# Integrable Polynomial Potentials in $N$-Body Problems on the Line

*Andrij VUS*

*Dept. of Mathematics and Mechanics, 1 Universytetska Str., Lviv 79000, Ukraine*
E-mail: *matmod@franko.lviv.ua*

Integrable natural systems of $n$ interacting particles on the line are investigated under assumption that the interacting potential is a polynomial. Restriction for degree of these potentials is obtained both for systems with pairwise interaction and for the case of lattices.

Dynamics of $n$ equal pair-interactive particles on the line is described by the Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i<j} V(x_i - x_j), \tag{1}$$

where the $x_i$ and $p_i$, $i = 1, \ldots, n$, are the coordinates and momenta of the particles. We henceforth call the function $V$ a potential. Complete integrability of this system was established in [1, 2] for the Weierstrass $\mathcal{P}$-function as the interaction potential. Moreover, this system possesses a complete collection of integrals which are polynomials in the momenta and are in involution. It is therefore natural to obtain a description of Hamiltonians (1) which admit integrals that are polynomials in the momenta. We are interested in considering the problem of integrability of such natural system of interacting particles in Liouville's sense for polynomial potential $V(z)$, such that $\deg V(z) = k > 2$.

**Theorem 1.** *Let the potential $V(z)$ admit an integral $F$, which is polynomial in the momenta. Then the potential $z^k$ admits a nontrivial integral, which is also polynomial.*

**Theorem 2.** *The 3-body problem with the Hamiltonian (1) is integrable if and only if $k \leqslant 4$.*

**Proof.** The total momentum $P = \sum p_i$ is the first integral of the system under consideration. Therefore this system can be reduced to the system with two degrees of freedom and the Hamiltonian

$$H = \frac{1}{2} \left( p_1^2 + p_2^2 \right) + V(x) + V\left( -\frac{x}{2} + \frac{y\sqrt{3}}{2} \right) + V\left( -\frac{x}{2} - \frac{y\sqrt{3}}{2} \right).$$

Now we shall use the Yoshida's theorem [3] on the nonintegrability of natural systems with homogeneous potential. According to his algorithm, we calculate the Kowalewski's indicators

$$\Delta \varrho_i = \left( 1 + 8k\lambda_i/(k-2)^2 \right)^{1/2},$$

where $\lambda_i$ are the eigenvalues of the matrix $\Gamma = \frac{\partial^2 W}{\partial x^2}(c)$, $c \in \mathbb{C}^n$ is a nontrivial solution of the system of equations

$$\frac{\partial W}{\partial x_j}(c) = c_j, \qquad 1 \leqslant j \leqslant n. \tag{2}$$

In our case

$$W = x^k + \left(-\frac{x}{2} + \frac{y\sqrt{3}}{2}\right)^k + \left(-\frac{x}{2} - \frac{y\sqrt{3}}{2}\right)^k.$$

The solution of the system (2) is

$$c_1 = \left(2^{k-1}/k\left(1 + 2^{k-1}\right)\right)^{1/(k-2)}, \tag{3}$$
$$c_2 = 0. \tag{4}$$

The Kowalewski's indicators are

$$\Delta\varrho_1 = \frac{3k-2}{k-2} \in \mathbb{Q},$$
$$\Delta\varrho_2 = \left(1 + \frac{24k(k-1)}{(k-2)^2(1+2^{k-1})}\right)^{1/2}. \tag{5}$$

To show that $\Delta\varrho_2 \notin \mathbb{Q}$ consider the Diophantine equation

$$1 + \frac{24k(k-1)}{(k-2)^2(1+2^{k-1})} = \left(\frac{l}{(k-2)(1+2^{k-1})}\right)^{1/2}.$$

One can easily prove that for $k > 10$ $l \notin \mathbb{N}$, and it is easy to calculate $l$ for $k \leqslant 10$ directly and check that $l$ also is not natural. ∎

The analogous result is established for the case of $n$ pair-interactive particles on the line.

**Theorem 3.** *The n-body problem with the Hamiltonian (1) is nonintegrable for $k > 2$.*

**Proof.** First of all we reduce the system of $n$ particles to the system with two degrees of freedom. Let the initial conditions of the dynamics are

$$x_1 = x_2 = \cdots = x_r = y, \tag{6}$$
$$x_{r+1} = \cdots = x_{2r} = -y, \tag{7}$$
$$x_{2r+1} = -x_{2r+2} = x, \tag{8}$$
$$\dot{x}_1 = \dot{x}_2 = \cdots = \dot{x}_r = p_y, \tag{9}$$
$$\dot{x}_{r+1} = \cdots = \dot{x}_{2r} = -p_y, \tag{10}$$
$$\dot{x}_{2r+1} = -\dot{x}_{2r+2} = p_x \tag{11}$$

for $n = 2(r+1)$, and additionally

$$x_{2r+3} = \dot{x}_{2r+3} = 0$$

for odd values of $n$ $(n = 2(r+1)+1)$. Then the reduced Hamiltonian can be written in the form

$$H = p_x^2 + rp_y^2 + W(x, y),$$

where

$$W(x, y) = 2r\left((x - y)^k + (x + y)^k\right) + (2x)^k + r^2(2y)^k$$

for $n = 2(r+1)$ and

$$W(x, y) = 2r\left((x - y)^k + (x + y)^k\right) + (2x)^k + r^2(2y)^k + 2x^k + 2ry^k$$

for $n = 2(r+1)+1$. The analogous Diophantine equations can be easily considered and one can prove that these equations do not have solutions for $k > 2$. ∎

Consider now the problem of integrability of the system of $(n+1)$ interactive particles with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n+1} p_i^2 + \sum_{i=1}^{n} V(x_i - x_{i+1}) + \lambda V(x_{n+1} - x_1), \tag{12}$$

where $\lambda$ can be equal to 0 or 1.

**Theorem 4.** *The system with the Hamiltonian (12) can be reduced to the system with two degrees of freedom and the Hamiltonian*

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x) + \lambda V\left(\frac{x}{n} + \frac{y\sqrt{n^2-1}}{n}\right) + (n-1)V\left(\frac{x}{n} - \frac{y\sqrt{n^2-1}}{n(n-1)}\right). \tag{13}$$

**Theorem 5.** *The systems with the Hamiltonians (13) for $\lambda \in \{0,1\}$ do not possess the additional first integral for $k > 2$.*

The proof of the Theorem 5 is based on considering the Kowalewski's indicators for the Hamiltonian (13). In this case also $\Delta\varrho_1 = \frac{3k-2}{k-2} \in \mathbb{Q}$ and it is proved that $\Delta\varrho_2 \notin \mathbb{Q}$ for values $k > 2$.

[1] Moser J., Three integrable Hamiltonian systems connected with semisimple Lie algebras, *Invent. Math.*, 1976, V.37, N 2, 93–108.

[2] Calogero F., Exactly solvable one-dimensional many body problems, *Letters al Nuovo Cimento*, 1975, V.13, N 11, 411–416.

[3] Yoshida H., A criterion for the non-existence of an additional analytic integral in Hamiltonian systems with n degrees of freedom, *Phys. Lett. A*, 1989, V.141, N 3–4, 108–112.