Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 2, 688–690

On Multi-Parameter Families of Hermitian Exactly Solvable Matrix Schrödinger Models

Stanislav SPICHAK

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01601 Kyiv-4, Ukraine
E-mail: *spichak@imath.kiev.ua*

Five multi-parameter families of Hermitian exactly solvable matrix Schrödinger operators in one variable was constructed.

1 Introduction

One of the principal aims of the present paper is developing a systematic algebraic procedure for constructing exactly solvable (ES) Hermitian matrix Schrödinger operators

$$\hat{H}[x] = \partial_x^2 + V(x). \tag{1}$$

Here $V(x)$ is an 2×2 matrix whose entries are smooth complex-valued functions of x. Hereafter we denote d/dx as ∂_x.

The well-known procedure of constructing a ES matrix (scalar) model is based on the concept of a Lie-algebraic Hamiltonian [1, 2] (the Turbiner–Shifman approach). We call a second-order operator in one variable Lie-algebraic if the following requirements are met:

- the Hamiltonian is a quadratic form with constant coefficients of first-order operators $Q_1, Q_2, \ldots, Q_n$ forming a Lie algebra g;
- the Lie algebra g has a finite-dimensional invariant subspace $\mathcal{I}$ of the whole representation space.

Now if a given Hamiltonian $H[x]$ is Lie-algebraic, then after being restricted to the space $\mathcal{I}$ it becomes a matrix operator $\mathcal{H}$ whose eigenvalues and eigenvectors are computed in a purely algebraic way. This means that the Hamiltonian $H[x]$ is exactly solvable.

In the paper [3] we have extended the Turbiner–Shifman approach to the construction of quasi-exactly solvable (QES) models on line for the case of matrix Hamiltonians. In this paper we suggested the method for construction of exactly solvable matrix models, which based on the idea explained in [3]. Let us remind, the method consists in supplementing a set of operators $Q_1, Q_2, \ldots, Q_n$, forming a representation of some algebra, so that the obtained set of operators left an appropriate subspace $\mathcal{I}$ invariant. However, there is a difference between the approaches suggested in this paper and in [3]. Namely, the obtained set of operators does not form a Lie algebra, in contrast to a set found in [3].

So, let us realize this method considering the set of the operators

$$Q_1 = A, \qquad Q_2 = Be^{-x}, \qquad Q_3 = c^x(\partial_x + C), \qquad Q_4 = \partial_x, \tag{2}$$

which form the representation of the algebra $L^2_{4,8}$, found in [4]. Here $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$.

The operators (2) belong to the class $\mathcal{L}$ of matrix differential operators of the form

$$\mathcal{L} = \{Q : \ Q = a(x)\partial_x + A(x)\}, \tag{3}$$

where $a(x)$ is a smooth real-valued function and $A(x)$ is an 2×2 matrix whose entries are smooth complex-valued functions of x.

The corresponding finite-dimensional invariant space has the form

$$\mathcal{G} = \langle e^{-cx}\vec{e}_1, e^{-(c+1)x}\vec{e}_1, \dots, e^{-(c+k+1)x}\vec{e}_1\rangle \oplus \langle e^{-cx}\vec{e}_2, e^{-(c+1)x}\vec{e}_2, \dots, e^{-(c+k)x}\vec{e}_2\rangle, \tag{4}$$

where k is an arbitrary natural number.

It is easy to verify that all operators from the class (3) and acting in the space (4), are

$$\begin{aligned} &R_1 = S_0, \qquad R_2 = S_+ e^x\partial_x, \qquad R_3 = S_+\partial_x, \qquad R_4 = S_0 e^x \partial_x,\\ &R_5 = S_0\partial_x, \qquad R_6 = S_+ e^{-x}\partial_x, \qquad R_7 = S_- e^x\partial_x, \end{aligned} \tag{5}$$

where $S_0 = \sigma_3/2$, $S_\pm = (i\sigma_2 \pm \sigma_1)/2$, σ_k are the 2×2 Pauli matrices.

Then we construct a Hamiltonian $H[x]$ of the form

$$H[x] = \xi(x)\partial_x^2 + B(x)\partial_x + C(x), \tag{6}$$

which can be obtained by using of all bilinear combinations of operators belonging to the linear span of the operators (2), (5).

Here we omitted a very cumbersome calculation and some technical methods to reduce an operator (6) to a standard Schrödinger operator

$$\hat{H}[y] = \partial_y^2 + V(y). \tag{7}$$

We give below the final results, namely, the restrictions on the choice of parameters and the explicit forms of the QES Hermitian Schrödinger operators (7). In the formulae below we denote the conjunction of two statements A and B as $[A]\bigvee[B]$.

Let complex-valued parameters $\tilde{\boldsymbol{\beta}} = (\beta_1, i\beta_2, \beta_3)$, $\tilde{\boldsymbol{\delta}} = (\delta_1, i\delta_2, \delta_3)$ and others satisfy the following conditions

$$\begin{aligned} &\left[\tilde{\boldsymbol{\beta}}^2 < 0,\ \varepsilon \neq 0,\ \beta_1 \neq \beta_2\right] \wedge \left[\{\alpha_0, \alpha_1, \alpha_2, \lambda, \gamma_0, \beta_0, \varepsilon(\beta_1-\beta_2),\ \delta_1(\beta_1-\beta_2)+\beta_3\delta_3, \delta_3\} \subset \mathbb{R}\right]\\ &\qquad \wedge \left[\mu = \alpha_0 = \alpha_2\lambda - \beta_0 + 2\alpha_2\frac{\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\delta}}}{\tilde{\boldsymbol{\beta}}^2} = -\gamma_0\lambda + 2\alpha_1\frac{\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\varepsilon}}}{\tilde{\boldsymbol{\beta}}^2}\right.\\ &\qquad \left. = \alpha_1\lambda - \beta_0\lambda - \gamma_0 + 2\alpha_1\frac{\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\delta}}}{\tilde{\boldsymbol{\beta}}^2} + 2\alpha_2\frac{\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\varepsilon}}}{\tilde{\boldsymbol{\beta}}^2} = 0\right]. \end{aligned}$$

Then, the following Schrödinger operator be hermitian:

$$\begin{aligned} \hat{H}[y] = {} & \partial_y^2 + \frac{1}{16\left(\alpha_2 e^{2x} + \alpha_1 e^x\right)}\left[\left(\alpha_1^2 + 8\alpha_2\gamma_0 - 4\beta_0^2 - 4\tilde{\boldsymbol{\beta}}^2\right)e^{2x}\right.\\ & \left. + \left(8\alpha_1\gamma_0 - 8\gamma_0\beta_0 - 8\lambda\tilde{\boldsymbol{\beta}}^{\mathbf{2}}\right)e^x - 4\left(\lambda^2\tilde{\boldsymbol{\beta}}^{\mathbf{2}} + \gamma_0^2\right)\right]\\ & + \left(P\cos\left(\theta(x)\sqrt{-\tilde{\boldsymbol{\beta}}^2} + \Omega\right) + \frac{\varepsilon(\beta_1-\beta_2)}{\sqrt{-\tilde{\boldsymbol{\beta}}^2}}e^{-x}\cos\left(\theta(x)\sqrt{-\tilde{\boldsymbol{\beta}}^2}\right)\right)\sigma_1\\ & + \left(P\sin\left(\theta(x)\sqrt{-\tilde{\boldsymbol{\beta}}^2} + \Omega\right) + \frac{\varepsilon(\beta_1-\beta_2)}{\sqrt{-\tilde{\boldsymbol{\beta}}^2}}e^{-x}\sin\left(\theta(x)\sqrt{-\tilde{\boldsymbol{\beta}}^2}\right)\right)\sigma_3\Bigg|_{x=z^{-1}(y)}, \end{aligned} \tag{8}$$

here

$$P = \sqrt{\delta_3^2 - \frac{(\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\delta}})^2}{\tilde{\boldsymbol{\beta}}^2}}, \quad \cos\Omega = \frac{\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\delta}}}{P\sqrt{-\tilde{\boldsymbol{\beta}}^2}}, \quad \sin\Omega = \frac{\delta_3}{P}, \quad \theta(x) = -\int\frac{e^x+\lambda}{\alpha_0 + \alpha_1 e^x + \alpha_2 e^{2x}}dx.$$

We denote the function $z^{-1}(y)$ as the inverse of the function

$$y = z(x) \equiv \int \frac{dx}{\sqrt{\alpha_2 x^2 + \alpha_1 x + \alpha_0}}. \tag{9}$$

Furthermore, the basis elements of the corresponding transformed invariant space take the form

$$\mathcal{G} = \langle \Lambda^{-1} e^{-cz^{-1}(y)} \vec{e}_1, \Lambda^{-1} e^{-(c+1)z^{-1}(y)} \vec{e}_1, \ldots, \Lambda^{-1} e^{-(c+k+1)z^{-1}(y)} \vec{e}_1 \rangle$$
$$\oplus \langle \Lambda^{-1} e^{-cz^{-1}(y)} \vec{e}_2, \Lambda^{-1} e^{-(c+1)z^{-1}(y)} \vec{e}_2, \ldots, \Lambda^{-1} e^{-(c+k)z^{-1}(y)} \vec{e}_2 \rangle,$$

where the constant matrix

$$\Lambda = \Lambda_1 \cdot \Lambda_2 = \exp\left(\frac{\beta_3}{2\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\epsilon}}} \tilde{\boldsymbol{\epsilon}}\boldsymbol{\sigma} \right) \cdot \exp(\nu\sigma_3), \qquad e^{2\nu} = \frac{\sqrt{-\tilde{\boldsymbol{\beta}}^2}}{\beta_1 - \beta_2}.$$

We give the particular example of a Hermitian model which has the important property. Namely, a corresponding invariant space is a Hilbert one. That is, one can define a scalar product

$$\langle f_1(y), f_2(y) \rangle = \int \vec{f}_1(y)^\dagger f_2(y) dy,$$

where $\vec{f}_1(y)^\dagger$ is a Hermitian conjugation of the vector $f_1(y)$. Let us put in the formula (8) $\alpha_2 = \beta_2 = \delta_3 = \gamma_0 = 1$, $\epsilon = \frac{1}{2}$, and the rest coefficients are equal zero. Then we have the following Hamiltonian

$$\hat{H}(y) = \partial_y^2 - \left(\sin y + \frac{1}{2} y \cos y \right) \sigma_1 + \left(\cos y - \frac{1}{2} y \sin y \right) \sigma_3 + \frac{3}{4}.$$

The corresponding invariant space for this operator $\mathcal{G}$ has $(2k+3)$-dimension and is generated by the vectors

$$\vec{f}_j = i e^{-y^2/4} y^j \exp\left(\frac{-i\sigma_2}{2} y \right) \vec{e}_1, \qquad \vec{g}_s = -i e^{-y^2/4} y^s \exp\left(\frac{-i\sigma_2}{2} y \right) \vec{e}_2,$$

where $j = 0, \ldots, k+1$, $s = 0, \ldots, k$, $\vec{e}_1 = (1,0)^T$, $\vec{e}_2 = (0,1)^T$, σ_i $(i = 1,2,3)$ are 2×2 Pauli matrices.

[1] Zaslavskii O.B. and Ulyanov V.V., New classes of exact solutions of the Schrödinger equation and potential-field description of spin systems, *Sov. Phys. JETF*, 1984, V.60, 991–996.

[2] Turbiner A.V., Quasi-exactly-solvable problems and sl(2) algebra, *Commun. Math. Phys.*, 1988, V.118, 467–474.

[3] Spichak S. and Zhdanov R., On algebraic classification of Hermitian quasi-exactly solvable matrix Schrödinger operators on line, *J. Phys. A: Math. Gen.*, 1999, V.32, 3815–3831.

[4] Abramenko A., Matrix realizations of four-dimensional Lie algebras and corresponding invariant spaces, in Proceedinds of Third International Conference "Symmetry in Nonlinear Mathematical Physics" (12–18 July, 1999, Kyiv), Editors A.G. Nikitin and V.M. Boyko, Kyiv, Institute of Mathematics, 2000, V.30, Part 2, 324–329.