

Conditional Symmetry Reduction and Invariant Solutions of Nonlinear Wave Equations

Ivan M. TSYFRA

Institute of Geophysics of NAS of Ukraine, 32 Palladina Avenue, Kyiv, Ukraine

E-mail: *itsyfra@imath.kiev.ua*

We obtain sufficient conditions for the solution found with the help of conditional symmetry operators to be an invariant one in classical Lie sense. Several examples of nonlinear partial differential equations are considered.

1 Introduction

It is well known that the symmetry reduction method is very efficient for construction of exact solutions for nonlinear partial differential equations of mathematical physics. With the help of symmetry operators one can find ansatze which reduce partial differential equation to the equation with smaller number of independent variables. Application of conditional symmetry operators essentially widens the class of ansatze reducing initial differential equation [1, 2, 3]. It turns out however that some of these ansatze result in the classical invariant solutions. It is obvious that the existence of conditional symmetry operator does not guarantee that the solution obtained with the help of corresponding ansatz is really new that it is not invariant solution in the classical Lie sense. We have proved theorem allowing us to exclude the operators that lead to the classical invariant solutions.

2 Basic theorem

Let us consider some partial differential equation

$$U(x, u, u_1, \dots, u_k) = 0, \quad (1)$$

where $u \in C^k(\mathbb{R}^n, \mathbb{R}^1)$, $x \in \mathbb{R}^n$, and u_k denotes all partial derivatives of k -th order. Suppose that the following conditions are fulfilled.

1. Equation (1) is conditionally invariant under involutive family of operators $\{Q_i\}$, $i = \overline{1, p}$

$$Q_i = \xi_i^l(x, u) \frac{\partial}{\partial x_l} + \eta_i(x, u) \frac{\partial}{\partial u} \quad (2)$$

and corresponding ansatz reduces this equation to ordinary differential equation.

2. There exists the general solution of reduced equation in the following form

$$u = f(x, C_1, \dots, C_t), \quad (3)$$

where f is arbitrary smooth function of its arguments, C_1, \dots, C_t are arbitrary real constants. The following theorem has been proved.

Theorem 1. *Let equation (1) be invariant under the m -dimensional Lie algebra AG_m with basis elements:*

$$X_j = \tilde{\xi}_j^l(x) \frac{\partial}{\partial x_l} + \tilde{\eta}_j(x, u) \frac{\partial}{\partial u}, \quad j = 1, \dots, m, \quad (4)$$

and conditionally invariant with respect to involutive family of operators $\{Q_i\}$ satisfying conditions 1, 2.

If the system

$$\xi_i^n \frac{\partial u}{\partial x_i} = \eta'_i(x, u) \tag{5}$$

is invariant under s -dimensional subalgebra AG_s with basis elements

$$Y_a = \xi_a^l(x) \frac{\partial}{\partial x_l}, \quad a = \overline{1, s}, \tag{6}$$

of algebra AG_m and $s \geq t + 1$, then conditionally invariant solution of equation (1) with respect to involutive family of operators $\{Q_i\}$ is an invariant solution in the classical Lie sense.

Proof. From the theorem conditions it follows that the system of equations (1), (5) is invariant with respect to AG_s algebra with basis elements Y_a . Consider one parameter subgroup of transformations of space $X \times U$ (variables x, u) with infinitesimal operator Y_j . These transformations map any solution from (3) into solution of system (1), (5). Thus the following relations

$$u - f(x', C_1, C_2, \dots, C_t) = u - f(x, C'_1, \dots, C'_t), \tag{7}$$

where $a \in \mathbb{R}^1$, C'_1, \dots, C'_t depend on C_1, \dots, C_t, a , are fulfilled in this case. Equality (7) is true for arbitrary group parameter $a \in \mathbb{R}^1$. Considering it in the vicinity of point $a = 0$ we obtain

$$\xi_a^l(x) \frac{\partial f}{\partial x_l} = - \frac{\partial f}{\partial C_1} \beta_{j1} - \dots - \frac{\partial f}{\partial C_t} \beta_{jt}, \quad j = \overline{1, s}, \tag{8}$$

where $\beta_{jk} = \frac{\partial C'_k}{\partial a}$ at the point $a = 0$. As far as the mentioned reasoning is valid for arbitrary operator Y_j then condition (8) is equivalent to the following system of s equations

$$Y_j f = - \sum_{k=1}^t \frac{\partial f}{\partial C_k} \beta_{jk}, \quad 1 \leq j \leq s. \tag{9}$$

From system (9) it follows that there exist such real constants γ_p that the the condition

$$\sum_{p=1}^s \gamma_p Y_p f = 0,$$

is true since $s \geq t + 1$. Therefore the solution $u = f(x, C_1, \dots, C_t)$ is invariant with respect to one-parameter Lie group with infinitesimal operator $Q = \sum_{p=1}^s \gamma_p Y_p$. ■

Note that theorem can be generalized for infinitesimal operators of the form

$$Y_a = \xi_a^l(x) \frac{\partial}{\partial x_l} + \eta_a(x, u), \tag{10}$$

where

$$\eta_a(x, u) = F_a(x)u + \Phi_a(x), \tag{11}$$

and $F_a(x)$ and $\Phi_a(x)$ are arbitrary smooth functions.

3 Examples

Now consider several examples. We first study nonlinear wave equation

$$u_{xt} = \sin u. \quad (12)$$

We prove that equation (12) is conditionally invariant with respect to the operator

$$X = \left(u_{xx} + \frac{1}{2} \tan uu_x^2 \right) \partial_u. \quad (13)$$

We use the definition of conditional symmetry for arbitrary differential equation given in [4]. Therefore we can use the following differential consequences

$$D_x(u_{xt} - \sin u) = 0, \quad D_x^2(u_{xt} - \sin u) = 0, \quad D_x(\eta) = 0, \quad (14)$$

and

$$u_t = -\frac{2 \cos u}{u_x}. \quad (15)$$

It is easy to verify that the equality

$$\frac{X}{2}(u_{xt} - \sin u) = 0,$$

where $\frac{X}{2}$ is the extended symmetry operator of the second order, is satisfied on the manifold given by relations (12), (14), (15). Thus equation (12) is conditionally invariant with respect to the Lie–Bäcklund vector field (13). That is why we can reduce it by means of the ansatz which is the solution of the following equation

$$u_{xx} + \frac{1}{2} \tan uu_x^2 = 0 \quad (16)$$

and has an implicit form

$$H(u) = C(t)x + \alpha(t), \quad (17)$$

where

$$H(u) = \int \frac{du}{\sqrt{\cos u}}.$$

Substituting (17) into (12) we receive the reduced system in the form

$$C'(t) = 0, \quad \frac{1}{2}C(t)\alpha'(t) + 1 = 0.$$

Finally by integrating this system we obtain solution of equation (12)

$$H(u) = C_1x - \frac{2}{C_1}t + C_2. \quad (18)$$

Both of equations (12) and (16) are invariant with respect to three-dimensional Lie algebra with basis elements

$$Q_1 = u_x \partial_u, \quad Q_2 = u_t \partial_u, \quad Q_3 = (tu_t - xu_x) \partial_u.$$

And also the solution depends on two constants. So, the theorem conditions are fulfilled. Thus we conclude that solution (18) is an invariant one in the classical Lie sense as a consequence of theorem. It is obvious that there exist the linear combination of operators Q_1, Q_2, Q_3 such that obtained solution is invariant under transformations generated by this operator.

Now consider equation

$$u_t - u_{xx} = \lambda \exp(u)u_x + u_x^2. \quad (19)$$

It has been proved that equation (19) is conditionally invariant with respect to operator

$$Q = (u_{xx} + u_x^2) \frac{\partial}{\partial u}.$$

The corresponding ansatz has the form

$$u = \ln(f(t)x + \phi(t)), \quad (20)$$

where f, ϕ are unknown functions. Substitution of (20) in (19) yields the system of two ordinary differential equations in the form

$$f' = \lambda f^2, \quad \phi' = \lambda f \phi.$$

Having integrated this system one can obtain exact solution of equation (19)

$$u = \ln \left(\frac{x + C_1}{C - \lambda t} \right). \quad (21)$$

Note, that equation

$$u_{xx} + u_x^2 = 0$$

is invariant with respect to three-dimensional algebra with basis elements

$$Q_1 = \frac{\partial}{\partial t}, \quad Q_2 = \frac{\partial}{\partial x}, \quad Q_3 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - \frac{\partial}{\partial u}.$$

Thus according to theorem the solution (21) is an invariant one.

It can be verified that solution (21) is invariant with respect to one-parameter transformation group with infinitesimal operator

$$Q = \alpha Q_1 + \beta Q_2 + \gamma Q_3$$

when $\alpha = \gamma C_1, \beta = -2\gamma C \lambda^{-1}$.

Finally consider equation

$$u_t - a(u)u_{xx} = u(1 - a(u)), \quad (22)$$

where $a(u)$ is arbitrary smooth function. We have proved that equation (22) is conditionally invariant with respect to operator

$$Q = (u_{xx} - u) \frac{\partial}{\partial u}.$$

The invariance surface condition leads to the following ansatz

$$u(t, x) = \phi_1(t) \exp x + \phi_2(t) \exp(-x),$$

which reduces considered equation. It is easy to construct an exact solution of equation (22) using this approach in the form

$$u = A \exp(t + x) + B \exp(t - x), \quad (23)$$

where A, B are real constants.

It should be noted that the maximal invariance Lie algebra of point transformations is two-dimensional algebra with basis operators ∂_t, ∂_x . But solution (23) depends on two constants. Therefore the theorem conditions are not satisfied. Really it is easy to prove, that solution (23) cannot be constructed by means of Lie point group technique.

4 Conclusion

Thus we obtain a sufficient condition for the solution found with the help of conditional symmetry operators to be an invariant solution in the classical sense. The theorem proved by means of infinitesimal invariance method allows us to optimize the algorithm for construction of conditional symmetry operators, a priori excluding the operators that lead to the classical invariant solutions. It is obvious that this theorem can be generalized and applicable to construction of exact solutions for partial differential equations by using the method of differential constraints, Lie–Bäcklund symmetry method and the approach suggested in [5].

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