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The Complete Set of Generalized Symmetries for the Calogero–Degasperis–Ibragimov–Shabat Equation

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We find the complete set of local generalized symmetries (including x, t-dependent ones) for the Calogero–Degasperis–Ibragimov–Shabat (CDIS) equation, and investigate the properties of these symmetries.

1 Introduction

All known today integrable scalar (1+1)-dimensional evolution equations with time-independent coefficients possess infinite-dimensional Abelian algebras of time-independent higher order symmetries (see e.g. [1, 2]). However, the equations of this kind usually do not have *local* time-dependent higher order symmetries. The only known exceptions from this rule seem to occur [3] for linearizable equations like e.g. the Burgers equation, for which the complete set of symmetries was found in [4]. In the present paper we confirm this for a third order linearizable equation (4), which is referred below as Calogero–Degasperis–Ibragimov–Shabat equation, and exhibit the complete set of its time-dependent local generalized symmetries. This equation was discovered by Calogero and Degasperis [5] and studied, among others, by Ibragimov and Shabat [6], Svinolupov and Sokolov [7], Sokolov and Shabat [8], Calogero [9], and by Sanders and Wang [10].

The paper is organized as follows. In Section 2 we recall some well known definitions and results on the symmetries of evolution equations. In Section 3 we present the main result – Theorem 1, giving the complete description of the set of all local generalized symmetries for CDIS equation.

2 Basic definitions and known results

Consider a $(1+1)$-dimensional evolution equation

$$\partial u/\partial t = F(x, u, u_1, \dots, u_n), \qquad n \geq 2, \qquad \partial F/\partial u_n \neq 0, \tag{1}$$

for a scalar function u, where $u_l = \partial^l u/\partial x^l$, $l = 0, 1, 2, \dots$, $u_0 \equiv u$, and its (local) *generalized symmetries* [1], i.e. the generalized vector fields $\mathcal{G} = G\,\partial/\partial u$, where $G = G(x, t, u, u_1, \dots, u_k)$, $k \in \mathbb{N}$, is such that the evolution equation $\partial u/\partial\tau = G$ is compatible with (1). Below we shall identify the symmetry $\mathcal{G} = G\,\partial/\partial u$ with its *characteristics* G.

Recall [2, 12] that for any function $H = H(x, t, u, u_1, \dots, u_q)$ the greatest m such that $\partial H/\partial u_m \neq 0$ is called its *order* and is denoted as $m = \operatorname{ord} H$. We assume that $\operatorname{ord} H = 0$ for any $H = H(x, t)$. A function f of $x, t, u, u_1, \dots$ is called *local* (cf. [11, 15]), if it has a finite order.

Denote by $S_F^{(k)}$ the space of local generalized symmetries of (1) that are of order not higher than k. Let also

$$S_F = \bigcup_{j=0}^{\infty} S_F^{(j)},\ \Theta_F = \{H(x,t) \mid H(x,t) \in S_F\},\ \mathrm{St}_F = \{G \in S_F \mid \partial G/\partial t = 0\},$$
$$S_{F,k} = S_F^{(k)}/S_F^{(k-1)} \text{ for } k \in \mathbb{N};\ S_{F,0} = S_F^{(0)}/\Theta_F.$$

The set S_F is a Lie algebra with respect to the Lie bracket (see e.g. [1, 15])

$$[H,R] = R_*(H) - H_*(R) = \nabla_H(R) - \nabla_R(H).$$

Here for any local Q we set

$$Q_* = \sum_{i=0}^{\operatorname{ord} Q} \frac{\partial Q}{\partial u_i} D^i, \qquad \nabla_Q = \sum_{i=0}^{\infty} D^i(Q)\,\frac{\partial}{\partial u_i},$$

and $D = \partial/\partial x + \sum_{i=0}^{\infty} u_{i+1}\partial/\partial u_i$ is the total derivative with respect to x.

Note (see e.g. [1]) that a local function G is a symmetry of (1) if and only if

$$\partial G/\partial t = -[F,G]. \tag{2}$$

Equation (2) implies [1, 11]

$$\partial G_*/\partial t \equiv (\partial G/\partial t)_* = \nabla_G(F_*) - \nabla_F(G_*) + [F_*, G_*], \tag{3}$$

where $\nabla_F(G_*) \equiv \sum_{j=0}^{\operatorname{ord} G} \nabla_F\left(\frac{\partial G}{\partial u_j}\right) D^j$ and likewise for $\nabla_G(F_*)$; $[\cdot\,,\cdot]$ stands for the usual commutator of linear differential operators.

Consider also (see e.g. [2, 11, 12] for more information) the set FS of formal series in powers of D, i.e., the expressions of the form $\mathfrak{H} = \sum_{j=-\infty}^{m} h_j D^j$, where h_j are local functions. The greatest k such that $h_k \neq 0$ is called the degree of $\mathfrak{H} \in FS$ and is denoted by $\deg \mathfrak{H}$. Recall that $\mathfrak{R} \in FS$ is called a *formal symmetry* of infinite rank for (1), if it satisfies the relation (see e.g. [2, 12])

$$\partial \mathfrak{R}/\partial t + \nabla_F(\mathfrak{R}) - [F_*, \mathfrak{R}] = 0.$$

3 Symmetries of the CDIS equation

The Calogero–Degasperis–Ibragimov–Shabat (CDIS) equation has the form [5, 6]

$$u_t = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1. \tag{4}$$

Let us mention that this is the only third order (1+1)-dimensional scalar polynomial λ-homogeneous evolution equation of the form $u_t = u_n + f(u, u_1, \ldots, u_{n-1})$ with $\lambda = 1/2$ which possesses infinitely many x,t-independent local generalized symmetries [13]. This equation is linearized into $v_t = v_3$ upon setting $v = \exp(\omega)u$, where $\omega = D^{-1}(u^2)$ [8]. It appears to possess only one local conserved density $\rho = u^2$ (see e.g. [7, 8] and references therein), but it has a Hamiltonian operator and infinitely many conserved densities explicitly dependent on the nonlocal variable ω [7].

In order to refer to the sets of symmetries of the CDIS equation, we shall use the subscript 'CDIS' instead of F, i.e., S_{CDIS} will denote the Lie algebra of all generalized symmetries of (4), etc. From now on F will stand for the right-hand side of the CDIS equation, that is, $u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1$.

Let G be a local generalized symmetry of order $k \geq 1$ for (4). Equating to zero the coefficient at D^{k+2} in (3) and solving the arising equation, we obtain (see e.g. [1]) that

$$\partial G/\partial u_k = c_k(t), \tag{5}$$

where $c_k(t)$ is a function of t.

Below we assume without loss of generality that any symmetry $G \in S_{\mathrm{CDIS},k}$, $k \geq 1$, vanishes if the relevant function $c_k(t)$ is identically equal to zero.

Equating to zero the coefficients at D^{k+1} and D^k in (3), we see that for $k \geq 3$ we have $\partial^2 G/\partial x \partial u_{k-1} = 0$ and

$$\partial^2 G/\partial x \partial u_{k-2} = \dot{c}_k(t)/3. \tag{6}$$

Repeatedly using (6) and taking into account that $G \in S_{\mathrm{CDIS}}$ implies $\tilde{G} = \partial^r G/\partial x^r \in S_{\mathrm{CDIS}}$, we find that $\operatorname{ord} \tilde{G} \leq k - 2r$ and

$$\partial \tilde{G}/\partial u_{k-2r} = (1/3)^r d^r c_k(t)/dt^r. \tag{7}$$

For $r = [k/2]$ we have $\operatorname{ord} \tilde{G} \leq 1$. As u_1 is the only generalized symmetry of CDIS equation from $S^{(1)}_{\mathrm{CDIS}}$, and u_1 is time-independent, we see that $c_k(t)$ satisfies the equation $d^m c_k(t)/dt^m = 0$ for $m = [k/2] + 1$. Therefore, $\dim S_{\mathrm{CDIS},k} \leq [k/2] + 1$ for $k \geq 1$.

As all symmetries from $S^{(2)}_{\mathrm{CDIS}}$ are exhausted by u_1, by Theorem 2 of [17] all generalized symmetries of the CDIS equation are polynomial in time t.

Now let us turn to the study of time-independent symmetries of CDIS equation. This equation has infinitely many x, t-independent generalized symmetries, hence [18] a formal symmetry of infinite rank of the form $\mathfrak{L} = D + \sum_{j=0}^{\infty} a_j D^{-j}$, where a_j are some x, t-independent local functions.

Since we have $\deg \nabla_G(F_*) \leq 2$ for any G, by (3) and Lemma 9 from [15] for any $G \in \mathrm{St}_{\mathrm{CDIS}}$, $k = \operatorname{ord} G \geq 2$, we can represent G_* in the form $G_* = \sum_{j=1}^{k} \alpha_j \mathfrak{L}^j + \mathfrak{B}$, where α_j are some constants and $\mathfrak{B}$ is some formal series with time-independent coefficients, $\deg \mathfrak{B} < 1$. We have $\partial \mathfrak{L}/\partial x = 0$, so $\partial G_*/\partial x = \partial \mathfrak{B}/\partial x$ and $\deg \partial G_*/\partial x < 1$.

Thus, any symmetry $G \in \mathrm{St}_{\mathrm{CDIS}}$, $k \equiv \operatorname{ord} G \geq 2$, can be represented in the form

$$G = G_0(u, \ldots, u_k) + Y(x, u). \tag{8}$$

It is obvious that $\partial Y/\partial x = \partial G/\partial x \in \mathrm{St}_{\mathrm{CDIS}}$ and $\operatorname{ord} \partial Y/\partial x = 0$. But the CDIS equation has no generalized symmetries of order zero, so $\partial Y/\partial x = 0$, and thus any time-independent symmetry G of order $k \geq 2$ for CDIS equation is x-independent as well. The straightforward computation shows that the same statement holds true for the symmetries of order lower than 2. Using the symbolic method, it is possible to show [13] that CDIS equation has no even order t, x-independent symmetries. Hence, it has no even order time-independent generalized symmetries at all.

Now let us show that the same is true for time-dependent generalized symmetries as well. Recall that the CDIS equation is invariant under the scaling symmetry $K = 3tF + xu_1 + u/2$. Hence, if a symmetry Q contains the terms of weight γ (with respect to the weighting induced by K, cf. [13, 14]), there exists a homogeneous symmetry $\tilde{Q}$ of the same weight γ. We shall write this as $\mathrm{wt}(\tilde{Q}) = \gamma$. Note that we have $[K, \tilde{Q}] = (\gamma - 1/2)\tilde{Q}$.

If $G \in S_{\mathrm{CDIS},k}$, $k \geq 1$, is a polynomial in t of degree m, then its leading coefficient $\partial G/\partial u_k = c_k(t)$ also is a polynomial in t of degree $m' \leq m$, i.e., $c_k(t) = \sum_{j=0}^{m'} t^j c_{k,j}$, where $c_{k,m'} \neq 0$.

Consider $\tilde{G} = \partial^{m'} G/\partial t^{m'} \in S^{(k)}_{\rm CDIS}$. We have $\partial \tilde{G}/\partial u_k = \mathrm{const} \neq 0$, hence $\tilde{G}$ contains the terms of the weight $k + 1/2$. Let P be the sum of all terms of weight $k + 1/2$ in $\tilde{G}$. Clearly, P is a homogeneous symmetry of weight $k+1/2$ by construction, $\operatorname{ord} P = k$ and $\partial P/\partial u_k$ is a nonzero constant. Next, $\partial P/\partial t = -[F, P] \in S_{\rm CDIS}$, and the symmetry $\partial P/\partial t$ is homogeneous of weight $k+7/2$. Obviously, $\operatorname{ord} \partial P/\partial t \leq k-1$. By the above, all symmetries in $S_{\rm CDIS}$ are polynomial in t, and thus for any homogeneous $B \in S_{\rm CDIS}$, $b \equiv \operatorname{ord} B \geq 1$, we have $\partial B/\partial u_b = t^r c_b$, $c_b = \mathrm{const}$ for some $r \geq 0$. Hence, $\mathrm{wt}(B) = b - 3r + 1/2 \leq b + 1/2$, and thus for $k \geq 1$ the set $S_{\rm CDIS}$ does not contain homogeneous symmetries B such that $\mathrm{wt}(B) = k + 7/2$ and $\operatorname{ord} B \leq k - 1$, so $\partial P/\partial t = 0$.

Taking into account the absence of generalized symmeries of order zero for CDIS equation, we conclude that existence of a time-independent generalized symmetry of order $k \geq 1$ is a necessary condition for the existence of a polynomial-in-time symmetry $G \in S_{\rm CDIS}$ of the same order. Moreover, by the above all symmetries from $S_{\rm CDIS}$ are polynomial in t. Hence, the absence of time-independent local generalized symmetries of even order for the CDIS equation immediately implies the absence of any *time-dependent* local generalized symmetries of even order.

Thus, we have shown that the CDIS equation has no (local) generalized symmetries of even order and that for any $k \geq 1$ $\dim S_{{\rm CDIS},k} \leq [k/2] + 1$. Therefore, if for all odd $k = 2l + 1$ we exhibit $l+1$ symmetries of order k, then these symmetries will span the whole Lie algebra $S_{\rm CDIS}$ of (local) generalized symmetries for the CDIS equation.

The symmetries in question can be constructed in the following way.

Let $\tau_{m,0} = x^m u_1 + m x^{m-1} u/2$, $m = 0, 1, 2 \ldots$, and $\tau_{1,1} = x(u_3 + 3u^2 u_2 + 9uu_1^2 + 3u^4 u_1) + 3u_2/2 + 5u_1 u^2 + u^5/2$. Note that $\tau_{1,1}$ is the first nontrivial master symmetry for the CDIS equation [10, 13]. It is easy to check that in accordance with Theorem 3.18 from [16] we have

$$[\tau_{m,j}, \tau_{m',j'}] = ((2j' + 1)m - (2j + 1)m')\tau_{m+m'-1,j+j'}, \tag{9}$$

where $\tau_{m,j}$ with $j > 0$ are defined inductively by means of (9), i.e. [16] $\tau_{0,j+1} = \frac{1}{2j+1}[\tau_{1,1}, \tau_{0,j}]$, $\tau_{m+1,j} = \frac{1}{2+4j-m}[\tau_{2,0}, \tau_{m,j}]$.

Thus, the CDIS equation, as well as the Burgers equation, represents a nontrivial example of a (1+1)-dimensional evolution equation possessing a hereditary algebra (9).

Using (9), it can be shown (cf. [16]) that $\mathrm{ad}^{m+1}_{\tau_{0,j}}(\tau_{m,j'}) = 0$, i.e. $\tau_{m,j'}$ are master symmetries of degree m for all equations $u_{t_j} = \tau_{0,j}$, $j = 0, 1, 2, \ldots$. Here $\mathrm{ad}_B(G) \equiv [B, G]$ for any (smooth) local functions B and G.

Let $\exp(\mathrm{ad}_B) \equiv \sum\limits_{j=0}^{\infty} \mathrm{ad}^j_B /j!$. As $\mathrm{ad}^{m+1}_{\tau_{0,j}}(\tau_{m,j'}) = 0$, it is easy to see (cf. [16]) that

$$G^{(k)}_{m,j}(t_k) = \exp(-t_k \,\mathrm{ad}_{\tau_{0,k}})\tau_{m,j} = \sum_{i=0}^{m} \frac{(-t_k)^i}{i!} \mathrm{ad}^i_{\tau_{0,k}}(\tau_{m,j}) = \sum_{i=0}^{m} \frac{((2k + 1)t_k)^i m!}{i!(m - i)!} \tau_{m-i,j+ik}$$

are time-dependent symmetries for the equation $u_{t_k} = \tau_{0,k}$ and $\operatorname{ord} G^{(k)}_{m,j} = 2(j + mk) + 1$. Note that $G^{(k)}_{m,j}$ obey the same commutation relations as $\tau_{m,j}$, that is

$$[G^{(k)}_{m,j}, G^{(k)}_{m',j'}] = ((2j' + 1)m - (2j + 1)m')G^{(k)}_{m+m'-1,j+j'}. \tag{10}$$

It is straightforward to verify that $\tau_{0,1} = F = u_3 + 3u^2 u_2 + 9uu_1^2 + 3u^4 u_1$ and thus $G_{m,j} \equiv G^{(1)}_{m,j}(t) = \exp(-t\,\mathrm{ad}_F)\tau_{m,j}$ are time-dependent symmetries for the CDIS equation.

It is easy to see that the number of symmetries $G_{m,j}$ of given odd order $k = 2l + 1$ equals $[k/2]+1 = l+1$. As $\dim S_{{\rm CDIS},k} \leq [k/2]+1$, these symmetries exhaust the space $S_{{\rm CDIS},k}$. Thus, we have proved the following theorem.

Theorem 1. *Any local generalized symmetry of the CDIS equation is a linear combination of the symmetries $G_{m,j}$ for $m = 0, 1, \cdots$ and $j = 0, 1, 2 \ldots$.*

Note that the technique of [10], based on the representation theory for the algebra $sl(2)$ generated by $\tau_{0,0} = u_1$, $2\tau_{1,0} = 2xu_1 + u$ and $\tau_{2,0} = x^2u_1 + xu$, enables one to obtain only a part of the symmetries, described in the above theorem. The reason for this is that $\langle \tau_{0,0}, \tau_{1,0}, \tau_{2,0} \rangle$ is a subalgebra of the algebra generated by $\tau_{m,0}, m = 0, 1, \ldots$. This is exactly the same phenomenon as in the case of Lie algebra of vector fields of the form $x^{m+1}\frac{d}{dx}$.

As a final remark, let us mention that, in complete analogy with the above, we can readily obtain the complete description of the set of local generalized symmetries for any of the equations $u_{t_k} = \tau_{0,k}$, $k = 2, 3, \ldots$. In this way we arrive at the following generalization of Theorem 1.

Theorem 2. *Any local generalized symmetry of the equation $u_{t_k} = \tau_{0,k}$, $k \in \mathbb{N}$, is a linear combination of the symmetries $G^{(k)}_{m,j}(t_k)$ for $m = 0, 1, \ldots$ and $j = 0, 1, 2 \ldots$.*

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