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The Soliton Content of the Camassa–Holm and Hunter–Saxton Equations

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The notion of a scalar equation describing pseudo-spherical surfaces is reviewed. It is shown that if an equation admits this structure, the existence of conservation laws, symmetries, and quadratic pseudo-potentials, can be studied by geometrical means. As an application, it is pointed out that the important Camassa–Holm and Hunter–Saxton equations possess features considered to be characteristic of standard "soliton" equations: an infinite number of local conservation laws, "Miura transformations", a zero curvature formulation, and nonlocal symmetries.

1 Introduction

In this contribution we review some recent developments linking differential geometry of surfaces and integrability of nonlinear partial differential equations. We concentrate on the notion of a scalar equation describing pseudo-spherical surfaces (or "of pseudo-spherical type") introduced by S.S. Chern and Keti Tenenblat [6, 18]: these equations share with the sine–Gordon equation the property that their (suitably generic) solutions determine two-dimensional surfaces equipped with metrics of constant Gaussian curvature -1.

Equations of pseudo-spherical type are introduced in Section 2, and we point out that equations possessing this structure are naturally the integrability condition of an $sl(2,\mathbb{R})$-valued linear problem. We then survey in Section 3 two standard aspects of the geometric theory of differential equations, conservation laws and symmetries: for equations describing pseudo-spherical surfaces, they can be understood by geometrical means. In Section 4 we consider our main application, the Camassa–Holm (Camassa and Holm [5]) and Hunter–Saxton (Hunter and Saxton [8], Hunter and Zheng [9]) equations. We show that for these important examples, the geometric approach reviewed in Sections 2 and 3 allows us to construct explicitly the following: quadratic pseudo-potentials, Miura transformations, "modified" equations, local conservation laws, zero curvature representations, and non-local symmetries.

2 Equations of pseudo-spherical type

This structure was introduced by S.S. Chern and K. Tenenblat in 1986 [6], motivated by the fact that [17] generic solutions of equations integrable by the Ablowitz, Kaup, Newell and Segur (AKNS) inverse scattering scheme determine – whenever their associated linear problems are real – pseudo-spherical surfaces, that is, Riemannian surfaces of constant Gaussian curvature -1.

Definition 1. A scalar differential equation $\Xi(x,t,u,u_x,\ldots,u_{x^n t^m}) = 0$ in two independent variables x, t is of pseudo-spherical type (or, it is said to describe pseudo-spherical surfaces) if there exist one-forms $\omega^\alpha \neq 0$,

$$\omega^\alpha = f_{\alpha 1}(x,t,u,\ldots,u_{x^r t^p})\,dx + f_{\alpha 2}(x,t,u,\ldots,u_{x^s t^q})\,dt, \qquad \alpha = 1,2,3 \tag{1}$$

whose coefficients $f_{\alpha\beta}$ are differential functions, such that the one-forms $\overline{\omega}^\alpha = \omega^\alpha(u(x,t))$ satisfy the structure equations

$$d\overline{\omega}^1 = \overline{\omega}^3 \wedge \overline{\omega}^2, \qquad d\overline{\omega}^2 = \overline{\omega}^1 \wedge \overline{\omega}^3, \qquad d\overline{\omega}^3 = \overline{\omega}^1 \wedge \overline{\omega}^2, \tag{2}$$

whenever $u = u(x,t)$ is a solution to $\Xi = 0$.

We recall that a differential function is a smooth function which depends on x, t, and a finite number of derivatives of u [13]. We sometimes use the expression "PSS equation" instead of "equation of pseudo-spherical type". Also, we exclude from our considerations the trivial case when the functions $f_{\alpha\beta}$ all depend only on x, t.

Example 1. Burgers' equation $u_t = u_{xx} + uu_x + h_x(x)$, is a PSS equation with

$$\omega^1 = ((1/2)u - (\beta/\eta))dx + \big((1/2)u_x + (1/4)u^2 + (1/2)h(x)\big)\,dt,$$
$$\omega^2 = -\omega^3 = \eta\,dx + ((\eta/2)u + \beta)dt,$$

in which $\eta \neq 0$ is a parameter, and β is a solution of the equation $\beta^2 - \eta\beta_x + \big(\eta^2/2\big)\,h(x) = 0$.

The geometric interpretation of Definition 1 is based on the following genericity notions ([15] and references therein):

Definition 2. Let $\Xi = 0$ be a PSS equation with associated one-forms ω^α, $\alpha = 1,2,3$. A solution $u(x,t)$ of $\Xi = 0$ is I-generic if $\big(\omega^3 \wedge \omega^2\big)\,(u(x,t)) \neq 0$, II-generic if $\big(\omega^1 \wedge \omega^3\big)\,(u(x,t)) \neq 0$, and III-generic if $\big(\omega^1 \wedge \omega^2)(u(x,t)\big) \neq 0$.

For instance, $u(x,t) = x+t$ is a I- and III-generic solution of the PSS equation $u_t = u_{xx}+u_x$ with associated one-forms $\omega^1 = udx + u_x dt$, $\omega^2 = dx$, and $\omega^3 = udx + u_x dt$.

Proposition 1. *Let $\Xi = 0$ be a PSS equation with associated one–forms ω^α.*

(a) *If $u(x,t)$ is a I–generic solution, $\overline{\omega}^2$ and $\overline{\omega}^3$ determine a Lorentzian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with connection one–form given by $\overline{\omega}^1$.*

(b) *If $u(x,t)$ is a II–generic solution, $\overline{\omega}^1$ and $-\overline{\omega}^3$ determine a Lorentzian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with connection one–form given by $\overline{\omega}^2$.*

(c) *If $u(x,t)$ is a III–generic solution, $\overline{\omega}^1$ and $\overline{\omega}^2$ determine a Riemannian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with connection one–form given by $\overline{\omega}^3$.*

As pointed out above, the main motivation for formulating Definition 1 is its relation with integrable equations. The following notion is implicit in [6]:

Definition 3. An equation is geometrically integrable if it describes a non-trivial one-parameter family of pseudo-spherical surfaces.

Proposition 2. *A geometrically integrable equation $\Xi = 0$ with associated one-forms ω^α, $\alpha = 1,2,3$, is the integrability condition of a one-parameter family of $sl(2,\mathbb{R})$-valued linear problems.*

Proof. The linear problem $d\psi = \Omega\psi$, in which

$$\Omega = Udx + Vdt = \frac{1}{2}\begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix}, \tag{3}$$

is integrable whenever $u(x,t)$ is a solution of $\Xi = 0$. ■

An important idea in integrable systems is that an equation $\Xi = 0$ is not just the integrability condition of a linear problem $\psi_x = X\psi$, $\psi_t = T\psi$, but that the zero curvature equation $X_t - T_x + [X, T] = 0$ is *equivalent* to $\Xi = 0$. It is a crucial problem to formalize this remark within the context of PSS equations. For evolutionary equations, we proceed thus [10, 15]: if $u_t = F(x, t, u, \ldots, u_{x^k})$ is a $k^{\rm th}$ order evolution equation, we consider the differential ideal I_F generated by the two-forms

$$du \wedge dx + F(x, t, u, \ldots, u_{x^k})\, dx \wedge dt, \qquad du_{x^l} \wedge dt - u_{x^{l+1}}\, dx \wedge dt, \qquad 1 \le l \le k-1,$$

on a manifold J with coordinates $x, t, u, u_x, \ldots, u_{x^k}$.

Definition 4. An evolution equation $u_t = F(x, t, u, \ldots, u_{x^k})$ is *strictly pseudo-spherical* if there exist one-forms $\omega^\alpha = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt$, $\alpha = 1, 2, 3$, whose coefficients $f_{\alpha\beta}$ are smooth functions on J, such that the two-forms

$$\Omega_1 = d\omega^1 - \omega^3 \wedge \omega^2, \qquad \Omega_2 = d\omega^2 - \omega^1 \wedge \omega^3, \qquad \Omega_3 = d\omega^3 - \omega^1 \wedge \omega^2, \tag{4}$$

generate I_F.

Note that local solutions of $u_t = F$ correspond to integral submanifolds of the exterior differential system $\{I_F, dx \wedge dt\}$. It follows that if $u_t = F$ is strictly pseudo-spherical, it is necessary and sufficient for the structure equations $\Omega_\alpha = 0$ to hold. The following lemma [14, 15], used in Section 3 below, allows us to *classify* strictly pseudo-spherical equations [6, 10, 14]:

Lemma 1. *Necessary and sufficient conditions for the $k^{\rm th}$ order equation $u_t = F$ to be strictly pseudo-spherical are the conjunction of* (a) *The functions $f_{\alpha\beta}$ satisfy $f_{\alpha 1, u_{x^a}} = 0$; $f_{\alpha 2, u_{x^k}} = 0$; $f_{11,u}^2 + f_{21,u}^2 + f_{31,u}^2 \neq 0$, in which $a \ge 1$ and $\alpha = 1, 2, 3$; and* (b) *F and $f_{\alpha\beta}$ satisfy the identities*

$$-f_{\alpha 1,u} F + \sum_{i=0}^{k-1} u_{x^{i+1}} f_{\alpha 2, u_{x^i}} + f_{\delta 1} f_{\gamma 2} - f_{\gamma 1} f_{\delta 2} + f_{\alpha 2, x} - f_{\alpha 1, t} = 0, \tag{5}$$

in which (α, δ, γ) is $(1, 2, 3)$, $(2, 3, 1)$, or $(3, 2, 1)$.

3 Conservation laws and symmetries for PSS equations

By local conservation laws of $\Xi = 0$ we mean one-forms $\theta = f dx + g dt$, f, g differential functions, such that $d_H\theta := (-D_t f + D_x g) dx \wedge dt = 0$ on solutions of $\Xi = 0$, where D_x and D_t denote the total derivatives operators with respect to x and t respectively [13]: cohomology questions [12] are beyond the scope of this paper. Nonlocal conservation laws can be also considered [12], and in fact, it is natural to study both cases simultaneously [18] when treating PSS equations. We begin with a purely geometric result [6, 18]:

Proposition 3. *Given a coframe $\left\{\overline{\omega}^1, \overline{\omega}^2\right\}$ and corresponding connection one-form $\overline{\omega}^3$ on a surface M, there exists a new coframe $\left\{\overline{\theta}^1, \overline{\theta}^2\right\}$ and new connection one-form $\overline{\theta}^3$ on M satisfying*

$$d\overline{\theta}^1 = 0, \qquad d\overline{\theta}^2 = \overline{\theta}^2 \wedge \overline{\theta}^1, \qquad \text{and} \qquad \overline{\theta}^3 + \overline{\theta}^2 = 0, \tag{6}$$

if and only if the surface M is pseudo-spherical.

Proof. Assume that the orthonormal frames dual to the coframes $\left\{\overline{\omega}^1, \overline{\omega}^2\right\}$ and $\left\{\overline{\theta}^1, \overline{\theta}^2\right\}$ possess the same orientation. The one-forms $\overline{\omega}^\alpha$ and $\overline{\theta}^\alpha$ are connected by means of

$$\overline{\theta}^1 = \overline{\omega}^1 \cos\rho + \overline{\omega}^2 \sin\rho, \qquad \overline{\theta}^2 = -\overline{\omega}^1 \sin\rho + \overline{\omega}^2 \cos\rho, \qquad \overline{\theta}^3 = \overline{\omega}^3 + d\rho. \tag{7}$$

It follows that $\overline{\theta}^1$, $\overline{\theta}^2$, $\overline{\theta}^3$ satisfying (6) exist if and only if the Pfaffian system

$$\overline{\omega}^3 + d\rho - \overline{\omega}^1 \sin\rho + \overline{\omega}^2 \cos\rho = 0 \tag{8}$$

on the space of coordinates (x, t, ρ) is completely integrable for $\rho(x, t)$, and this happens if and only if M is pseudo-spherical. ■

Equations (6) and (8) determine geodesic coordinates on M. Now, if the equation $\Xi = 0$ describes pseudo-spherical surfaces with associated one-forms $\omega^\alpha = f_{\alpha 1} dx + f_{\alpha 2} dt$, (6) and (8) imply that

$$\omega^3(u(x,t)) + d\rho - \omega^1(u(x,t)) \sin\rho + \omega^2(u(x,t)) \cos\rho = 0 \tag{9}$$

is completely integrable for $\rho(x, t)$ whenever $u(x, t)$ is a local solution of $\Xi = 0$. Equations (6) and (7) then imply that for each solution $u(x, t)$ and a corresponding solution $\rho(x, t)$ of (9), the one-form $\theta^1 = \overline{\omega}^1 \cos\rho + \overline{\omega}^2 \sin\rho$ is closed. If the functions $f_{\alpha\beta}$ can be expanded as power series in a parameter η, so can $\rho(x, t)$ and θ^1. Thus, in principle, geometrically integrable equations possess an infinite number of conservation laws. They may well be nonlocal, however, since they depend on solutions of the Pfaffian system (9), see [18]. The following lemma [14] allows us to construct them explicitly.

Lemma 2. *Let $\Xi = 0$ be a PSS equation with associated one-forms ω^α. Under the changes of variables $\Gamma = \tan(\rho/2)$ and $\hat{\Gamma} = \cot(\rho/2)$, equation (9) and the one-form θ^1 become,*

$$-2d\Gamma = \left(\overline{\omega}^3 + \overline{\omega}^2\right) - 2\Gamma\overline{\omega}^1 + \Gamma^2\left(\overline{\omega}^3 - \overline{\omega}^2\right), \tag{10}$$

$$\Theta = \overline{\omega}^1 - \Gamma\left(\overline{\omega}^3 - \overline{\omega}^2\right), \qquad (\textit{up to an exact differential form}) \tag{11}$$

and

$$2d\hat{\Gamma} = \left(\overline{\omega}^3 - \overline{\omega}^2\right) - 2\hat{\Gamma}\overline{\omega}^1 + \hat{\Gamma}^2\left(\overline{\omega}^3 + \overline{\omega}^2\right), \tag{12}$$

$$\hat{\Theta} = -\overline{\omega}^1 + \hat{\Gamma}\left(\overline{\omega}^3 + \overline{\omega}^2\right), \qquad (\textit{up to an exact differential form}). \tag{13}$$

We now turn to (generalized) symmetries. For ease of exposition, we restrict ourselves to strictly pseudo-spherical equations. We recall that a differential function G is a generalized symmetry of $u_t = F$ if and only if $u(x, t) + \tau G(u(x, t))$ is – to first order in τ – a solution of $u_t = F$ whenever $u(x, t)$ is a solution of $u_t = F$.

Let $u_t = F$ be an m^{th} order strictly pseudo-spherical equation with associated one-forms ω^α. Let $u(x, t)$ be a local solution of $u_t = F$, and set $\overline{G} = G(u(x, t))$, in which G is a differential function. We expand $\omega^\alpha(u(x, t) + \tau G(u(x, t))$ about $\tau = 0$, thereby obtaining an infinitesimal deformation $\overline{\omega}^\alpha + \tau\overline{\Lambda}_\alpha$, $\overline{\Lambda}_\alpha = \overline{g}_{\alpha 1} dx + \overline{g}_{\alpha 2} dt$, of the one-forms $\overline{\omega}^\alpha = \omega^\alpha(u(x, t))$. Lemma 1 implies that $\overline{g}_{\alpha 1} = f_{\alpha 1, u}(u(x, t))\,\overline{G}$, and $\overline{g}_{\alpha 2} = \sum\limits_{i=0}^{m-1} f_{\alpha 2, u_{x^i}}(u(x, t))\left(\partial^i \overline{G}/\partial x^i\right)$.

Theorem 1. *Suppose that $u_t = F(x, t, u, \ldots, u_{x^m})$ is strictly pseudo-spherical with associated one-forms $\omega^\alpha = f_{\alpha 1} dx + f_{\alpha 2} dt$, $\alpha = 1, 2, 3$, and let G be a differential function. The deformed one-forms $\overline{\omega}^\alpha + \tau\overline{\Lambda}_\alpha$ satisfy the structure equations of a pseudo-spherical surface up to terms of order τ^2 if and only if G is a generalized symmetry of $u_t = F$.*

Thus, generalized symmetries of strictly pseudo-spherical equations $u_t = F$ are identified with infinitesimal deformations of the pseudo-spherical structures determined by $u_t = F$ which preserve the Gaussian curvature to first order in the deformation parameter. The proof of Theorem 1 appears in [14, 15]. We note, finally, that nonlocal symmetries (see [4, 12] for a formal definition and applications of this important concept) can be also included in this geometrical framework [15], and that Theorem 1 can be used (see [14, 15]) to show the *existence* of (generalized, nonlocal) symmetries of evolutionary PSS equations.

4 The Camassa–Holm and Hunter–Saxton equations

Several facts about the important Camassa–Holm [5] and Hunter–Saxton equations [8] (the former derivated as a shallow water equation, the later describing weakly nonlinear unidirectional waves) are already known: for example, their inverse scattering solutions have been found (Beals, Sattinger and Szmigielski [2, 3]), their bi-Hamiltonian character has been discussed (Camassa and Holm [5], Hunter and Zheng [9]) and, it has been proven that the Korteweg-de Vries, Camassa–Holm and Hunter–Saxton equations exhaust (in a precise sense) the bi-Hamiltonian equations which can be modeled as geodesic flows on homogeneous spaces related to the Virasoro group (Khesin and Misiołek [11]). It is shown in this section that these three equations are of pseudo-spherical type, and that therefore they can be studied using the results summarized in Sections 2 and 3. We begin with the classical KdV:

Example 2. The KdV equation $u_t = u_{xxx} + 6uu_x$ describes pseudo-spherical surfaces [17, 6] with associated one-forms $\omega^\alpha = f_{\alpha 1}dx + f_{\alpha 2}dt$, in which

$$\omega^1 = (1-u)\,dx + \left(-u_{xx} + \eta u_x - \eta^2 u - 2u^2 + \eta^2 + 2u\right)\,dt, \tag{14}$$

$$\omega^2 = \eta\,dx + \left(\eta^3 + 2\eta u - 2u_x\right)\,dt, \tag{15}$$

$$\omega^3 = (-1-u)\,dx + \left(-u_{xx} + \eta u_x - \eta^2 u - 2u^2 - \eta^2 - 2u\right)\,dt, \tag{16}$$

and η is an arbitrary parameter. After rotating the coframe $\{\omega^1, \omega^2\}$ and changing Γ for $-\Gamma$, we can write the Pfaffian system (10) as

$$\text{(a) } \Gamma_x = -u - \eta\Gamma - \Gamma^2, \qquad \text{(b) } \Gamma_t = \left(\Gamma_{xx} - 3\Gamma^2\eta - 2\Gamma^3\right)_x.$$

It follows from the fact that KdV is strictly pseudo-spherical that if Γ solves (b), u as given by (a) solves KdV. We thus recover the Miura transformation and the modified KdV equation. Now take a solution $u(x,t)$ of KdV and compute $\Gamma(x,t,\eta)$ from (a). Equation (b) is invariant under the transformation $(\Gamma \mapsto -\Gamma, \eta \mapsto -\eta)$, and therefore (a) implies that $\overline{u}(x,t,\eta) = \Gamma_x(x,t,\eta) - \Gamma(x,t,\eta)\eta - \Gamma(x,t,\eta)^2$ is a one-parameter family of solutions of KdV. It follows that $\overline{u}(x,t,\eta) = u(x,t) + 2\Gamma_x(x,t,\eta)$, and therefore we also recover the classical Darboux transformation!

We now consider the Camassa–Holm (CH)

$$m = u_{xx} - u, \qquad m_t = -m_x\,u - 2\,m\,u_x, \tag{17}$$

and Hunter–Saxton (HS) equations

$$m = u_{xx}, \qquad m_t = -m_x\,u - 2\,m\,u_x. \tag{18}$$

Below and henceforth, we let ϵ be equal to 1 for CH and 0 for HS.

Theorem 2. *The Camassa–Holm and Hunter–Saxton equations, (17) and (18) respectively, describe pseudo-spherical surfaces.*

Proof. We consider one-forms σ^α, $\alpha = 1,2,3$, given by

$$\begin{aligned}\sigma^1 &= \left(m - \beta + \epsilon\,\eta^{-2}(\beta-1)\right)\,dx \\ &\quad + \left(-u_x\beta\,\eta^{-1} - \beta\,\eta^{-2} - u\,m - 1 + u\beta + u_x\,\eta^{-1} + \eta^{-2}\right)\,dt,\end{aligned} \tag{19}$$

$$\sigma^2 = \eta\,dx + \left(-\beta\,\eta^{-1} - \eta\,u + \eta^{-1} + u_x\right)\,dt, \tag{20}$$

$$\sigma^3 = (m+1)\,dx + \left(\epsilon\,u\,\eta^{-2}(\beta-1) - u\,m + \eta^{-2} + \frac{u_x}{\eta} - u - \frac{\beta}{\eta^2} - \frac{u_x\beta}{\eta}\right)\,dt, \tag{21}$$

in which the parameters η and β are constrained by the relation

$$\eta^2 + \beta^2 - 1 = \epsilon \left[\frac{\beta - 1}{\eta} \right]^2 . \tag{22}$$

It is not hard to check that the structure equations (2) are satisfied whenever $u(x,t)$ is a solution of (17) (if $m = u_{xx} - u$) and whenever $u(x,t)$ is a solution of (18) (if $\epsilon = 1$ and $m = u_{xx} - u$) and whenever $u(x,t)$ is a solution of (18) (if $\epsilon = 0$ and $m = u_{xx}$). ■

The fact that the CH equation is of pseudo-spherical type first appeared in [16]. A natural way to dispense with the constraint (22) is by using a parameterization of the curve $\eta^2 + \beta^2 - 1 = \epsilon\,[(\beta-1)/\eta]^2$. We take

$$\eta = \sqrt{\epsilon + 1 - s^2}, \qquad \beta = \frac{\epsilon}{s-1} - s. \tag{23}$$

It follows that the CH and HS equations are geometrically integrable, and it is not difficult to write down $sl(2,\mathbb{R})$-valued linear problems associated to them, simply by applying Proposition 2.

We turn to the quadratic pseudo-potential (12) associated with the CH and HS equations. After parameterizing the one-forms σ^α using (23), rotating the resulting forms via (7), applying the transformation $\hat{\Gamma} \mapsto \gamma + \sqrt{\epsilon + 1 - s^2}/(1-s)$, and setting $s - 1 = 1/\lambda$, we obtain the following result:

Theorem 3. *The CH equation (17) and the HS equation (18) admit quadratic pseudo-potentials γ determined by*

$$m = \gamma_x + \frac{1}{2\lambda}\gamma^2 - \frac{1}{2}\lambda\,\epsilon, \qquad \gamma_t = \frac{\gamma^2}{2}\left[1 + \frac{1}{\lambda}u\right] - u_x\gamma - u\,m + \epsilon\left[\frac{1}{2}u\lambda - \frac{1}{2}\lambda^2\right], \tag{24}$$

in which $\lambda \neq 0$ is a parameter. Moreover, equations (17) and (18) possess the parameter-dependent conservation law

$$\gamma_t = \lambda\left(u_x - \gamma - \frac{1}{\lambda}u\gamma\right)_x . \tag{25}$$

In view of Example 2, it is natural to postulate the first equation of (24) as the analog of the Miura transformation for the CH and HS equations, and (25) as the corresponding "modified" equation. Note that, in contradistinction with the KdV case, the modified CH and HS equations are nonlocal equations for γ. We also remark that equations (24) determine very simple linear problems for the CH and HS equations: setting $\gamma = \psi_1/\psi_2$ and replacing into (24), we find that the compatibility condition of the linear problem $d\psi = (X dx + T dt)\psi$, in which $\psi = (\psi_1, \psi_2)^t$, and

$$X = \frac{1}{2}\begin{bmatrix} 0 & \epsilon\,\lambda + 2\,m \\ \lambda^{-1} & 0 \end{bmatrix}, \qquad T = \frac{1}{2}\begin{bmatrix} -u_x & -2u\,m + \epsilon\,\lambda u - \epsilon\,\lambda^2 \\ -1 - u\lambda^{-1} & u_x \end{bmatrix}, \tag{26}$$

is precisely the CH equation (17) (if $m = u_{xx} - u$) and the HS equation (18) (if $m = u_{xx}$). It is not hard to check that this linear problem is related to the one obtained from (3) and (19)–(21) by a $gl(2,\mathbb{R})$-valued gauge transformation.

We now use (24) and (25) to construct conservation laws for the CH and HS equations. Setting $\gamma = \sum_{n=1}^{\infty} \gamma_n \lambda^{n/2}$ yields the conserved densities

$$\gamma_1 = \sqrt{2}\,\sqrt{m}, \qquad \gamma_2 = -\frac{1}{2}\,\ln(m)_x, \qquad \gamma_3 = \frac{1}{2\sqrt{2}\,\sqrt{m}}\left[\epsilon - \frac{m_x^2}{4\,m^2} + \ln(m)_{xx}\right], \tag{27}$$

$$\gamma_{n+1} = -\frac{1}{\gamma_1}\,\gamma_{n,x} - \frac{1}{2\gamma_1}\sum_{j=2}^{n} \gamma_j\,\gamma_{n+2-j}, \qquad n \geq 3, \tag{28}$$

while the expansion $\gamma = \epsilon\,\lambda + \sum\limits_{n=0}^{\infty} \gamma_n \lambda^{-n}$ implies

$$\gamma_{0,x} + \epsilon\,\gamma_0 = m, \qquad \gamma_{n,x} + \epsilon\,\gamma_n = -(1/2)\sum_{j=0}^{n-1} \gamma_j\,\gamma_{n-1-j}, \quad n \geq 1. \tag{29}$$

It is not hard to see [16] that, in the CH case, the local conserved densities γ_n determined by (27) and (28) correspond to the ones found by Fisher and Schiff [7] using an "associated Camassa–Holm equation", while (29) yields the local conserved densities u, $u_x^2 + u^2$, and $uu_x^2 + u^3$, and a sequence of nonlocal conservation laws.

We finish with a theorem on nonlocal symmetries for the Camassa–Holm and Hunter–Saxton equations:

Theorem 4. *Let γ, δ and β be defined by the equations*

$$\gamma_x = m - (1/2\lambda)\,\gamma^2 + \epsilon\,(1/2)\,\lambda, \qquad \gamma_t = \lambda\,(u_x - \gamma - (1/\lambda)u\gamma)_x; \tag{30}$$

$$\delta_x = \gamma, \qquad \delta_t = \lambda\,(u_x - \gamma - (1/\lambda)u\gamma); \tag{31}$$

$$\beta_x = m\,\mathrm{e}^{(1/\lambda)\,\delta}, \qquad \beta_t = \mathrm{e}^{(1/\lambda)\,\delta}\left(-(1/2)\,\gamma^2 + \epsilon\,(1/2)\,\lambda^2 - u\,m\right); \tag{32}$$

which are compatible on solutions of (17) and (18). The systems of equations (17), (30)–(32) and (18), (30)–(32), possess the classical symmetry

$$W = \gamma\,\mathrm{e}^{(1/\lambda)\,\delta}\frac{\partial}{\partial u} + \left(m_x + \frac{2}{\lambda}\gamma\,m\right)\mathrm{e}^{(1/\lambda)\,\delta}\frac{\partial}{\partial m} + m\,\mathrm{e}^{(1/\lambda)\,\delta}\frac{\partial}{\partial \gamma} + \beta\frac{\partial}{\partial \delta} + \left(m\,\mathrm{e}^{(2/\lambda)\,\delta} + \frac{1}{2\lambda}\beta^2\right)\frac{\partial}{\partial \beta}. \tag{33}$$

Thus, in particular, the evolutionary vector field

$$V = \gamma\,\mathrm{e}^{(1/\lambda)\,\delta}\frac{\partial}{\partial u} + \left(m_x + \frac{2}{\lambda}\gamma\,m\right)\mathrm{e}^{(1/\lambda)\,\delta}\frac{\partial}{\partial m}$$

is a one-parameter family of nonlocal symmetries for (17) and (18).

Theorem 4 can be verified using the MAPLE package VESSIOT developed by I. Anderson and his coworkers, see [1]. We remark that it is certainly possible (see [16] for the CH case) to find the flow of W, and therefore Theorem 4 gives us a method to construct solutions to the CH and HS equations!

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