Transformation of Scientific System of Knowledge in Educational: Symmetry Analysis of Equations of Mathematical Physics

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The paper presents an outline for introduction of symmetry ideas and techniques for solution of partial differential equations into the curriculum for training of teachers of mathematics. The main goal of such special course would be integration of the previously gained knowledge in calculus, algebra and ordinary differential equations together with showing students some developments in modern mathematical science.

1 Introduction

Physical or mathematical theory as the most developed form of scientific knowledge cannot be transferred into a learning process completely, as the system of knowledge used in learning and the scope of scientific knowledge are not identical. The scientific system of knowledge is based on research process directly, and learning system is based on research only indirectly.

In this respect the following questions arise: 1) Whether the system of scientific knowledge that is being studied, is a theory in a strictly logical sense? 2) Whether this system of scientific knowledge would enable formation of theoretical thinking of students and give them comprehensive image of the learning object?

The same learning material can be structured in different ways depending on aims and techniques of learning. In the process of advanced studying physics and mathematics the main attention is concentrated on presentation of the core of fundamental physical theories and enabling students to master their deep essence. In this way the teaching system that employs scientific system of knowledge develops productive forms of reasoning, provides theoretical knowledge for understanding of the physical picture of the world by means of a mathematical model.

Real physical processes are mostly described by nonlinear partial differential equations. The search for exact solutions of such equations is one of the most important stages for mathematical description of nature. At present many efficient methods for solving PDEs: separation of variables, Poisson method, decomposition into Fourier series, inverse problem and others. All these methods are based on the ideas of symmetry and are efficiently employed for solving those problems that have implicit or explicit symmetry.

Mathematical foundation for the theory of symmetry of differential equations was created by prominent Norwegian mathematician Sophus Lie as far as in 1881–1885. Modern development for this theory was provided, in particular, by Ovsiannikov [1], Olver [2], Bluman and Kumei [3], Fushchych and his collaborators [4]. These books together with numerous manuals targeted towards post-graduate students may provide a basis for development of courses on application of symmetry methods to partial differential equations. There is some advancement in Ukraine as to introduction of symmetry methods into university courses, mostly as special courses for graduate students. However, basic calculus and algebra courses, especially for mathematical education students, underwent little changes through the recent half a century.

All that stimulated the author into development of a course “Group Theoretical Techniques for Investigation and Solution of Partial Differential Equations”. The main goal of the course is formation of modern perception of symmetry and ensuring fundamental mathematical background for future teachers of mathematics. The curriculum for this course determines the scope of knowledge needed for professional formation at Specialist’s and Master’s degrees in Mathematics and Mathematical Education. Added value of the course is integration of knowledge and skills gained in other mathematical courses – calculus, algebra, ordinary and partial differential equations, together with being a good foundation for preparation of degree theses.

2 Plan for the course on group theoretical techniques for investigation and solution of partial differential equations

The course is targeted at mathematical education students that are interested in gaining advanced knowledge in mathematics. It is assumed that students’ background includes calculus, algebra and ordinary differential equations. As a preparation to studying symmetry techniques, the course includes also basics of partial differential equations. The curriculum for the course includes 16 hours of lectures and 16 hours of practical seminars. Evaluation of the course results is planned on the basis of the individual written paper. Layout of the course is suggested as follows:

	Topic of the lecture	No. of hours
1.	Partial differential equations. Main Definitions. Examples. Cauchy problem and boundary problem. First order uniform linear partial differential equations. First order quasilinear equations. Examples.	2
2.	Classification of second order PDEs. Reduction of an equation to the canonical form. Survey of methods for solution of differential equations: d’Alembert method and Fourier method for separation of variables.	2
3.	Introduction to the theory of Lie groups. Historical survey of development of the symmetry theory. One-parameter transformation groups. Definitions, examples. The problem of construction of a group. Lie theorem. Operators of translation, rotation, scale transformation.	2
4.	Infinitesimal operator. Prolongation of an operator. Invariance criterion. Lie algorithm for investigation of symmetry of PDEs.	2
5.	Symmetry of the d’Alembert equation. The conformal group $C(1,3)$. Basis operators and invariance transformations.	2
6.	Group analysis of a nonlinear wave equation. Euler–Lagrange system. Construction of a maximal invariance algebra.	2
7.	Reduction and exact solutions of nonlinear wave equation. Derivation of new solutions.	2
8.	Symmetry of a first-order nonlinear equation. Symmetry properties and exact solutions for the eikonal equation.	2

The course accounts for the fact that partial differential equations are not included into basis courses for graduate students in Mathematical Education. First two lectures of the course present to the students main concepts of partial differential equations. Core part or the course is example-based explanation of the main concepts and techniques of the group analysis of PDEs with special consideration given to application and strengthening of the skills obtained in the previous basic courses.

3 A practical problem with solution

Here we show an example that may be presented to students with the reference to the needed basic skills. The task is to show that the the heat equation

$$u_t - u_{xx} = 0 \tag{1}$$

is invariant under the Galilei transformations

$$t' = t, \qquad x' = x + 2at, \qquad u' = ue^{-(ax+a^2t)}. \tag{2}$$

Invariance of the equation (1) can be checked by the formulae of substitution of variables in the partial derivatives:

$$u'_{t'} = \left(u_t + a^2u - 2au_x\right) e^{-(ax+a^2t)}, \qquad u'_{x'} = (u_x - au)e^{-(ax+a^2t)},$$
$$u'_{x'x'} = \left(u_{xx} + a^2u - 2au_x\right) e^{-(ax+a^2t)}.$$

From here we derive an equality

$$u'_{t'} - u'_{x'x'} = (u_t - u_{xx})e^{-(ax+a^2t)}$$

showing that under the action of the transformations (2) the equation (1) is transformed into the same equation in the transformed variables, so the invariance in proved.

The next task is to find an infinitesimal operator corresponding to the transformations (2). This task requires differentiation of the expressions for the transformations (2) and substitution of the condition $a = 0$:

$$X = \left(\frac{dt'}{da}\frac{\partial}{\partial t} + \frac{dx'}{da}\frac{\partial}{\partial x} + \frac{du'}{da}\frac{\partial}{\partial u}\right)\Big|_{a=0}$$
$$= \left(0 \cdot \frac{\partial}{\partial t} + 2t\frac{\partial}{\partial x} + ue^{-(ax+a^2t)}(-(x+2at))\frac{\partial}{\partial u}\right)\Big|_{a=0} = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}.$$

We see that appropriate breaking of the group analysis techniques into steps makes them accessible for mathematical education students with reasonable basic background in mathematics, providing them with practice in their calculation skills together with understanding of the value of this skills for practical problem solving and research.

4 Conclusions

Such special course contributes to widening of scientific horizons of students who are mainly future teachers of mathematics, deeper understanding of the role of modern mathematics and gives insight to development of modern Ukrainian mathematical school of thought.

The author also used simplified version of the course with more links to school geometry and physics in the optional secondary school course, with student research papers prepared as a result. One of these papers by Julia Lashkevych received a 2nd award at the 1998 Ukrainian Competition of Secondary School Student Research Papers.

Another aspect that provides particular benefits of introduction of the course into advanced mathematics teachers' training curriculum is wide use of symmetry in the secondary school courses, mainly in geometry and physics. School courses deal mostly with discrete symmetries in the Euclidean space (translation, rotation, mirror and central symmetries), and widening of the notion of symmetry seems appropriate for teachers' training. Further development of the course, if more hours could be used, may be towards introduction of the notions of conservation laws and the relevant links with the course of physics.

[1] Ovsyannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.

[2] Olver P., Application of Lie groups to differential equations, New York, Springer Verlag, 1987.

[3] Bluman G.W. and Kumei S., Symmetries and differential equations, New York, Springer Verlag, 1989.

[4] Fushchych W.I., Shtelen W.M. and Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kyiv, Naukova Dumka, 1989 (in Russian); Kluwer Publishers, 1993 (in English).