence, we may state that all equations of the form $F(I_1, I_2, I_3, I_4) = 0$ are conditionally invariant with respect to the operator A_1 (18) with the additional condition (15).

Finding similar representations for all elements of the functional basis (20) of the second-order differential invariants of the Galilei algebra (12) extended by the dilation operator (13) and the projective operator (14), we can construct functional basis of conditional differential operators. Such basis would allow to describe all Poincaré-invariant equations for the scalar complex-valued functions that are conditionally invariant under the operator A_1 (18).

5 Conclusion

The procedure for finding conditional differential invariants outlined above may be used for other cases when the additional condition (8) has the general solution that may be used as ansatz, and when a functional basis of the operator (9) in the variables involved in such reduction is already known.

Besides finding new conditionally invariant equations, further developments of the ideas presented in this paper may be description of all equations reducible by means of a certain ansatz, and search of methods for restoration of original equations from the reduced equations.

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