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On Involutions which Preserve Natural Filtration

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In this work we study involutions in finitely presented $*$-algebras which preserve the natural filtration.

1 Introduction

Introducing additional structures is often useful in a study of algebraic objects, in particular finitely presented algebras and their representations, – introducing topology in algebras gives a comprehensive theory of Banach algebras or, more generally, a theory of locally convex algebras; introducing an involution, which we can consider as some inner symmetry, calls into being the theory of $*$-algebras; considering an involution together with the corresponding norm gives the theory of C^*-algebras. Moreover, on the one hand, studying not all representations but only those which "conserve" this additional structure (for example, $*$-representations) is simple (for example, $*$-representations are indecomposable if and only if they is irreducible, see [1]) on the other hand, this is often sufficient for applications.

In [1] the theory of $*$-representations of finitely presented $*$-algebras is studied, and the involution in the considered $*$-algebras often preserves filtration (see Definition 1). In this article we consider the following question. Let $\mathbb{F}_n$ be a free algebra with n generators $x_1,\ldots,x_n$ and an identity e, and lets us also have a unital finitely presented algebra

$$\mathbf{A} = \mathbb{C}\langle\, x_1,\ldots,x_n \,|\, q_1 = 0,\ldots,q_m = 0\,\rangle,$$

where $q_k \in \mathbb{F}$, $k = 1,\ldots,m$. We can assume, without loss of generality, that all relations q_k are nonlinear, for otherwise, the algebra $\mathbf{A}$ is isomorphic to an algebra with a smaller number of generators (roughly speaking, we can exclude generators that are linear combinations of the others). We will denote by $V(\mathbf{A})$ the linear subspace of $\mathbf{A}$ generated by the elements $x_0 = e, x_1,\ldots,x_n$. Then the question is how many involutions which map $V(\mathbf{A})$ into itself exist in the algebra $\mathbf{A}$ such that the corresponding $*$-algebras are not $*$-isomorphic.

The answer is that such an involution is unique and so we can always suppose that the generators are self-adjoint (see Theorem 1 and Proposition 1). Moreover, in some cases there is a $*$-isomorphism between the corresponding $*$-algebras such as it "conserves" the relations (see Theorem 1 and examples).

2 Main result

We will denote the free $*$-algebra with n self-adjoint generators z_k by $\mathbb{F}_n^*$. Some other involution will be denoted by $\star$. It is given by defining its values on generators. We will denote the free $*$-algebra with such an involution by

$$\mathbb{F}_n^\star = \mathbb{C}\langle\, x_1,\ldots,x_n \,|\, x_k^\star = p_k, k = 1,\ldots,n\,\rangle,$$

where $p_k \in \mathbb{F}_n$.

Definition 1. We say that an involution $\star$ of a $*$-algebra $\mathbf{A}^\star$ preserves the natural filtration iff the involution maps $\mathrm{V}(\mathbf{A}^\star)$ into itself.

Theorem 1. *Let an involution $\star$ of the $*$-algebra $\mathbb{F}_n^\star$ preserve the natural filtration. Then there is a $*$-isomorphism $\varphi : \mathbb{F}_n^* \to \mathbb{F}_n^\star$. Moreover, $\varphi(V(\mathbb{F}_n^*)) = V(\mathbb{F}_n^\star)$.*

Proof. We can assume that the first $n-l$ generators are self-adjoint and the others are not, such otherwise, we can renumber the generators. We will prove the theorem by induction on the number l of the generators that are not self-adjoint.

If $l = 0$ then there is nothing to prove, since all the generators are self-adjoint.

Let $1 \leqslant l \leqslant n$. Put

$$y_k = \frac{x_k + x_k^\star}{2}, \qquad k = 0, \dots, n.$$

It is evident that $y_k^\star = y_k$. Because the involution preserves the filtration, $x_k^\star \in V(\mathbb{F}_n^\star)$ and so $y_k \in V(\mathbb{F}_n^\star)$.

If $y_0 = e, y_1, \dots, y_n$ are linearly independent then we define $\varphi : \mathbb{F}_n^* \to \mathbb{F}_n^\star$ on the generators by $\varphi(z_k) = y_k$, $k = 0, \dots, n$, $z_0 = e$. Since $\dim V(\mathbb{F}_n^\star) = n + 1$ and $y_0, y_1, \dots, y_n$ are linearly independent and lie in $V(\mathbb{F}_n^\star)$, y_k, $k = 0, \dots, n$, is a linear basis of $V(\mathbb{F}_n^\star)$ and so

$$x_k = \sum_{j=0}^{n} \alpha_k^j y_j, \qquad \alpha_k^j \in \mathbb{C}.$$

Then the homomorphism inverse to φ is defined on the generators by

$$\varphi^{-1}(x_k) = \sum_{j=0}^{n} \alpha_k^j z_j.$$

So φ is an isomorphism of the algebras $\mathbb{F}_n^\star$ and $\mathbb{F}_n^*$. It is evident that φ is also a $*$-homomorphism and $\varphi(V(\mathbb{F}_n^*)) = V(\mathbb{F}_n^\star)$.

Let now $y_0 = e, y_1, \dots, y_n$ be linearly dependent. Then, since the first $n - l$ generators are self-adjoint, $y_j = x_j$ for $j = 0, \dots, n-l$ and, consequently, y_j are linearly independent. Then there exists k $(n - l < k \leqslant n)$ such that

$$y_k = \sum_{j \neq k} \lambda_j y_j, \qquad \lambda_j \in \mathbb{C}.$$

And since y_j are self-adjoint,

$$y_k = \sum_{j \neq k} \overline{\lambda}_j y_j, \qquad \lambda_j \in \mathbb{C}.$$

If we put $a_j = (\lambda_j + \overline{\lambda}_j)/2$ then we get

$$y_k = \sum_{j \neq k} a_j y_j, \qquad a_j \in \mathbb{R}.$$

Renumbering the generators we can suppose that $k = n - l + 1$.

Put

$$\mathbb{F}_n^{\star_1} = \mathbb{C}\langle\, v_1, \dots, v_n \,|\, v_j^{\star_1} = v_j, j = 1, \dots, k, v_j^{\star_1} = q_j, j > k \,\rangle,$$

where

$$q_j = p_j\left(v_1,\ldots,v_{k-1},-2iv_k+\sum_{j\neq k}a_jv_j,v_{k+1},\ldots,v_n\right),\qquad j>k.$$

Define $\psi:\mathbb{F}_n^{\star_1}\to\mathbb{F}_n^{\star}$ on generators by the formula $\psi(v_j)=x_j$, if $j\neq k$, and

$$\psi(v_k)=\frac{i}{2}\left(x_k-\sum_{j\neq k}a_jx_j\right).$$

It is evident that ψ is an isomorphism of the algebras $\mathbb{F}_n^{\star_1}$ and $\mathbb{F}_n^{\star}$. Let us show that ψ is a $*$-homomorphism.

If $j<k$, then $\psi(v_j)^\star = x_j^\star = x_j = \psi(v_j) = \psi(v_j^{\star_1})$.

If $j>k$, then $\psi(v_j)^\star = x_j^\star = p_j$ and again

$$\psi(v_j^{\star_1}) = \psi(q_j) = \psi\left(p_j\left(v_1,\ldots,v_{k-1},-2iv_k+\sum_{j\neq k}a_jv_j,v_{k+1},\ldots,v_n\right)\right)$$

$$= p_j\left(x_1,\ldots,x_{k-1},x_k-\sum_{j\neq k}a_jx_j+\sum_{j\neq k}a_jx_j,x_{k+1},\ldots,x_n\right) = p_j = \psi(v_j^{\star_1}).$$

Finally,

$$\psi(v_k)^\star = -\frac{i}{2}\left(x_k^\star-\sum_{j\neq k}a_jx_j^\star\right)\quad\text{and}\quad \psi(v_k^{\star_1})=\psi(v_k)=\frac{i}{2}\left(x_k-\sum_{j\neq k}a_jx_j\right).$$

So

$$\psi(v_k^{\star_1})-\psi(v_k)^\star = i\left(y_k-\sum_{j\neq k}a_jy_j\right)=0,$$

i.e., $\psi(v_k^*)=\psi(v_k)^*$.

We have proved that $\mathbb{F}_n^{\star}$ and $\mathbb{F}_n^{\star_1}$ are $*$-isomorphic. Further, by the definition of ψ we again have $\psi(V(\mathbb{F}_n^{\star_1}))=V(\mathbb{F}_n^{\star})$. And now we have $l-1$ generators in $\mathbb{F}_n^{\star_1}$ that are not self-adjoint and so, by the inductive assumption, $\mathbb{F}_n^{\star_1}$ is $*$-isomorphic to $\mathbb{F}_n^{*}$ and, consequently, $\mathbb{F}_n^{\star}$ is $*$-isomorphic to $\mathbb{F}_n^{*}$. ■

3 Corollary and examples

In this section we will obtain a corollary of Theorem 1 and consider some examples.

Consider the $*$-algebra

$$\mathbf{A}=\mathbb{C}\langle\, x_1,\ldots,x_n \,|\, x_k^\star = p_k, k=1,\ldots,n,\ r_1=0,\ldots,r_m=0\,\rangle,$$

where $r_k\in\mathbb{F}_n$, $k=0,\ldots,m$. Let I be a $*$-ideal generated by $r_1,\ldots,r_m$, i.e., $\mathbf{A}$ is a $*$-isomorphic to the factor $\mathbb{F}_n^{\star}/\mathrm{I}$.

By increasing the number of generators (not more than two times) and adding new relations we always can construct a $*$-algebra which is $*$-isomorphic to $\mathbf{A}$ such that its generators are self-adjoint. The corollary of Theorem 1 claims that if the involution is "good" then we can leave the number of the generators and relations the same as in $\mathbf{A}$ and the length of words in the relations does not grow.

Proposition 1. *Let the involution $\star$ preserves the filtration. Then the $*$-algebra* $\mathbf{A}$ *is $*$-isomorphic to the $*$-algebra*

$$\mathbf{B} = \mathbb{C}\langle\, z_1, \ldots, z_n \,|\, z_k^* = z_k, k = 1, \ldots, n,\ s_1 = 0, \ldots, s_m = 0 \,\rangle,$$

where s_k have the same degrees as r_k, $k = 1, \ldots, m$.

Proof. Since the involution $\star$ preserves the filtration then, there exists a $*$-isomorphism $\varphi : \mathbb{F}_n^\star \to \mathbb{F}_n^*$. Denote by $\mathrm{J} = \varphi(\mathrm{I})$ the $*$-ideal generated by the relations $s_1 = \varphi(r_1), \ldots, s_m = \varphi(r_m)$. It is evident that so defined s_k have the same degrees as r_k. Then we can put $\mathbf{B} = \mathbb{F}_n^*/\mathrm{J}$.

Let i be an injection of I into $\mathbb{F}_n^\star$ and π a projection of the latter into $\mathbf{A}$. Similarly, let i_0 be an injection of J into $\mathbb{F}_n^*$ and π_0 a projection into $\mathbf{B}$. The restriction of φ to I will be denoted by φ_0. Then we get a commutative diagram of $*$-homomorphisms,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{I} & \xrightarrow{i} & \mathbb{F}_n^\star & \xrightarrow{\pi} & \mathbf{A} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \varphi_0} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & \mathrm{J} & \xrightarrow{i_0} & \mathbb{F}_n^* & \xrightarrow{\pi_0} & \mathbf{B} & \longrightarrow & 0
\end{array}$$

where ψ is defined by the formula $\psi(\pi(a)) = \pi_0(\varphi(a))$, for any $a \in \mathbb{F}_n^\star$.

Now we show that ψ is well-defined. Indeed, since π is surjective, ψ is defined for all elements of $\mathbf{A}$. If $\pi(a) = 0$ then $a \in \mathrm{I}$ and so $\varphi(a) \in \mathrm{J}$, consequently, $\psi(\pi(a)) = \pi_0(\varphi(a)) = 0$.

It is evident that ψ is surjective. Now we show that it is injective.Indeed, if $\psi(\pi(a)) = 0$, then it means that $\pi_0(\varphi(a)) = 0$ and so $\varphi(a) \in \mathrm{J}$, consequently, $a \in \mathrm{I}$, from where we get $\pi(a) = 0$. It is also evident that ψ is a $*$-homomorphism.

So we have constructed a $*$-isomorphism of the $*$-algebras $\mathbf{A}$ and $\mathbf{B}$. ■

Actually we have "changed" the generators in $\mathbf{A}$ so that the new generators are self-adjoint. But the next example shows that, generally speaking, the relations are changed too.

Example 1. Consider the $*$-algebra

$$\mathbf{Q_2} = \mathbb{C}\langle\, q_1, q_2 \,|\, q_1^\star = q_2, q_2^\star = q_1, q_1^2 = q_1, q_2^2 = q_2 \,\rangle.$$

A $*$-isomorphism $\varphi : \mathbb{F}_2^\star \to \mathbb{F}_2^*$ is defined by the formulas

$$\varphi(q_1) = z_1 + iz_2, \qquad \varphi(q_2) = z_1 - iz_2.$$

Then

$$\varphi(q_1^2 - q_1) = (z_1 + iz_2)^2 - z_1 - iz_2 = z_1^2 - z_2^2 + i\{z_1, z_2\} - z_1 - iz_2,$$

similarly

$$\varphi(q_2^2 - q_2) = z_1^2 - z_2^2 - i\{z_1, z_2\} - z_1 + iz_2,$$

where $\{\,,\}$ is the anticommutator.

It is evident that the ideal generated by these relations is also generated by the relations

$$z_1^2 - z_2^2 = z_1 \quad \text{and} \quad \{z_1, z_2\} = z_2.$$

So $\mathbf{Q_2}$ is $*$-isomorphic to the $*$-algebra

$$\mathbb{C}\langle\, z_1, z_2 \,|\, z_1^* = z_1, z_2^* = z_2, z_1^2 - z_2^2 = z_1, \{z_1, z_2\} = z_1 \,\rangle.$$

On the other hand, it is not difficult to show that there is no $*$-isomorphisms between $\mathbf{Q_2}$ and the $*$-algebra

$$\mathbb{C}\langle\, x_1, x_2 \,|\, x_1^* = x_1, x_2^* = x_2, x_1^2 = x_1, x_2^2 = x_2 \,\rangle.$$

The next two examples show that there are algebras that are not free for which an analogue of Theorem 1 is also true.

Example 2. Consider the $*$-algebra of polynomials in n variables, P_n. It is a factor of the free algebra by the ideal I generated by the relations

$$[x_j, x_k] = 0, \qquad j, k = 1, \dots n,$$

where $[\,,]$ is the commutator. All elements of the ideal I can be written as $[p_1, p_2]$, where $p_1, p_2 \in \mathbb{F}_n$. Then, for any involution in $\mathbb{F}_n$, $[p_1, p_2]^\star = [p_2^\star, p_1^\star] \in \mathrm{I}$ so I is a $*$-ideal. Let $\star$ preserves the filtration. Then the $*$-ideal $\varphi(\mathrm{I})$ consists of all elements which can be written as $[\varphi(p_1), \varphi(p_2)]$. So it is generated by the relations

$$[z_j, z_k] = 0, \qquad j, k = 1, \dots n,$$

And we have the $*$-isomorphism of P_n^* and $P_n^\star$.

Example 3. Consider one more algebra for which a theorem analogous to Theorem 1 holds. Let

$$\mathbf{A} = \mathbb{C}\langle\, p, q \,|\, [[p, q], p] = 0, [[p, q], q] = 0 \,\rangle.$$

Let I be an ideal generated by the corresponding relations. Then it is evident that for any $a, b, c \in V(\mathbb{F}_n^\star)$ we have $[[a, b], c] \in \mathrm{I}$.

Now, let us introduce in $\mathbf{A}$ an involution $\star$ which preserves the filtration. Let us show that the ideal I is a $*$-ideal,

$$-[[p, q], p]^\star = [p, [p, q]]^\star = [[p, q]^\star, p^\star] = [[q^\star, p^\star], p^\star],$$

but $p^\star, q^\star \in V(\mathbb{F}_n^\star)$ so $[[p, q], p]^\star \in \mathrm{I}$. Similarly, $[[p, q], q]^\star \in \mathrm{I}$.

Since $\star$ preserves the filtration, by Theorem 1 there is a $*$-isomorphism $\varphi : \mathbb{F}_2^\star \to \mathbb{F}_2^*$ and there exist elements $a_1, a_2 \in V(\mathbb{F}_2^\star)$ such that $\varphi(a_1) = z_1$ and $\varphi(a_2) = z_2$, where z_1 and z_2 are generators of $\mathbb{F}_2^*$. Then the $*$-ideal $\varphi(\mathrm{I})$ is generated by the relations

$$[[z_1, z_2], z_1] = 0, \qquad [[z_1, z_2], z_2] = 0.$$

So we have a $*$-isomorphism of $\mathbf{A}^\star$ and the $*$-algebra

$$\mathbb{C}\langle\, z_1, z_2 \,|\, z_1^* = z_1, z_2^* = z_2, [[z_1, z_2], z_1] = 0, [[z_1, z_2], z_2] = 0 \,\rangle.$$

[1] Ostrovskyĭ V.L. and Samoĭlenko Yu.S., Introduction to the theory of representations of finitely presented $*$-algebras. I. Representations by bounded operators, *Rev. Math. and Math. Phys.*, 1999, V.11, 1–261.