On the Localized Invariant Traveling Wave Solutions in Relaxing Hydrodynamic-Type Model

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There are presented results of the investigations of a modeling system describing long nonlinear waves propagation in structured media with two relaxing components. A set of traveling wave invariant solutions is analyzed. We determine the conditions assuring the existence of quasiperiodic solutions and show that such analysis is helpful in looking for the localized wave patterns, since the destruction of quasiperiodic regime very often is realized via the homoclinic bifurcation of saddle-focus, which corresponds to the many-hump soliton appearance.

1 Introduction

Actually it is well known [1, 2, 3, 4], that continual models of structured and hierarchic media possess more rich set of solutions than those of structureless media. In this work we analyze maybe the simplest nonlinear hydrodynamic–type system describing long waves propagation in structured media with two relaxing processes on microscopic level.

It is rather difficult to make any general statements concerning the whole family of solutions of a non-linear differential equation, yet, using the group theory and qualitative methods one is able to analyze a set invariant solutions, providing that equation under consideration possesses sufficiently large symmetry group. Very often this set contains interesting and physically meaningful solutions, reporting to attract nearby, not necessarily invariant ones [5, 6, 7]. In this work attention is paid to the problem of extracting localized traveling wave solutions associated with the homoclinic loops of corresponding dynamical system, being obtained from the initial PDE system via the group theory reduction. Actually there does not exist any regular analytical method enabling to predict the appearance of a homoclinic-type solution of a multidimensional dynamical system, yet some information about the possibility of homoclinic bifurcation could be obtained through the analysis of Poincaré canonical form (CPF), corresponding to some type of degeneracy of the linear part of the system. Aside of the case when CPF is Hamiltonian, or close to Hamiltonian [8], this information is incomplete, therefore answering the question on whether or not the homoclinic bifurcation does take place one finally should resort to the numerical simulation.

The contents of this work is following. We consider relatively simple modeling system describing high-rate processes in structured media with two relaxing components. Using the trivial symmetry inherent to any evolution system which does not contain independent variables, we pass to the three-dimensional system of ODE, describing a set of invariant traveling wave solutions. We analyze this system with the help of qualitative theory methods and state the conditions assuring the existence of quasiperiodic solutions in proximity of some stationary point. Since one of the possible scenarios of the quasiperiodic regime destruction is associated with the homoclinic bifurcation, we use the above results to localize the domain of parameter space where the homoclinic bifurcation could take place. In order to capture a family of homoclinic-type solutions we finally use the numerical algorithm outlined in [4].

2 Soliton-like invariant solutions of relaxing hydrodynamics model

The modeling system we are going to analyze is as follows:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \chi\,\rho^{\sigma-1}\frac{\partial \rho}{\partial x} = \alpha_1\,(\rho-1) + \alpha_2\,(\rho-1)^2 - q_1\kappa\,\eta \\
&\qquad + q_2\left[\nu\,\eta - (\sigma-1)\,\varphi\,(\rho-1) - \kappa\,(\lambda-\lambda_0)\right], \\
&\frac{\partial \lambda}{\partial t} = -\kappa\,\eta, \\
&\frac{\partial \eta}{\partial t} = \nu\,\eta - \varphi\,\left(\rho^{\sigma-1}-1\right) + \beta\,(\lambda-\lambda_0)^2 - \kappa\,(\lambda-\lambda_0)\,.
\end{aligned} \tag{1}$$

Let us consider ansatz

$$\rho = R(\omega), \qquad \omega = x - D\,t, \qquad \lambda = L(\omega), \qquad \eta = N(\omega), \tag{2}$$

that describes a family of invariant traveling wave solutions. Inserting (1) into the system (1) we obtain dynamical system

$$\begin{aligned}
&D\left(\chi R^{\sigma-1} - D\right)\dot{R} = D\left[\alpha_1\,(R-1) + \alpha_2\,(R-1)^2\right] - qN\kappa \\
&\qquad + p\,\left[\nu\,N - \theta\,(\rho-1) - \kappa\,(L-L_0)\right], \\
&D\left(\chi R^{\sigma-1} - D\right)\dot{L} = \kappa\,\left(\chi R^{\sigma-1} - D\right)N,
\end{aligned} \tag{3}$$

$$\begin{aligned}
&D\left(\chi R^{\sigma-1} - D\right)\dot{N} = \left(\chi R^{\sigma-1} - D\right)\Big(-\left(R^{\sigma-1}-1\right)\,\varphi - \nu\,N \\
&\qquad -\beta\,(L-L_0)^2 + \kappa\,(L-L_0)\Big)\,.
\end{aligned} \tag{4}$$

where $\theta = (\sigma-1)\varphi$, $q = q_1\,D$, $p = q_2\,D$. One easily gets convinced that linear part of system (3) in reference frame $X = R-1$, $Y = L-L_0$, $Z = N$, centered at the critical point $A(1,\,L_0,\,0)$, is as follows:

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{bmatrix} \alpha_1 D - p\,\theta & -p\,\kappa & p\,\nu - q\,\kappa \\ 0 & 0 & \kappa\,\Delta \\ \theta\,\Delta & \kappa\,\Delta & -\nu\,\Delta \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{5}$$

where $\Delta = \chi - D$, $h = \theta/\nu$. We are going to employ the methods of local nonlinear analysis to the investigation of quasiperiodic and soliton-like solutions appearance in vicinity of critical point $A(1,\,L_0,\,0)$. In order for such analysis be effective, restrictions on the parameters should be imposed, assuring that the eigenvalues of the linearization matrix $\hat{M}$ standing at the RHS of system (5) have the $(0,\,\pm i\,\Omega)$ degeneracy [9, 4]. This leads to the following conditions:

$$\alpha_1 = 0, \qquad D = \chi + p\,h, \qquad \Omega^2 = -p\,h^2\kappa\,(p\,\kappa + \nu\,q) > 0. \tag{6}$$

In order to avoid considering the case that is obviously unstable [10], we should chose the parameters in such a way that the nonlinear wave pack velocity D be greater than the acoustic sound velocity χ. From this requirement we immediately obtain inequality

$$p\,h > 0. \tag{7}$$

Assuming that conditions (6)–(7) are satisfied, let us construct canonical Poincaré form corresponding to system (3). To do that we first make a transition to new variables

$$\left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right) = \frac{p}{-\Omega^3 h}\left[\begin{array}{ccc} 0 & 0 & \kappa\, h^2\,(q\,\nu+\kappa\, p) \\ -h\,\theta\,\Omega & -\kappa\, h\,\Omega & \theta\,\Omega \\ \Omega\,\kappa\, p\, h^3 & -q\,\kappa\,\Omega h^2 & -h^2\kappa\, p\,\Omega \end{array}\right]\left(\begin{array}{c} X \\ Y \\ Z \end{array}\right),$$

chosen in such a way that linear part of the system is written in standard quasi-diagonal form:

$$\begin{aligned} x_1' &= -\Omega\, x_2 + \sum_{i+j=2 i\le j} A_{ij}x_i x_j + \sum_{i+j+k=2 i\le j\le k} A_{ijk}x_i x_j x_k + \cdots, \\ x_2' &= \Omega\, x_1 + \sum_{i+j=2 i\le j} B_{ij}x_i x_j + \sum_{i+j+k=2 i\le j\le k} B_{ijk}x_i x_j x_k + \cdots, \\ x_3' &= \sum_{i+j=2 i\le j} C_{ij}x_i x_j + \sum_{i+j+k=2 i\le j\le k} C_{ijk}x_i x_j x_k + \cdots, \end{aligned} \tag{8}$$

where $(\cdot)' = D\left(\chi\, R^{\sigma-1} - D\right)\, d\,(\cdot)\,/d\,\omega$. Passage from the representation (8) to CPF

$$\begin{aligned} r' &= a\, r\, z + o\left(r^3,\, |z|^3\right), \\ z' &= b_1 r^2 + b_2 z^2 + f\, z^3 + o\left(r^3,\, |z|^3\right). \end{aligned} \tag{9}$$

is based on standard method that can be found e.g. in [9]. Connections between the coefficients of the CPF and those of system (8) prove to be as follows [11, 12]:

$$a = \frac{(A_{13}+B_{23})}{2}, \qquad b_1 = \frac{(C_{11}+C_{22})}{2}, \qquad b_2 = C_{33}. \tag{10}$$

CPF (9) is obtained from the initial dynamical system through the series of asymptotic transformations [9], followed by the passage to the cylindric coordinates (r, φ, z) and averaging over the "fast" angular variable φ. Therefore the limit cycle, appearing in (9) corresponds to the quasiperiodic solution of the initial PDE system, while the homoclinic trajectory corresponds to the non-classical many-hump solitary wave pack.

As it is shown in [12], stability of the periodic solution of system (9) is determined by the sign of the coefficient f. General expression on this coefficient, given in [11, 12], is too cumbersome to be of any use in analytical treatment, but it drastically simplifies when $C_{33} = 0$. This is so when the following expression holds

$$\alpha_2 = \frac{h\, p\,\nu\left(2\, h\,\beta\,\nu - \kappa^2\right)}{2\,\kappa^2\,(h\, p + \chi)}. \tag{11}$$

Under this condition

$$f = C_{333} + \frac{1}{\Omega}\,(A_{33}C_{23} - B_{33}C_{13}). \tag{12}$$

Here and henceforth we put calculation with $\sigma = 3$.

We must stress in this place that, although the CPF (9) serves as a basement for the classification of regimes appearing after the removal of degeneracy, its investigation will not be of any use until the coefficients (10), (12) remain unknown. This problem is rather cumbersome unless one uses some tools for symbolic calculus. Below we give the exact expressions on the CPF coefficients, obtained with the help of the program "Mathematica 4.0":

$$\begin{aligned} a &= \frac{2\,(\nu^2 h^2 p\,\beta\, - \nu\,\kappa^2\chi) - \nu\,\kappa^2 h\, p}{2\,\kappa}, \qquad b_1 = -\frac{h^4 p^2\beta}{2\,\kappa}\left[p\left(\kappa^2+\nu^2\right) - q\,\kappa\,\nu\right], \\ f &= \frac{h^4 p^3\nu^2\beta}{\Omega^2}\left(\kappa^2 - 2\, h\,\beta\,\nu\right). \end{aligned} \tag{13}$$

In order to remove the degeneracy of the linear part of system (3) two parameter family of small perturbation is introduced:

$$F \to \alpha_1 (R-1) + \alpha_2 (R-1), \qquad D \to \chi + h\,p + \epsilon.$$

This induces the following change of the CPF:

$$\begin{aligned} r' &= \mu_1 r + a\,r\,z + o\left(r^3,\,|z|^3,\,|\mu_i|\right),\\ z' &= \mu_2 z + b_1 r^2 + b_2 z^2 + f\,z^3 + o\left(r^3,\,|z|^3,\,|\mu_i|\right), \end{aligned} \tag{14}$$

where

$$\mu_1 = \frac{\nu}{2}\left(\epsilon - \frac{Dh^2 p\,q\kappa}{\Omega^2}\,\alpha_1\right), \qquad \mu_2 = -\frac{Dh^2p^2\kappa^2}{\Omega^2}\alpha_1. \tag{15}$$

As it is shown in our previous work [12], for sufficiently small $\mu_1\,\mu_2$ the limit cycle appears in system (14) along the manifold

$$\mu_2 + 3f\,(\mu_1/a)^2 = 0, \tag{16}$$

providing that $a\,b_1 < 0$. Here we have two possibilities:

- $f > 0$ and $\mu_1\,\mu_2 < 0$; stable periodic regime exists if the following inequality holds

$$\mu_2 < -3f\,(\mu_1/a)^2 < 0; \tag{17}$$

- $f < 0$ and $\mu_1\,\mu_2 < 0$; unstable periodic regime exists if the following inequality holds

$$\mu_2 > -3f\,(\mu_1/a)^2 > 0 \tag{18}$$

(note that in proximity of the critical point $A(1,\,L_0,\,0)$ the factor $\left(\chi R^{\sigma-1} - D\right)$ is negative and this circumstance has been taken into account when determining the stability type of the limit cycle).

Numerical simulations show that one of the scenarios of destruction of the limit cycle of system (14) is associated with the appearance of homoclinic loop (for $a = 2$ it can be shown analytically [9]). These regimes correspond to the soliton-like solutions of the initial PDE system. In the first case, corresponding to formula (17), homoclinic trajectory comes out from the critical point $A(1,\,L_0,\,0)$ (which is a saddle-focus) along the one-dimensional unstable invariant manifold W^u and returns into the critical point along the two-dimensional stable invariant manifold W^s. In the second case, corresponding to formula (18), the direction of movement is opposite. Analysis based on the equations (13)–(18) shows that both of these cases could be realized in the system (1). In the first case we obtain a soliton-like solution with oscillating front while in the second one – a localized pack with oscillating "tail".

As it was mentioned above, a homoclinic bifurcation is not the only scenario leading to the destruction of quasiperiodic movement appearing in system (3). We cannot exclude another scenarios, associated e.g. with a pair of tori appearance (i.e. the case when the Lyapunov indices cross the unit circle having non-zero imaginary part). In this situation final answer on whether or not the soliton-like regimes appear in system could give the numerical experiments.

We employed the numerical procedure put forward in [4] (cf. also [13]). The procedure enables to find out a set of points belonging to parameter plane $(\chi,\,\alpha_1)$, for which trajectory going out of the origin along the one-dimensional unstable invariant manifold W^u returns to the origin along the two-dimensional stable invariant manifold W^s. Having fixed the rest of parameters as follows $\varphi = -1$, $\nu = \kappa = q = 1$, $p = -0.25$, we defined numerically a distance between the

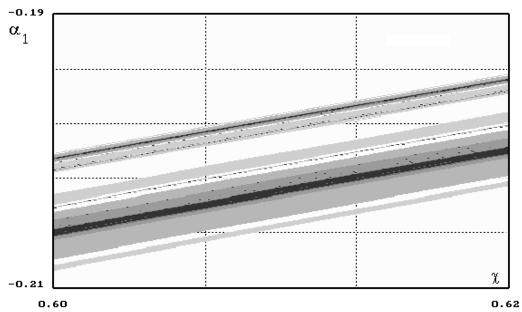

Figure 1.

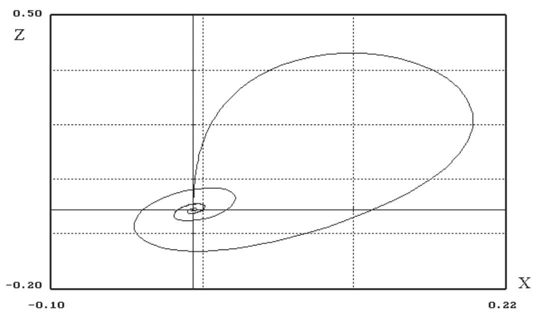

Figure 2.

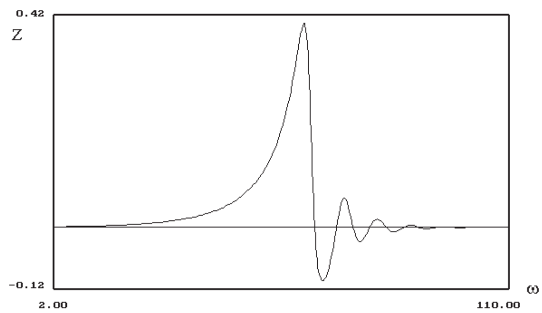

Figure 3.

origin (which coincide with the critical point $A(1,\, L_0,\, 0)$) and the point $X^\Gamma(\omega)$, $Y^\Gamma(\omega)$, $Z^\Gamma(\omega)$ of the phase trajectory $\Gamma(\cdot;\, \chi,\, \alpha_1)$:

$$f^\Gamma(\chi,\, \alpha_1; \omega) = \sqrt{[X^\Gamma(\omega)]^2 + [Y^\Gamma(\omega)]^2 + [Z^\Gamma(\omega)]^2}, \tag{19}$$

starting from the fixed Cauchy data lying on the unstable manifold W^u, close to the origin. Next we determine minimum $f^\Gamma_{\min}(\chi,\, \alpha_1)$ of the function (19) for that part of the trajectory that lies beyond the point at which the distance gains its first local maximum, providing that it remains all the time inside the ball of a fixed (sufficiently large) radius. The results is presented on Fig. 1. Here white color marks values of the parameters χ, α_1 for which $f^\Gamma_{\min} > 1.2$, light grey correspond to the values $0.6 < f^\Gamma_{\min} < 1.2$, grey color - to the values $0.3 < f^\Gamma_{\min} < 0.6$, deep grey – to $0.01 < f^\Gamma_{\min} < 0.3$, black - to $f^\Gamma_{\min} < 0.01$. It is seen on Fig. 1, that the points corresponding to homoclinic loop appearance form a connected set and this is in agreement with the general statements [14]. Let us note that employment of the same procedure for another systems gives a Cantor set instead of the connected curve [4, 13].

Solving numerically system (3) for proper values of the parameters, one is able to obtain a soliton-like solution. Because of the numerical error it is rather impossible to read our form Fig. 1 exact value of the parameters χ, α_1, corresponding to homoclinic loop. Therefore we put $\alpha_1 = -0.2$ and, using the Bolzano–Weierstrass method, varied χ until the homoclinic trajectory was attained at $\chi = 0.619803$. This approach proves to be effective because bifurcation values of the parameters form a smooth curve in the plane $(\chi,\ \alpha_1)$. Projection of the homoclinic trajectory onto the $(X,\ Z)$ plane is shown on Fig. 2; while the corresponding solution $R\,(\omega) = R\,(x - D\,t)$ is shown on Fig. 3.

So we have get convinced that system (1) possesses invariant soliton-like solutions, that look like a many-hump wave pack moving with uniform speed D. Let us note in conclusion that till now we did not touch upon the crucial problem of stability of the soliton-like regimes and their attractive features, but we hope to face this problem it in the nearest future.

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