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Asynchronous Development of the Growing-and-Decaying Mode

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The solution to the Davey–Stewartson I equation is analyzed to show that the resonance between periodic soliton and growing-and-decaying mode exists. Under the quasi-resonant condition, the mode develops first in the one side region of the periodic soliton. The periodic soliton is accelerated as a result of the growth and decay of the mode existed in the region and the wave field shifts to the intermediate state, where only the periodic soliton exists. This intermediate state persists over a comparatively long time interval. After sufficiently long time, the mode starts to grow in the opposite side of the periodic soliton.

1 Introduction

A uniform train of weakly nonlinear deepwater waves is unstable to long wave modulational perturbations of the envelope, which is known as the Benjamin–Feir instability [1]. It is well known that the long time evolution of the unstable wave train is described by the nonlinear Schrödinger (NLS) equation [2, 3, 4]. The extension to the two-dimensional case was examined by Zakharov, Benney and Roskes and Davey and Stewartson [2, 5, 6]. The long time evolution of a two-dimensional wave-packet is described by the Davey–Stewartson (DS) equation [6]

$$\begin{aligned} &iu_t + pu_{xx} + u_{yy} + r|u|^2u - 2uv = 0,\\ &v_{xx} - pv_{yy} - r(|u|^2)_{xx} = 0, \end{aligned} \tag{1}$$

where $p = \pm 1$, r is constant. Equation (1) with $p = 1$ and $p = -1$ are called the DS I and DS II equations, respectively. The time evolution of the solution of the 1D-NLS equation with periodic boundary condition and with Benjamin–Feir unstable initial condition was studied numerically by Lake et al. [7]. They found that a modulated unstable wave train achieves a state of maximum modulation and returns to an unmodulated initial state. The nonlinear evolution of an unstable mode is described by the growing-and-decaying mode soluiton to the 1D-NLS equation [8].

The DS I equation has also the growing-and-decaying mode solution, which is given by [9]

$$u = u_0 e^{i\zeta}\frac{g}{f}, \qquad v = -2\,(\ln f)_{xx} \tag{2}$$

with

$$\begin{aligned} &f = 1 - e^{-\Omega t+\sigma}\cos\eta + \frac{M}{4}e^{-2\Omega t+2\sigma},\\ &g = 1 - e^{-\Omega t+\sigma+i\phi}\cos\eta + \frac{M}{4}e^{-2\Omega t+2\sigma+2i\phi}, \end{aligned}$$

where

$$\begin{aligned} &\zeta = kx + ly - \omega t, \qquad \omega = k^2 + l^2 - ru_0^2, \qquad \eta = \beta x + \delta y - \gamma t + \theta,\\ &\Omega = (\beta^2+\delta^2)\cot\frac{\phi}{2}, \qquad \gamma = 2k\beta + 2l\delta, \qquad M = \frac{2}{1+\cos\phi} > 1, \qquad \sin^2\frac{\phi}{2} = \frac{\delta^2-\beta^2}{2ru_0^2}, \end{aligned}$$

σ and θ are arbitrary phase constants. The existence condition for the nonsingular solution is given by $M > 1$ for real ϕ, which is satisfied for

$$0 < \left(\delta^2 - \beta^2\right) < 2ru_0^2,$$

which is in agreement with the Benjamin–Feir unstable condition. This solution grows exponentially at initial stage, and reaches a state of maximum modulation and after reaching maximum modulation, demodulates and finally returns to an unmodulated initial state. Therefore, the solution (2) describes the nonlinear evolution of monochromatic perturbation with the Benjamin–Feir unstable condition in two-dimension.

The interactions between two-periodic solitons, between periodic soliton and line soliton and between periodic soliton and algebraic soliton to the DS equation have been investigated in detail [10, 11, 12]. It was shown that the periodic soliton resonances exist in each case. Pelinovsky pointed out the existence of the resonance between line soliton and growing-and-decaying mode [10]. The growing-and-decaying mode exists substantially only a finite period in time, but the resonance between line soliton and growing-and-decaying mode brings about the infinite phase shift to the line soliton. If the growing-and-decaying mode exists within only a finite time in reality, the mechanism bringing about the infinite phase shift to the line soliton is puzzle. Recently, we have investigated the time evolution of the quasi-resonant interaction between line soliton and growing-and-decaying mode and found the existence of an asynchronous development of the growing-and-decaying mode [13].

In this paper, it is shown that under the quasi-resonant condition for the interaction between periodic soliton and growing-and-decaying mode, the asynchronous development of the growing-and-decaying mode also exists.

2 Quasi-resonance between periodic soliton and growing-and-decaying mode

The interaction between periodic soliton and growing-and-decaying mode to the DS I equation is studied in this section. The solution describing the interaction can be obtained by the N-soliton solution of Satsuma and Ablowitz [14]. The solution consisting of a periodic soliton and growing-and-decaying mode is given by

$$u = u_0 e^{i\zeta}\frac{g}{f}, \qquad v = -2\left(\ln f\right)_{xx} \tag{3}$$

with

$$\begin{aligned} f = {} & 1 - \frac{1}{L_1L_2}e^{\xi_1}\cos\eta_1 - e^{\xi_2}\cos\eta_2 + \frac{M_1}{4L_1^2L_2^2}e^{2\xi_1} + \frac{M_2}{4}e^{2\xi_2} \\ & - \frac{1}{4}e^{\xi_1+\xi_2}\left\{\frac{M_1}{L_1L_2}e^{\xi_1}\cos(\eta_2+\Psi_1-\Psi_2) + M_2e^{\xi_2}\cos(\eta_1+\Psi_1+\Psi_2)\right\} \\ & + \frac{1}{2L_1L_2}e^{\xi_1+\xi_2}\Big\{L_1\cos(\eta_1+\eta_2+\Psi_1) + L_2\cos(\eta_1-\eta_2+\Psi_2)\Big\} + \frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)}, \end{aligned} \tag{4}$$

$$\begin{aligned} g = {} & 1 - \frac{1}{L_1L_2}e^{\xi_1+i\phi_{1r}}\cos(\eta_1+i\phi_{1i}) - e^{\xi_2+i\phi_2}\cos\eta_2 + \frac{M_1}{4L_1^2L_2^2}e^{2\xi_1+2i\phi_{1r}} + \frac{M_2}{4}e^{2\xi_2+2i\phi_2} \\ & - \frac{1}{4}e^{\xi_1+\xi_2+i(\phi_{1r}+\phi_2)}\left\{\frac{M_1}{L_1L_2}e^{\xi_1+i\phi_{1r}}\cos(\eta_2+\Psi_1-\Psi_2)\right. \\ & \left. + M_2e^{\xi_2+i\phi_2}\cos(\eta_1+i\phi_{1i}+\Psi_1+\Psi_2)\right\} \end{aligned}$$

$$+\frac{1}{2L_1L_2}e^{\xi_1+\xi_2+i(\phi_{1r}+\phi_2)}\Big\{L_1\cos(\eta_1+\eta_2+i\phi_{1i}+\Psi_1)$$
$$+L_2\cos(\eta_1-\eta_2+i\phi_{1i}+\Psi_2)\Big\}+\frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)+2i(\phi_{1r}+\phi_2)}, \tag{5}$$

where

$$\xi_1=\alpha x+\kappa y-\Omega_1 t+\sigma_1, \qquad \xi_2=-\Omega_2 t+\sigma_2,$$
$$\eta_1=\beta_1 x+\delta_1 y-\gamma_1 t+\theta_1, \qquad \eta_2=\beta_2 x+\delta_2 y-\gamma_2 t+\theta_2,$$
$$\sin^2\frac{\phi_1}{2}=\frac{(\alpha+i\beta_1)^2-(\kappa+i\delta_1)^2}{2ru_0^2}, \qquad \sin^2\frac{\phi_2}{2}=\frac{\delta_2^2-\beta_2^2}{2ru_0^2},$$
$$\Omega_1=2k\alpha+2l\kappa-\Re\left\{\left\{(\alpha+i\beta_1)^2+(\kappa+i\delta_1)^2\right\}\cot\frac{\phi_1}{2}\right\},$$
$$\gamma_1=2k\beta_1+2l\delta_1-\Im\left\{\left\{(\alpha+i\beta_1)^2+(\kappa+i\delta_1)^2\right\}\cot\frac{\phi_1}{2}\right\},$$
$$\Omega_2=(\beta_2^2+\delta_2^2)\cot\frac{\phi_2}{2}, \qquad \gamma_2=2k\beta_2+2l\delta_2,$$
$$M_1=\frac{2ru_0^2|\sin\frac{\phi_1}{2}|^2\cosh\phi_{1i}-(\alpha^2+\beta_1^2)+(\kappa^2+\delta_1^2)}{2ru_0^2|\sin\frac{\phi_1}{2}|^2\cos\phi_{1r}-(\alpha^2+\beta_1^2)+(\kappa^2+\delta_1^2)}, \qquad M_2=\frac{2}{1+\cos\phi_1},$$
$$L_1e^{i\Psi_1}=\frac{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2}-i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}}{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1+\phi_2}{2}-i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}},$$
$$L_2e^{i\Psi_2}=\frac{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2}+i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}}{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1+\phi_2}{2}+i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}},$$

where we have assumed that ϕ_2 is real and θ_1, θ_2, σ_1 and σ_2 are arbitrary constants. When we consider the case: $0<\Omega_2$, $0<\alpha$, $0<\kappa$ and $0<\Omega_1$, the solutions long before and after the mode growing are given by

$$f=\frac{M_2}{4}e^{2\xi_2}\left\{1-e^{\xi_1}\cos(\eta_1+\Psi_1+\Psi_2)+\frac{M_1}{4}e^{2\xi_1}\right\}, \tag{6}$$
$$g=\frac{M_2}{4}e^{2(\xi_2+i\phi_2)}\left\{1-e^{\xi_1+i\phi_{1r}}\cos(\eta_1+i\phi_{1i}+\Psi_1+\Psi_2)+\frac{M_1}{4}e^{2(\xi_1+i\phi_{1r})}\right\}, \tag{7}$$

and

$$f=1-\frac{1}{L_1L_2}e^{\xi_1}\cos\eta_1+\frac{M_1}{4L_1^2L_2^2}e^{2\xi_1}, \tag{8}$$
$$g=1-\frac{1}{L_1L_2}e^{\xi_1+i\phi_{1r}}\cos(\eta_1+i\phi_{1i})+\frac{M_1}{4L_1^2L_2^2}e^{2(\xi_1+i\phi_{1r})}, \tag{9}$$

respectively, which are periodic soliton solutions. It is shown that the phase shift of the periodic soliton due to the growing-and-decaying mode is given by the amount $\ln(L_1L_2)$ (or $-\ln(L_1L_2)$). $(L_1L_2)=\infty$ and 0 may be thought of as resonance between periodic soliton and growing-and-decaying mode, the conditions of which are obtained by equating the denominator and numerator of L_1 or L_2 to zero, respectively: We now investigate the condition which L_1 becomes infinity, namely

$$D=2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1+\phi_2}{2}-i\left\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\right\}=0. \tag{10}$$

When we express α, κ, β_1, δ_1, β_2 and δ_2 in term of ϕ_1, ϕ_2, θ_1 and θ_2 as follows,

$$\alpha + i\beta_1 = i\sqrt{2ru_0^2}\sin\frac{\phi_1}{2}\sinh\theta_1, \qquad \kappa + i\delta_1 = i\sqrt{2ru_0^2}\sin\frac{\phi_1}{2}\cosh\theta_1,$$
$$\beta_2 = \sqrt{2ru_0^2}\sin\frac{\phi_2}{2}\sinh\theta_2, \qquad \delta_2 = \sqrt{2ru_0^2}\sin\frac{\phi_2}{2}\cosh\theta_2.$$

Equation (10) is rewritten as

$$D = 2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\left\{\cos\frac{\phi_1+\phi_2}{2} - \cosh(\theta_1-\theta_2)\right\}.$$

Therefore, the resonant condition is given by

$$\phi_2 = 2\theta_{1i} - \phi_{1r}, \qquad \theta_2 = \theta_{1r} + \frac{\phi_{1i}}{2}.$$

We study the time evolution of soliton in the following five periods in time. The solutions (4) and (5) are approximated in each period as follows:

(**p$_1$**) $t \to -\infty$ (before the mode grows). The solution is given by equations (6) and (7), only the periodic soliton exists in the wave field.

(**p$_2$**) $t \sim \dfrac{\sigma_2}{\Omega_2}$; $(e^{-\Omega_2 t+\sigma_2} \sim O(1))$. The solutions in the backward region and forward region of the periodic soliton are given by

$$f \simeq 1 - e^{\xi_2}\cos\eta_2 + \frac{M_2}{4}e^{2\xi_2}, \tag{11}$$
$$g \simeq 1 - e^{\xi_2+i\phi_2}\cos\eta_2 + \frac{M_2}{4}e^{2(\xi_2+i\phi_2)}, \tag{12}$$

and

$$f \simeq \frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)}, \tag{13}$$
$$g \simeq \frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)+2i\phi_{1r}+\phi_2)}, \tag{14}$$

respectively. The solutions corresponding to equations (11)–(12) and (13)–(14) denote the growing-and-decaying mode and uniform state, respectively. Therefore, in this period, the mode is growing only in the backward region of the periodic soliton, but the mode has not grown as yet in the forward region.

(**p$_3$**) $t \sim \dfrac{\sigma_2 + \frac{1}{2}\ln L_1L_2}{\Omega_2}$; $(\sqrt{L_1L_2}e^{-\Omega_2 t+\sigma_2} \sim O(1))$

$$f \simeq 1 + \frac{1}{2L_2}e^{\xi_1+\xi_2}\cos(\eta_1+\eta_2+\Psi_1) + \frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)},$$
$$g \simeq 1 + \frac{1}{2L_2}e^{\xi_1+\xi_2+i(\phi_{1r}+\phi_2)}\cos(\eta_1+\eta_2+i\phi_{1i}+\Psi_1) + \frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)+2i(\phi_{1r}+\phi_2)}.$$

Only the periodic soliton in the resonant state exists in the wave field.

(**p$_4$**) $t \sim \dfrac{\sigma_2 + \ln L_1L_2}{\Omega_2}$; $(L_1L_2e^{\xi_2} \sim O(1))$. The solutions in the backward region and forward region of the periodic soliton are given by

$$f \simeq 1, \qquad g \simeq 1,$$

and

$$f \simeq \frac{M_1}{4L_1^2L_2^2}e^{2\xi_1}\left\{1 - L_1L_2e^{\xi_2}\cos(\eta_2 + \Psi_1 - \Psi_2) + \frac{M_2L_1^2L_2^2}{4}e^{2\xi_2}\right\},$$
$$g \simeq \frac{M_1}{4L_1^2L_2^2}e^{2(\xi_1+i\phi_{1r})}\left\{1 - L_1L_2e^{\xi_2+2i\phi_2}\cos(\eta_2 + \Psi_1 - \Psi_2) + \frac{M_2L_1^2L_2^2}{4}e^{2(\xi_2+i\phi_2)}\right\},$$

respectively. In this period, the mode is developed only in the forward region of the periodic soliton.

($\mathbf{p_5}$) $t \to +\infty$. The solution is given by equations (8) and (9) which is the periodic soliton after the grow and decay of the mode.

3 Conclusions

We have investigated the time evolution of the quasi-resonant interaction between periodic soliton and growing-and-decaying mode. Under the quasi-resonant condition, the mode develops first in the one side region of the periodic soliton. The periodic soliton is accelerated as a result of the grow and decay of the mode existed in the region and the wave field shifts to the intermediate state, where only the periodic soliton in the resonant state exists. This intermediate state persists over a comparatively long time interval. After sufficient long time, the mode starts to grow in the opposite side of the periodic soliton. The existence of soliton changes the evolution of the growing-and-decaying mode drastically as if the periodic soliton dominated the evolution of the instability in whole region of the wave field.

[1] Benjamin T.B. and Feir J.E., The disintegration of wave trains in deep water. Part 1. Theory, *J. Fluid Mech.*, 1967, V.27, 417–430.

[2] Zakharov V.E., Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech. Tech. Phys.*, 1968, V.9, 86–94.

[3] Hasimoto H. and Ono H., Nonlinear modulation of a gravity waves, *J. Phys. Soc. Jpn.*, 1972, V.33, 805–811.

[4] Davey A., The propagation of a weak nonlinear wave, *J. Fluid Mech.*, 1972, V.53, 769–781.

[5] Benny D. J. and Roskes G., Wave instabilites, *Stud. Appl. Math.*, 1969, V.48, 377–388.

[6] Davey A. and Stewartson K., On three-dimensional packets of surface waves, *Proc. R. Soc. London A*, 1974, V.338, 101–110.

[7] Lake B.M., Yuen H.C., Rungaldier H. and Ferguson W.E., Nonlinear deep-water waves: theory and experiment, *J. Fluid Mech.*, 1977, V.83, 49–74.

[8] Tajiri M. and Watanabe Y., Breather solutions to the focusing nonlinear Schrödinger equation, *Phys. Rev. E*, 1998, V.57, 3510–3519.

[9] Tajiri M. and Arai T., Growing-and-decaying mode solution to the Davey–Stewartson eqution, *Phys. Rev. E*, 1999, V.60, 2297–2305.

[10] Pelinovsky D., Instability and decay of solitary waves in the Davey–Stewartson 1 system, *Phys. Lett. A*, 1994, V.196, 181–186.

[11] Watanabe Y. and Tajiri M., Periodic soliton resonance: solution to the Davey–Stewartson equation, *J. Phys. Soc. Jpn.*, 1998, V.67, 705–708.

[12] Arai T., Takeuchi K. and Tajiri M., Note on periodic soliton resonance: solutions to the Davey–Stewartson II equation, *J. Phys. Soc. Jpn.*, 2001, V.70, 55–59.

[13] Tajiri M., Takeuchi K. and Arai T., Asynchronous development of the Benjamin–Feir unstable mode: solution of the Davey–Stewartson equation, *Phys. Rev. E*, to appear.

[14] Satsuma J. and Ablowitz M.J., Two-dimensional lumps in nonlinear dispersive system, *J. Math. Phys.*, 1979, V.20, 1496–1503.