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Time-Dependent Supersymmetry and Parasupersymmetry in Quantum Mechanics

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Concepts of supersymmetry and parasupersymmetry known for the one-dimensional stationary Schrödinger equation are generalized to the time-dependent equation. Our approach is based on differential transformation operators for the non-stationary Schrödinger equation called Darboux transformation operators and on chains of such operators. As an illustration new exactly solvable time-dependent potentials are derived.

1 Introduction

Supersymmetry has been introduced in quantum mechanics by Nicolai [1] and later by Witten [2]. It was realized afterwards that this approach is really a particular case of transformation operators method well known in mathematics (see e.g. [3]) when it is applied to the stationary Schrödinger equation and when the transformation operator has a differential form [4]. In particular, when the transformation operator is a first order differential operator this approach is equivalent to the one studied by Darboux in 1882 [5]. When the same method is applied to the transformed equation one gets a chain of transformations. We shall see further that the algebraic structure underlying such a chain is parasupersymmetry.

In this lecture I am planning to show how this approach may be generalized to the time-dependent case, i.e. to the time-dependent Schrödinger equation. This generalization is straightforward. Therefore I will develop the time-dependent constructions in parallel lines with the time-independent ones. The left-hand lines of the most formulae will be devoted to the stationary (known) results and the right-hand lines will show their time-dependent generalization.

2 Time-dependent Darboux transformations and time-dependent supersymmetry

The main idea of the transformation operators method is so called *intertwining relation* (see e.g. [4]). Let us suppose that one knows the solutions of the Schrödinger equation (stationary or non-stationary)

$$\begin{aligned} &h_0\psi_E = E\psi_E, \qquad (i\partial_t - h_0)\psi = 0, \\ &h_0 = -\partial_x^2 + V_0(x), \qquad x \in [a,b]. \end{aligned} \tag{1}$$

For the stationary case they are supposed to be known for all real and if necessary complex values of the parameter E.

To solve another Schrödinger equation

$$\begin{aligned} &h_1\varphi_E = E\varphi_E, \qquad (i\partial_t - h_1)\psi = 0, \\ &h_1 = -\partial_x^2 + V_1(x), \qquad x \in [a,b] \end{aligned} \tag{2}$$

one may introduce so called *transformation operator* which I will denote by L. The defining relation for this operator is the *intertwining relation*

$$Lh_0 = h_1 L, \qquad L(i\partial_t - h_0) = (i\partial_t - h_1)L. \tag{3}$$

Therefore it is also called *intertwiner*. It is clear from (3) that $\varphi = L\psi$ is a solution to (2) provided ψ is a solution to (1). The equation (1) is called the initial equation, the Hamiltonian h_0 is the initial Hamiltonian and the potential V_0 is the initial potential. The equation (2), Hamiltonian h_1 and the potential V_1 are transformed entities.

In the simplest case one can try to find the operator L as a first order differential operator

$$L = L_0(x) + L_1(x)\partial_x, \qquad L = L_0(x,t) + L_1(x,t)\partial_x.$$

Note that for the time-dependent case I do not include to L the derivative with respect to time. If it would be included to it, L should become a second order operator since it follows from (1) that $i\partial_t = -\partial_x^2 + V_0$ but we want to have the operator L only as a first order differential operator.

If one introduces the potential difference $A = h_1 - h_0 = V_1(x) - V_0(x)$ then the intertwining relation reduces to the system of differential equations for A and for the coefficients of the operator L. Note that this system can be integrated both in stationary and in non-stationary cases. I give here only the final result

$$L = -\frac{u_x(x)}{u(x)} + \partial_x, \qquad L = L_1(t)\left(-\frac{u_x(x,t)}{u(x,t)} + \partial_x\right). \tag{4}$$

Here u is a solution to the initial equation

$$h_0 u(x) = \alpha u(x), \qquad (i\partial_t - h_0)u(x,t) = 0.$$

The main difference between time-dependent and time-independent cases is that for the time-independent case the coefficient L_1 is an arbitrary constant which always may be put equal to equal to 1, but for the time-dependent case it is an arbitrary function of time.

The potential difference depends on the function u but for the time-dependent case it depends also on the function $L_1(t)$. For the time-independent case the function u can always be chosen real for all real values of the parameter E whereas for the time-dependent case this function takes essentially complex values. Our main idea for the time-dependent case is to dispose of the arbitrary function $L_1(t)$ for satisfying the reality condition for the potential difference. As it happens this is possible only if the function u is subject to an additional condition which we call *the reality condition of the new potential* or simply *the reality condition*

$$[\log u/\overline{u}]_{xxx} = 0.$$

The bar means the complex conjugation. Under this condition the function $L_1(t)$ becomes real

$$L_1(t) = \exp\left[2\int dt\ \mathrm{Im}\,(\log u)_{xx}\right] \tag{5}$$

and for the potential difference one gets

$$A(x) = -\left[\log u^2(x)\right]_{xx}, \qquad A(x,t) = -\left[\log |u(x,t)|^2\right]_{xx}. \tag{6}$$

We see from (4), (5), (6) that the potential difference and the transformation operator are defined only by the function u. Therefore we call it the *transformation function*. As it follows from (6) a sole condition which should be imposed on u is the absence of zeros for x belonging to the interval (a, b) where the initial Schrödinger equation is solved. No boundary or asymptotic condition should be imposed on it.

Note, that when the potential V_0 is independent of time, one can take $u(x, t)$ in the form

$$u(x, t) = u(x)e^{-i\alpha t}.$$

In this case the reality condition is satisfied and the function $L_1 = \text{const}$. The time-dependent transformation reduces just to the known time-independent one.

Once one knows the operator L one can introduce so called Laplace adjoint to it which is defined by the formal relations

$$(c\partial_x)^+ = -\overline{c}\partial_x, \qquad c \in \mathbb{C}, \qquad (AB)^+ = B^+A^+.$$

Then

$$L^+ = -\frac{u_x(x)}{u(x)} - \partial_x, \qquad L^+ = -L_1(t)\left(\frac{\overline{u}_x(x,t)}{\overline{u}(x,t)} + \partial_x\right)$$

and

$$(i\partial_t - h_0)^+ = i\partial_t - h_0.$$

The conjugation of the intertwining relations gives us corresponding relations for L^+

$$h_0L^+ = L^+h_1, \qquad (i\partial_t - h_0)L^+ = L^+(i\partial_t - h_1).$$

These relations mean that the operator L^+ realizes the transformation in the inverse direction, i.e. from the solutions of the transformed equation to the solutions of the initial one. It is clear now that the superposition L^+L transforms solutions of the initial equation into solutions of the same equation and hence this is a symmetry operator for the initial Schrödinger equation. By the same reason the operator LL^+ is a symmetry operator for the transformed equation. For the stationary case there exists only one second order differential symmetry operator, this is the Hamiltonian (may be displaced by a constant). For the non-stationary case the Hamiltonian in general is not an integral of motion. So, our transformation is possible only for such systems which have symmetry operators either of the second order in ∂_x or of the first order in ∂_x and ∂_t. In other words

$$L^+L = h_0 - \alpha, \qquad L^+L = g_0 - \alpha \tag{7}$$

and

$$LL^+ = h_1 - \alpha, \qquad LL^+ = g_1 - \alpha. \tag{8}$$

We denote by g_0 and g_1 corresponding symmetry operators for the nonstationary Schrödinger equation. These relations may be treated as factorizations of the operators g_0 (h_0) and g_1 (h_1).

It follows from (7) and (8) that for the non-stationary case the following intertwining relations take place

$$Lg_0 = g_1L, \qquad g_0L^+ = L^+g_1. \tag{9}$$

Moreover, when a Hilbert space is introduced and formally adjoint operator coincides with the adjoint with respect to an inner product, the operators L^+L and LL^+ are nonnegative. Hence, the symmetry operators g_0 (h_0) and g_1 (h_1) are bounded from below. Furthermore, by constructions one has $Lu = 0$. It follows from here and (7) that $g_0u = \alpha u$.

The intertwining relations and factorization properties may be rewritten in another form. Let us introduce the following matrices

$$\mathcal{H} = \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix}, \qquad \mathcal{G} = \begin{pmatrix} g_0 & 0 \\ 0 & g_1 \end{pmatrix}$$

and

$$\mathcal{Q}^+ = \begin{pmatrix} 0 & L^+ \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{Q} = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}.$$

It is easy to see now that the factorization property may be rewritten in the form

$$\mathcal{Q}^+\mathcal{Q} + \mathcal{Q}\mathcal{Q}^+ = \mathcal{H} - \alpha\mathcal{I}, \qquad \mathcal{Q}^+\mathcal{Q} + \mathcal{Q}\mathcal{Q}^+ = \mathcal{G} - \alpha\mathcal{I},$$

where $\mathcal{I}$ is the unity 2×2 matrix, and the intertwining relations result in

$$\mathcal{Q}\mathcal{H} - \mathcal{H}\mathcal{Q} = 0, \qquad \mathcal{Q}\mathcal{G} - \mathcal{G}\mathcal{Q} = 0.$$

We see from here that the operators $\mathcal{H}$, Q, Q^+ or $\mathcal{G}$, Q, Q^+ form a simplest superalgebra. In the time-dependent case the operators $\mathcal{G}$, Q, Q^+ depend on time. Therefore we have a *time-dependent superalgebra*.

The operators L and L^+ have non-trivial kernels. Nevertheless if one introduces the space of the solutions of the initial equation, T_0, and the space of the solutions of the transformed equation, T_1, one can establish a one-to one correspondence between these spaces. Let us decompose the spaces $T_{0,1}$ into a direct sums

$$T_{0,1} = T_{0,1}^0 \oplus T_{0,1}^1, \qquad T_0^1 = \ker L^+L, \qquad T_1^1 = \ker LL^+.$$

The spaces $T_{0,1}^1$ are two-dimensional. It is clear by constructions that $u \in T_0^1$. The equation $L^+L = 0$ except for u has another solution linearly independent with u which has the form

$$\tilde{u} = uL_1^{-2} \int \frac{dx}{u\overline{u}}\,.$$

It is easy to show that the function $v = L\tilde{u} = 1/(L_1\overline{u}\,)$ is such that $L^+v = 0$. This means that $v \in \ker LL^+$. Another solution of the equation $LL^+ = 0$ linearly independent with v has the form $\tilde{v} = vL_1^{-2}\int 1/(v\overline{v}\,)\,dx$ and $L^+\tilde{v} = u$. Once we know the basis functions $u, \tilde{u} \in T_0^1$ and $v, \tilde{v} \in T_1^1$ we can define a linear one-to-one correspondence between T_0^1 and T_1^1 by defining the correspondence between the bases: $u \longleftrightarrow \tilde{v}$ and $\tilde{u} \longleftrightarrow v$. The equations $L^+L\psi = 0$ and $LL^+\varphi = 0$ has no solutions when solving on T_0^0 and T_0^1 respectively. These operators are hence invertible on these spaces. This means that they establish the one-to-one correspondence between T_0^0 and T_0^1. So, we have established the one-to one correspondence between T_0 and T_1.

This correspondence is very useful for finding all square integrable solutions of the transformed equation. It is easy to see that the function $\varphi = L\psi$ is square integrable provided so is $\psi \in T_0^0$ and when ψ is not square integrable φ is not either. Hence, to find all square integrable solutions of the transformed equation it remains to analyze the functions v and $\tilde{v}$.

As I have mentioned, u should be a nodeless solution of the initial Schrödinger equation and the operator g_0 is bounded from below. Let E_0 be its lower bound. Then, according to the oscillator theorem u may be nodeless only if $\alpha \le E_0$. When g_0 has a discrete spectrum, E_0 may be associated with the ground state level. If we take $\alpha = E_0$ then neither v nor $\tilde{v}$ are square integrable and this level will be absent in the spectrum of g_1. All other levels of g_0 are unchanged in the course of the Darboux transformation. When $\alpha < E_0$ there are two possibilities. The first one corresponds to the case when the function $v = 1/(L_1\overline{u}\,)$ is square integrable. In this case the operator g_1 has an additional discrete spectrum level with respect to g_0. In the second case neither v nor $\tilde{v}$ are square integrable and g_1 has exactly the same spectrum as g_0, i.e. they are strictly isospectral.

3 Chains of transformations and parasupersymmetry

Once we know the potential V_1 we can take it as V_0 and realize the Darboux transformation once again, etc. In such a way we obtain a chain of exactly solvable symmetry operators

$$h_0 \to h_1 \to \cdots \to h_N, \qquad g_0 \to g_1 \to \cdots \to g_N \tag{10}$$

and a chain of first order transformation operators

$$L_{0,1} \to L_{1,2} \to \cdots \to L_{N-1,N}.$$

If one is not interested in the intermediate operators, one can expunge all the intermediate transformation functions from the final result and express it only in terms of solutions of the initial equation. Moreover, in this case one does not have to impose the reality condition on the intermediate potentials. This leads to the following formulae for solutions of the transformed equation

$$\varphi = L_{0N}(t) W(u_1, u_2, \ldots, u_N) \begin{vmatrix} u_1 & u_2 & \cdots & \psi \\ u_{1x} & u_{2x} & \cdots & \psi_x \\ \cdots & \cdots & \cdots & \cdots \\ u_{1x}^{(N)} & u_{2x}^{(N)} & \cdots & \psi_x^{(N)} \end{vmatrix}. \tag{11}$$

Here $g_0 u_k = \alpha_k u_k$. For the stationary case $L_{0N}(t) = 1$ and this formula reduces to the known Krum–Krein formula [6, 7]. The formula (11) defines an N-order transformation operator $\varphi = L_{0,N}\psi$, $L_{0,N} = L_{N-1,N} L_{N-2,N-1} \cdots L_{0,1}$. This operator is an intertwiner for the symmetry operators g_0 and g_N. The operators $L_{0,N}$ and its adjoint $L_{0,N}^+$ factorize now a polynomial of the operators g_0 and g_N

$$L_{0,N}^+ L_{0,N} = \prod_{k=1}^{N} (h_0 - \alpha_k), \qquad L_{0,N}^+ L_{0,N} = \prod_{k=1}^{N} (g_0 - \alpha_k), \tag{12}$$

$$L_{0,N} L_{0,N}^+ = \prod_{k=1}^{N} (h_N - \alpha_k), \qquad L_{0,N} L_{0,N}^+ = \prod_{k=1}^{N} (g_N - \alpha_k). \tag{13}$$

Let us consider a chain in which all elements are good. Such chains are known as *completely reducible* ones. For this chain one can consider nth order transformation operators

$$L_{p,p+n} = L_{p+n-1,p+n} L_{p+n-2,p+n-1} \cdots L_{p,p+1}, \qquad n \le N$$

and their adjoint. They factorize polynomials of the symmetry operators g_p and g_{p+n}

$$L_{p,p+n}^+ L_{p,p+n} = \prod_{k=1}^{n} (g_p - \alpha_{p+k}), \qquad L_{p,p+n} L_{p,p+n}^+ = \prod_{k=1}^{n} (g_{p+n} - \alpha_{p+k})$$

and they are intertwiners for g_p and g_{p+n} and for the Schrödinger equations with the Hamiltonians h_p and h_{p+n}.

Let us introduce now the diagonal matrix operators

$$\mathcal{H} = \operatorname{diag}(h_0, h_1, \ldots, h_N), \qquad \mathcal{G} = \operatorname{diag}(g_0, g_1, \ldots, g_N)$$

and nilpotent supercharges

$$\mathcal{Q}_{p,q}^+ = L_{p,q} e_{p,q}, \qquad \mathcal{Q}_{p,q} = L_{p,q}^+ e_{q,p},$$

where $e_{p,q}$ is $(N+1)\times(N+1)$ matrix with a single non-zero entry which is equal to one and stands at the intersection of pth column and qth row.

Instead of the chain of the Schrödinger equations one can write now the single equation (supersymmetric Schrödinger equation)

$$(i\mathcal{I}\partial_t - \mathcal{H})\Psi(x,t) = 0.$$

Intertwining relations between transformation operators and $i\partial_t - h_p$ are equivalent to the commutation of the supercharges $\mathcal{Q}_{p,q}$ with $i\mathcal{I}\partial_t - \mathcal{H}$. This means that all $\mathcal{Q}_{p,q}$ are integrals of motion for the system with the superhamiltonian $\mathcal{H}$. The condition of the complete reducibility leads to the following non-linear algebra

$$\begin{aligned}
&\mathcal{Q}_{s,p}\mathcal{Q}_{p,q} = \mathcal{Q}_{s,q}, \qquad N+1 \geq q > p > s,\\
&\mathcal{Q}^{+}_{p,p+n}\mathcal{Q}_{p,p+n+m} = \prod_{i=1}^{n}(\mathcal{G}_0 - \alpha_{p+i})\mathcal{Q}_{p+n,p+n+m}, \qquad p+n+m \leq N+1,\\
&\mathcal{Q}_{p-n-m,p}\mathcal{Q}^{+}_{p-n,p} = \prod_{i=1}^{n}(\mathcal{G}_0 - \alpha_{p+i-1})\mathcal{Q}_{p-n-m,p-n}, \qquad p-n-m \geq 0, \quad p \leq N+1,\\
&\mathcal{Q}_{p,p+n}\mathcal{Q}^{+}_{p,p+n}\mathcal{Q}_{p,p+n} = \prod_{i=1}^{n}(\mathcal{G}_0 - \alpha_{p+i})\mathcal{Q}_{p,p+n}, \qquad p+n \leq N+1, \qquad n,m = 1,2,\ldots
\end{aligned}$$

Similar non-linear algebras are known for the stationary Schrödinger equation as parasuperalgebras (see e.g. [8, 9]). The operators involved in this algebra depend on time. Hence one has here a time-dependent parasuperalgebra.

4 Time-dependent exactly solvable potentials

4.1 Harmonic oscillator with a time varying frequency

Consider first a time-dependent generalization of the harmonic oscillator

$$h_0 = -\partial_x^2 + \omega^2(t)x^2. \tag{14}$$

Some solutions of the Schrödinger equation with such a Hamiltonian are well-known but we will need other solutions for using as transformation functions. To get them we will use the method of separation of the variables in its general formulation as R-separation of variables well-described in the book by Miller [10]. This method is based on classification of orbits in adjoint representation of a symmetry group for a given equation.

Symmetry algebra of the Schrödinger equation with the Hamiltonian (14) is the well-known Schrödinger algebra. Consider first representation of this algebra suitable for our purpose.

Operators $a = \epsilon\partial_x - \frac{i}{2}\dot{\epsilon}x$, $a^+ = \bar{\epsilon}\partial_x + \frac{i}{2}\dot{\bar{\epsilon}}x$, $aa^+ - a^+a = \frac{1}{4}$, where $\epsilon = \epsilon(t)$ is a solution of a classical equation of motion for the Harmonic oscillator with a time-varying frequency $\ddot{\epsilon}(t) + 4\omega^2(t)\epsilon(t) = 0$ are creation and annihilation operators and together with the identity operator close the Heisenberg–Weil algebra. All operators of the Schrödinger algebra are constructed in terms of a and a^+

$$\begin{aligned}
&K_1 = a - a^+, \qquad K_{-1} = -i(a + a^+), \qquad K_0 = i,\\
&K_{-2} = -i(a + a^+)^2, \qquad K_2 = -i(a - a^+)^2, \qquad K^0 = -2\left[a^2 - (a^+)^2\right].
\end{aligned}$$

Symmetry operators are classified by the orbits of adjoint representation of the symmetry group. It is well-known that in the case under consideration there exist five different orbits. We shall consider every orbit successively.

Two orbits with representatives $J_1 = K_1$ and $J_1 = K_2$ give the same solution of the Schrödinger equation

$$\psi = \gamma^{-1/2} \exp\left[\frac{i\lambda x}{8\gamma} + \frac{ix^2\dot{\gamma}}{4\gamma} - \frac{i\lambda^2\delta}{64\gamma}\right], \tag{15}$$
$$2\gamma = \epsilon + \bar{\epsilon}, \qquad 2i\delta = \epsilon - \bar{\epsilon}, \qquad \dot{\epsilon}\bar{\epsilon} - \epsilon\dot{\bar{\epsilon}} = \frac{i}{2}.$$

Using the function (15) we construct the transformation function

$$u = \gamma^{-1/2} \cosh\frac{\lambda x}{8\gamma} \exp\left[\frac{ix^2\dot{\gamma}}{4\gamma} - \frac{i\lambda^2\delta}{64\gamma}\right], \qquad L_1(t) = \gamma = (\epsilon + \bar{\epsilon})/2,$$

which gives us the following potential

$$V_1 = \omega^2(t)x^2 - \frac{\lambda^2}{32\gamma^2}\cosh^{-2}\frac{\lambda x}{8\gamma}.$$

When $\omega = 0$ it reduces to the well-known one soliton potential. Therefore it may be considered as a non-stationary generalization of the one soliton potential. The Fig. 1 shows the behavior of this potential for $\omega = 1/2$ (stationary case) and $\gamma = \frac{1}{2}\cos t$. At the bottom of the harmonic oscillator parabola one can see an additional minimum of the varying depth.

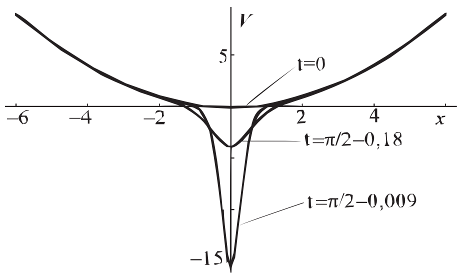

Figure 1. Potential with a time-dependent anharmonic member.

Using the same function (15) one can construct the transformation function of a more general form

$$u = u_\lambda + u_{\bar{\lambda}} = \gamma^{-1/2}\cosh\left(\frac{\nu x}{8\gamma} + \mu\nu\frac{\delta}{32\gamma}\right)\exp\left[\frac{ix^2\dot{\gamma}}{4\gamma} - \frac{i\mu x}{8\gamma} + i\left(\nu^2 - \mu^2\right)\frac{\delta}{64\gamma}\right],$$
$$\lambda = -\mu - i\nu, \qquad L_1(t) = \gamma$$

which gives the following potential

$$V_1 = \omega^2(t)x^2 - \frac{\nu^2}{32\gamma^2}\cosh^{-2}\left(\frac{\nu x}{8\gamma} + \mu\nu\frac{\delta}{32\gamma}\right). \tag{16}$$

When $\omega = 0$ this potential reduces to the known non-stationary soliton potential which gives rise to a one soliton solution to the Kadomtsev–Petviashili (two-dimensional KdV) nonlinear equation. The next figure shows the plot of this potential at different time-moments. Here an additional minimum of a varying depth oscillates between the parabola walls.

The next orbit is presented by the operator $J_2 = K_2 - K_1$. Corresponding solution of the Schrödinger equation has the form

$$\psi = \delta^{-1/2}\exp\left(ix^2\frac{\dot{\delta}}{4\delta} - ix\frac{\gamma}{2\delta^1} + i\frac{\gamma^3}{6\delta^3} + i\lambda\frac{\gamma}{\delta}\right) Q\left(2^{-1/2}\left(\frac{x}{\delta} - \frac{\gamma^2}{2\delta^2}\right) - 2^2/3\lambda\right),$$

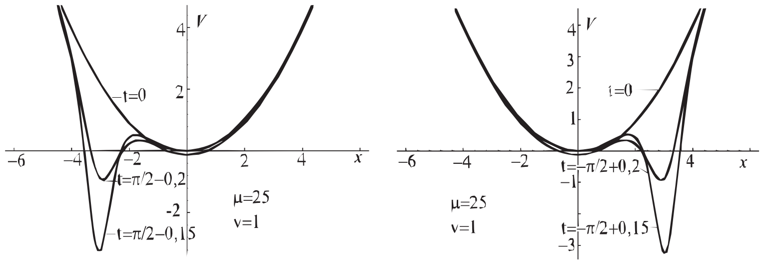

Figure 2. Potential with a time-dependent anharmonic member at different time moments.

where $\gamma = \epsilon+\bar{\epsilon}$, $i\delta = \epsilon-\bar{\epsilon}$, and $Q(z)$ is the Airy function satisfying the equation $Q_{zz}(z) = zQ(z)$. Exactly solvable potential is expressed in this case in terms of the Airy function. To obtain a real and regular on the whole real line potential one can realize a second order transformation with the mutually conjugated transformation functions u_λ and $u_{\bar{\lambda}}$. For $\omega = 0$ the plot of one of these potentials is shown by the Fig. 3.

The fourth orbit has the representative $J_3 = K_2 - K_{-2}$ and creates the following solution of the Schrödinger equation

$$\psi = \gamma^{-1/4}\left(\frac{\epsilon}{\bar{\epsilon}}\right)^{\lambda/2} \exp\left(i\frac{\dot{\gamma}x^2}{8\gamma}\right) Q\left(\frac{x}{2\sqrt{\gamma}}\right), \qquad \gamma = \epsilon\bar{\epsilon},$$

where $Q(z)$ is the parabolic cylinder function satisfying the equation $Q_{zz}(z) - \left(z^2/4+\lambda\right)Q(z) = 0$. At $\lambda = -n-1/2$ one gets the discrete basis functions of corresponding Hilbert space

$$\psi_n = N_n\gamma^{-1/4}\left(\frac{\bar{\epsilon}}{\epsilon}\right)^{n/2+1/4} \exp\left(\frac{2i\dot{\gamma}-1}{16\gamma}x^2\right) He_n\left(\frac{x}{2\sqrt{\gamma}}\right),$$

where $He_n(z) = 2^{-n/2}H_n(z/\sqrt{2})$ are Hermite polynomials.

The same functions with $\lambda = n+1/2$ are suitable for the Darboux transformations and they generate the following potential differences

$$A_2^{m,l} = \frac{1}{2\gamma}\left[1 + \frac{f''_{ml}(z)}{f_{ml}(z)} - \left(\frac{f'_m l(z)}{f_{ml}(z)}\right)^2\right],$$
$$f_{ml}(z) = q_m(z)q_{l+1}(z) - q_l(z)q_{m+1}(z), \qquad z = x/(2\sqrt{\gamma})$$

which are well-defined for $m = 0, 2, 4, \ldots$, $l = m+1, m+3, \ldots$. For $m = 2$ and $l = 5$ the behavior of the transformed potential is shown by the Fig. 4.

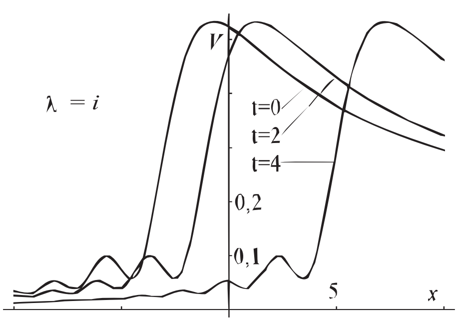

Figure 3. Potential generated with the help of the Airy function.

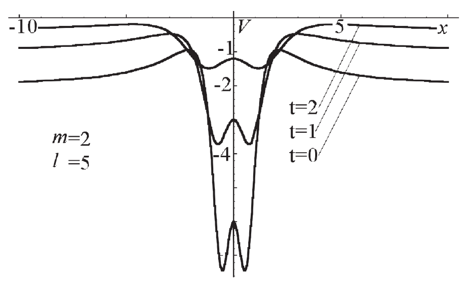

Figure 4. Potentials $V_2^{m,l}(x,t)$ at $m = 2$ and $l = 5$.

One can take a general solution of the equation for the parabolic cylinder functions as transformation function. For example, when $\lambda = 1/2$ one has

$$u = \gamma^{-1/4} \left(\frac{\epsilon}{\bar{\epsilon}}\right) \exp\left(\frac{2i\dot{\gamma} + 1}{16\gamma} x^2\right) \left[C + \operatorname{erf}\left(\frac{x}{2\sqrt{2\gamma}}\right)\right],$$

which gives the following potential

$$V_1 = \omega^2(t)x^2 - \frac{1}{4\gamma}\left[1 - 2zQ^{-1}(z)e^{-z^2/2} - 2Q^{-2}(z)e^{-z^2}\right],$$
$$Q(z) = \sqrt{\frac{\pi}{2}}\left[C + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right], \qquad z = \frac{x}{2\sqrt{\gamma}}, \qquad |C| > 1.$$

For $\omega = \mathrm{const} \neq 0$ these potentials reduce to the known isospectral potentials with an equidistant spectrum. For $\omega = 0$ their behavior is shown by the Fig. 5. The cases a) and b) differ by the values of parameters the potential depends on.

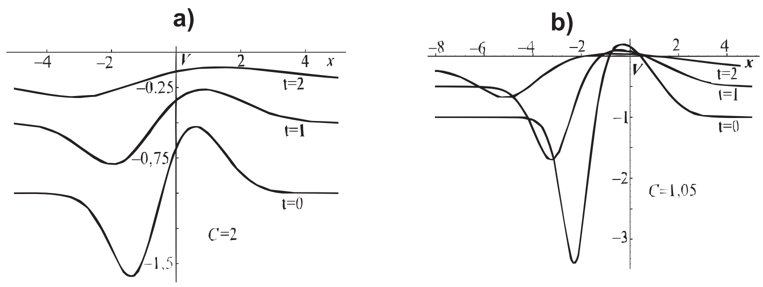

Figure 5. Time-dependent generalization of isospectral potentials.

4.2 Singular oscillator with a time dependent frequency

Consider now the following Hamiltonian:

$$h_0 = -\partial_x^2 + \omega^2(t)\,x^2 + gx^{-2}.$$

Symmetry algebra of the Schrödinger equation with this Hamiltonian is $su(1.1) \sim sl(2,\mathbb{R})$. We use the following representation for this algebra:

$$[K_+ = 2\left[\left(a^+\right)^2 - \bar{\varepsilon}^2 g x^{-2}\right], \quad K_- = 2\left[a^2 - \varepsilon^2 g x^{-2}\right],$$
$$K_0 = \frac{1}{2}\left(K_- K_+ - K_+ K_-\right) = \frac{1}{2}\left[K_-, K_+\right].$$

Consider solutions of the Schrödinger equation which are eigenstates of K_0: $K_0\varphi_\lambda(x,t) = \lambda\varphi_\lambda(x,t)$. When $\lambda = n + k$, $n = 0, 1, 2, \ldots$ we have a discrete basis of the Hilbert space

$$\varphi_n(x,t) = 2^{1/2-3k}\sqrt{\frac{n!}{\Gamma(n+2k)}}\,\gamma^{-k}\left(\frac{\bar{\varepsilon}}{\varepsilon}\right)^{n+k} x^{2k-1/2}$$
$$\times \exp\left[i\frac{x^2\dot{\gamma}}{8\gamma} - \frac{x^2}{16\gamma}\right] L_n^{2k-1}\left(\frac{x^2}{8\gamma}\right), \qquad k = \frac{1}{2} + \frac{1}{4}\sqrt{1+4g}, \qquad \gamma = \varepsilon\bar{\varepsilon}.$$

To construct spontaneously broken supersymmetric model we need transformation functions $u(x,t)$ such that neither $u(x,t)$ nor $u^{-1}(x,t)$ are from the Hilbert space and $u(x,t)$ is nodeless for all real values of t and $x > 0$. These conditions are fulfilled for the functions

$$u_p(x,t) = \gamma^{-k}\left(\frac{\overline{\varepsilon}}{\varepsilon}\right)^{-p-k} x^{2k-1/2}\exp\left[i\frac{x^2\dot{\gamma}}{8\gamma}+\frac{x^2}{16\gamma}\right]L_p^{2k-1}\left(\frac{-x^2}{8\gamma}\right),$$
$$K_0u_p(x,t) = -(p+k)u_p(x,t).$$

These transformation functions create the following exactly solvable family of potential differences $A(x,t) = \omega^2(t)x^2 + gx^{-2} - V_1(x,t)$:

$$A(x,t) = A_p(x,t) = \frac{1}{4\gamma} - \frac{4k-1}{x^2} - \frac{1}{8}\left(\frac{xL_{p-1}^{2k}(z)}{\gamma L_p^{2k-1}(z)}\right)^2 + \frac{x^2L_{p-2}^{2k+1}(z)+4\gamma L_{p-1}^{2k}(z)}{8\gamma^2L_p^{2k-1}(z)}, \qquad z = -\frac{x^2}{8\gamma}.$$

To construct a model with exact supersymmetry we need transformation functions $u(x,t)$ such that $u^{-1}(x,t)$ is square integrable on semiaxis $x \geq 0$ and satisfies the zero boundary condition at the origin for all values of t. The following solution of the Schrödinger equation may be chosen in this case:

$$u_p(x,t) = \gamma^{k-1}\left(\frac{\overline{\varepsilon}}{\varepsilon}\right)^{k-p-1} x^{3/2-2k}\exp\left[i\frac{x^2\dot{\gamma}}{8\gamma}+\frac{x^2}{16\gamma}\right]L_p^{1-2k}\left(\frac{-x^2}{8\gamma}\right),$$
$$K_0u_p(x,t) = (k-p-1)\,u_p(x,t).$$

It is not difficult to establish the possible values of p. If p is even it may takes the values $p < 2k-1$ and $p = [2k]+1, [2k]+3, \ldots$. For odd p values we may use only $p = [2k], [2k]+2, \ldots$, where $[2k] \equiv \operatorname{entire}(2k)$. For regular potential differences we obtain

$$A_p(x,t) = \frac{1}{4\gamma} + \frac{4k-3}{x^2} - \frac{1}{2}\left(\frac{xL_{p-1}^{2-2k}(z)}{2\gamma L_p^{1-2k}(z)}\right)^2 + \frac{x^2L_{p-2}^{3-2k}(z)+4\gamma L_{p-1}^{2-2k}(z)}{8\gamma^2L_p^{1-2k}(z)}.$$

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