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Fractional Supersymmetry and F−fold Lie Superalgebras

M. RAUSCH de TRAUBENBERG † *and M.J. SLUPINSKI* ‡

† *Laboratoire de Physique Théorique, Université Louis-Pasteur, and CNRS*
3 rue de l'Université, 67084 Strasbourg, France
E-mail: *rausch@lpt1.u-strasbg.fr*

‡ *Institut de Recherches en Mathématique Avancée, Université Louis-Pasteur, and CNRS*
7 rue R. Descartes, 67084 Strasbourg Cedex, France
E-mail: *slupins@irmasv2.u-strasbg.fr*

We give infinite dimensional and finite dimensional examples of F−fold Lie superalgebras. The finite dimensional examples are obtained by an inductive procedure from Lie algebras and Lie superalgebras.

1 Introduction

It is generally held that supersymmetry is the only non-trivial extension of the Poincaré algebra. This point of view is based on the two theorems [1, 2]. However, as usual, if some of the assumptions of these two *no-go* theorems are relaxed symmetry beyond supersymmetry can be constructed [3–23] In all these possible extensions of the Poincaré symmetry, new generators are introduced. The basic structure underlying these extensions is related to algebraic structures which are neither Lie algebras, nor Lie superalgebras. In this contribution we would like to give some results concerning fractional supersymmetry (FSUSY) [6–23], one of the possible extensions of supersymmetry, and the associated algebraic structure, the so-called F−Lie algebra [21]. Such a structure admits a $\mathbb{Z}_F$ grading, the zero-graded part defining a Lie algebra, and an F−fold symmetric product (playing the role of the anticommutator in the case $F = 2$) allows one to express the zero graded part in terms of generators of the non zero graded part. In Section 2 the basic definition of F−Lie algebras will be given. In Section 3, some examples of infinite dimensional F−Lie algebras will be explicitly constructed. In Section 4, some examples of finite dimensional F−Lie algebras will be explicitly constructed with the classification of the usual Lie (super)algebras as a guideline.

2 F−Lie algebras

A natural mathematical structure, generalizing the concept of Lie superalgebras and relevant for the algebraic description of fractional supersymmetry was introduced in [21] and called an F−Lie algebra. We do not want to go into the detailed definition of this structure here and will only recall the basic points, useful for our purpose. More details can be found in [21].

Let F be a positive integer and $q = e^{2i\frac{\pi}{F}}$. We consider now a complex vector space S which has an automorphism ε satisfying $\varepsilon^F = 1$. We set $A_k = S_{q^k}$, $1 \le k \le F-1$ and $B = S_1$ (S_{q^k} is the eigenspace corresponding to the eigenvalue q^k of ε). Hence,

$$S = B \oplus A_1 \oplus \cdots \oplus A_{F-1}.$$

We say that S is an F−Lie algebra if:

1. B, the zero graded part of S, is a Lie algebra.
2. A_i ($i = 1, \ldots, F-1$), the i graded part of S, is a representation of B.
3. There are symmetric multilinear $B-$equivariant maps

$$\{\ ,\ldots,\ \}:\ \mathcal{S}^F(A_k) \to B.$$

In other words, we assume that some of the elements of the Lie algebra B can be expressed as $F-$th order symmetric products of "more fundamental generators". It is easy to see that

$$\{\varepsilon(a_1), \ldots, \varepsilon(a_F)\} = \varepsilon\left(\{a_1, \ldots, a_F\}\right), \quad \forall\, a_1, \ldots, a_F \in A_k.$$

The generators of S are assumed to satisfy Jacobi identities ($b_i \in B$, $a_i \in A_k$, $1 \le k \le F-1$):

$$\begin{aligned}
&\left[\left[b_1, b_2\right], b_3\right] + \left[\left[b_2, b_3\right], b_1\right] + \left[\left[b_3, b_1\right], b_2\right] = 0,\\
&\left[\left[b_1, b_2\right], a_3\right] + \left[\left[b_2, a_3\right], b_1\right] + \left[\left[a_3, b_1\right], b_2\right] = 0,\\
&\left[b, \{a_1, \ldots, a_F\}\right] = \{\left[b, a_1\right], \ldots, a_F\} + \cdots + \{a_1, \ldots, \left[b, a_F\right]\},\\
&\sum_{i=1}^{F+1} \left[a_i, \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{F+1}\}\right] = 0.
\end{aligned} \tag{1}$$

The first three identities are consequences of the previously defined properties but the fourth is an extra constraint.

More details (unitarity, representations, *etc.*) can be found in [21]. Let us first note that no relation between different graded sectors is postulated. Secondly, the sub-space $B \oplus A_k \subset S$ ($k = 1, \ldots, F-1$) is itself an $F-$Lie algebra. From now on, $F-$Lie algebras of the types $B \oplus A_k$ will be considered.

3 Examples of infinite dimensional $F-$Lie algebras

It is possible to construct an $F-$Lie algebra starting from a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}-$module $\mathcal{D}$. The basic idea is the following. We consider $\mathfrak{g}$ a semi-simple Lie algebra of rank r. Let $\mathfrak{h}$ be a Cartan sub-algebra of $\mathfrak{g}$, let $\Phi \subset \mathfrak{h}^\star$ be the corresponding set of roots and let $\mathfrak{f}_\alpha$ be the one-dimensional root space associated to $\alpha \in \Phi$. We choose a basis $\{H_i, i = 1, \ldots, r\}$ of $\mathfrak{h}$ and elements $E^\alpha \in \mathfrak{f}_\alpha$ such that the commutation relations become

$$\begin{aligned}
&\left[H_i, H_j\right] = 0, \qquad \left[H_i, E^\alpha\right] = \alpha^i E^\alpha,\\
&\left[E^\alpha, E^\beta\right] = \begin{cases} \epsilon\{\alpha, \beta\} E^{\alpha+\beta} & \text{if } \ \alpha + \beta \in \Phi,\\ \dfrac{2\alpha.H}{\alpha.\alpha} & \text{if } \ \alpha + \beta = 0,\\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{2}$$

We now introduce $\{\alpha_{(1)}, \ldots, \alpha_{(r)}\}$, a basis of simple roots. The weight lattice $\Lambda_W(\mathfrak{g}) \subset \mathfrak{h}^\star$ is the set of vectors μ such that $\frac{2\alpha.\mu}{\alpha.\alpha} \in \mathbb{Z}$ and, as is well known, there is a basis of the weight lattice consisting of the fundamental weights $\{\mu_{(1)}, \ldots, \mu_{(r)}\}$ defined by $\frac{2\mu_{(i)}\cdot\alpha_{(j)}}{\alpha_{(j)}\cdot\alpha_{(j)}} = \delta_{ij}$. A weight $\mu = \sum\limits_{i=1}^{r} n_i \mu_{(i)}$ is called dominant if all the $n_i \ge 0$ and it is well known that the set of dominant weights is in one to one correspondence with the set of (equivalence classes of) irreducible finite dimensional representations of $\mathfrak{g}$.

Recall briefly how one can associate a representation of $\mathfrak{g}$ to $\mu \in \mathfrak{h}^\star$, $\mu = \sum\limits_{i=1}^{r} n_i \mu_i$, $n_i \in \mathbb{C}$. We start with a vacuum $|\mu\rangle$ such that

$$\begin{aligned} &E^\alpha |\mu\rangle = 0, \qquad \alpha > 0, \\ &2\frac{\alpha_{(i)}.H}{\alpha^2}|\mu\rangle = n_i|\mu\rangle, \qquad i = 1,\dots,r. \end{aligned} \tag{3}$$

The space obtained from $|\mu\rangle$ by the action of elements of $\mathfrak{g}$:

$$\mathcal{V}_\mu = \left\{E^{-\alpha_{(i_1)}}\cdots E^{-\alpha_{(i_k)}}|\mu\rangle, \alpha_{(i_1)},\dots,\alpha_{(i_k)} > 0\right\},$$

clearly defines a representation of $\mathfrak{g}$. Taking the quotient of $\mathcal{V}_\mu$ by its maximal $\mathfrak{g}-$stable subspace, the representation $\mathcal{D}_\mu$ of highest weight μ is obtained. If the n_i are positive integers, this is the irreducible finite dimensional representation of $\mathfrak{g}$ corresponding to the dominant weight μ.

To come back to our original problem, consider a finite dimensional irreducible representation $\mathcal{D}_\mu$, with highest weight $\mu = \sum\limits_{i=1}^{r} n_i\mu_i, n_i \in \mathbb{N}$. The basic idea is to try to define a structure of an $F-$Lie algebra on $S = B \oplus A_1 = (\mathfrak{g} \oplus \mathcal{D}_\mu) \oplus \mathcal{D}_{\frac{\mu}{F}}$ since, roughly speaking, the representations $\mathcal{D}_\mu$ and $\mathcal{D}_{\frac{\mu}{F}}$ can be related:

$$\mathcal{S}^F\left(\mathcal{D}_{\mu/F}\right) \sim \mathcal{D}_\mu. \tag{4}$$

Indeed $|\frac{\mu}{F}\rangle^{\otimes F} \in \mathcal{S}^F\left(\mathcal{D}_{\frac{\mu}{F}}\right)$ and $|\mu\rangle \in \mathcal{D}_\mu$ both satisfy equation (3). However, the representation $\mathcal{D}_\mu$ is finite dimensional but the sub-representation of $\mathcal{S}^F\left(\mathcal{D}_{\frac{\mu}{F}}\right)$ generated by $|\frac{\mu}{F}\rangle^{\otimes F}$ is infinite dimensional [21, 22]. Thus, the main difficulty in such a construction is to do with the requirement of relating an infinite dimensional representation $\mathcal{D}_{\mu/F}$ to a finite dimensional representation $\mathcal{D}_\mu$ in an equivariant way, *i.e.* respecting the action of $\mathfrak{g}$. One possible way of overcoming this difficulty is to embed $\mathcal{D}_\mu$ into an infinite dimensional (reducible but indecomposable) representation [21, 22, 23]. Another possibility is to embed $\mathfrak{g}$ into an infinite dimensional algebra (dubbed $V(\mathfrak{g})$) [21, 23, 24]) and extend the representations $\mathcal{D}_\mu$ and $\mathcal{D}_{\frac{\mu}{F}}$ to representations of $V(\mathfrak{g})$.

There is another difficulty related to such a construction. If one starts with $\mathfrak{D}_{\mu_1}$, the vector representation of $\mathfrak{so}(1, d-1)$, the representation $\mathfrak{D}_{\frac{\mu_1}{F}}$ cannot be exponentiated (see *e.g.* [25]) and does not define a representation of the Lie group $\overline{SO(1, d-1)}$ (the universal covering group of $SO(1, d-1)$) except when $d = 3$, where such representations describe relativistic anyons [18, 26].

4 Example of finite dimensional $F-$Lie algebras

In the previous section, we indicated how one can construct infinite dimensional examples of $F-$Lie algebras. In this section, with the classification of Lie (super)algebras as a guideline, we will give an inductive construction of finite dimensional $F-$Lie algebras.

In what follows S is a $1-$Lie algebra means:

1. $S = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with $\mathfrak{g}_0$ a Lie algebra and $\mathfrak{g}_1$ is a representation of $\mathfrak{g}_0$ isomorphic to the adjoint representation;
2. there is a $\mathfrak{g}_0-$ equivariant map $\mu : \mathfrak{g}_1 \to \mathfrak{g}_0$ such that $[f_1, \mu(f_2)] + [f_2, \mu(f_1)] = 0, f_1, f_2 \in \mathfrak{g}_1$.

The basic result is the following theorem:

Theorem 1. *Let $\mathfrak{g}_0$ be a (complex) Lie algebra and $\mathfrak{g}_1$ a representation of $\mathfrak{g}_0$. Suppose given*
(i) the structure of an F_1-Lie algebra on $S_1 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$;
(ii) the structure of an F_2-Lie algebra on $S_2 = \mathbb{C} \oplus \mathfrak{g}_1$.
Then $S = \left(\mathfrak{g}_0 \otimes \mathbb{C}\right) \oplus \mathfrak{g}_1$ can be given the structure of an $(F_1 + F_2)-$Lie algebra.

Proof. There exists (i) a $\mathfrak{g}_0-$equivariant map $\mu_1 : \mathcal{S}^{F_1}(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_0$ and (ii) a $\mathfrak{g}_0-$equivariant map $\mu_2 : \mathcal{S}^{F_2}(\mathfrak{g}_1) \longrightarrow \mathbb{C}$, the second map is just a symmetric $F_2^{\text{th}}-$order invariant form on $\mathfrak{g}_1$. Now, consider $\mu : \mathcal{S}^{F_1+F_2}(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_0 \otimes \mathbb{C} \cong \mathfrak{g}_0$ defined by $\forall\ f_1, \dots, f_{F_1+F_2} \in \mathfrak{g}_1$:

$$\begin{aligned}&\mu(f_1, \dots, f_{F_1+F_2})\\ &\qquad = \frac{1}{F_1!}\frac{1}{F_2!}\sum_{\sigma\in S_{F_1+F_2}} \mu_1(f_{\sigma(1)}, \dots, f_{\sigma(f_{F_1})}) \otimes \mu_2(f_{\sigma(f_{F_1+1})}, \dots, f_{\sigma(f_{F_1+F_2})}).\end{aligned} \tag{5}$$

with $S_{F_1+F_2}$ the group of permutations of $F_1 + F_2$ elements. By construction, this map is a $\mathfrak{g}_0-$equivariant map from $\mathcal{S}^{F_1+F_2}(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_0$, thus the first three Jacobi identities (1) are clearly satisfied. The last Jacobi identity is more difficult to check and is directly related to the last Jacobi identity for the F_1-Lie algebra S_1 by a factorisation property. Indeed (with $F = F_1 + F_2$) if one calculates:

$$\sum_{i=0}^{F} \left[f_i, \mu\left(f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_F\right)\right],$$

and selects terms of the form (with $\sigma \in S_{F_1+F_2+1)}$)

$$\mu_1\left(f_{\sigma(1)}, \dots, f_{\sigma(f_{F_1})}\right) \otimes \mu_2\left(f_{\sigma(f_{F_1+1})}, \dots, f_{\sigma(f_{F_1+F_2})}\right),$$

using $\mu_2\left(f_{\sigma(f_{F_1+1})}, \cdots, f_{\sigma(f_{F_1+F_2})}\right) \in \mathbb{C}$ the identity reduces to

$$\sum_{i=0}^{F_1}\left[f_{\sigma(i)}, \mu_1\left(f_{\sigma(1)}, \dots, f_{\sigma(i-1)}, f_{\sigma(i+1)}, \dots, f_{\sigma(f_{F_1})}\right)\right] \otimes \mu_2\left(f_{\sigma(f_{F_1+1})}, \dots, f_{\sigma(f_{F_1+F_2})}\right) = 0.$$

This follows from the corresponding Jacobi identity for the F_1-Lie algebra S_1. Now proceeding along the same lines for the other terms, a similar factorisation works. Thus the fourth Jacobi identity is satisfied and S is an $(F_1 + F_2)-$Lie algebra. ∎

Here there are some families of examples:

1. $S_1 = \mathfrak{g} \oplus \mathrm{Ad}(\mathfrak{g})$(a $1-$Lie algebra); $S_2 = \mathbb{C} \oplus \mathrm{Ad}(\mathfrak{g})$ (a Lie superalgebra if $\mathfrak{g}$ admits an equivariant quadratic form).
2. If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra (basic of type I or II or $Q(n)$ [27]) we associate to $\mathfrak{g}$ an "augmented" Lie superalgebra as follows:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \longrightarrow \left\{\begin{array}{ll} S = \mathcal{B} \oplus \mathcal{F} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 & \text{if } \mathfrak{g} \text{ is of type I},\\ S = \mathcal{B} \oplus \mathcal{F} = \mathfrak{g}_0 \oplus (\mathfrak{g}_1 \oplus \mathfrak{g}_1) & \text{if } \mathfrak{g} \text{ is of type II},\\ S = \mathcal{B} \oplus \mathcal{F} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 & \text{if } \mathfrak{g} = Q(n). \end{array}\right. \tag{6}$$

The non-zero graded part of these "augmented" Lie superalgebras always admits a $\mathfrak{g}_0$ invariant quadratic form and hence $S_2 = \mathbb{C} \oplus \mathcal{F}$ is a Lie superalgebra:

For the type I superalgebras we have $\mathfrak{g}_1 = \mathfrak{D} \oplus \mathfrak{D}^*$ (see [27]), and so one has a natural map: $\mathcal{S}^2(\mathfrak{D} \oplus \mathfrak{D}^*) \longrightarrow \mathbb{C}$.

For the type II superalgebras, we recall that $\mathfrak{g}_1$ admits an invariant antisymmetric bilinear form and hence $\mathfrak{g}_1 = \mathcal{D}$ is self-dual [27]. Therefore, there is an invariant quadratic form on $\mathcal{F} = \mathfrak{g}_1 \oplus \mathfrak{g}_1$.

For the strange superalgebra $Q(n)$, $\mathfrak{g}_0 = \mathfrak{sl}(n + 1)$, the representation $\mathfrak{g}_1$ is the adjoint representation of $\mathfrak{g}_0$ (see [27]) and hence admits an invariant quadratic form (the Killing form).

The existence of an invariant bilinear form on $\mathfrak{g}_1$ (*i.e.* before the "augmentation" (6)) means that there is a $\mathfrak{g}_0-$equivariant mapping $\mathcal{S}_\pm{}^2(\mathfrak{g}_1) \longrightarrow \mathbb{C}$ (where $\mathcal{S}_+{}^2(\mathfrak{g}_1)$ (resp. $\mathcal{S}_-{}^2(\mathfrak{g}_1)$) represent the two-fold symmetric (antisymmetric) tensor product of $\mathfrak{g}_1$). We denote generically this tensor by $\delta_{\alpha\beta}$ when it is symmetric and $\Omega_{\alpha\beta}$ when it is antisymmetric. This can equivalently be rewritten in a basis of $\mathfrak{g}_1, F_\alpha \in \mathfrak{g}_1$

$$\begin{aligned} \{F_\alpha, F_\beta\} &= \delta_{\alpha\beta}, \quad \text{for the type I superalgebras and for } Q(n), \\ [F_\alpha, F_\beta] &= \Omega_{\alpha\beta}, \quad \text{for the type II superalgebras.} \end{aligned} \tag{7}$$

with $\{\ ,\ \}$ (resp. $[\ ,\ ]$) the symmetric (resp. antisymmetric) bilinear forms.

However, after the "augmentation "(6) in the case of Lie superalgebra of type II, the mapping $\mathcal{S}^2(\mathcal{F}) \longrightarrow \mathbb{C}$ (*i.e.* the quadratic form on $\mathcal{F}$) reads

$$\{F_{i\alpha}, F_{j\beta}\} = \varepsilon_{ij}\Omega_{\alpha\beta}, \tag{8}$$

with $F_{i\alpha} \in \mathfrak{g}_1 \oplus \mathfrak{g}_1$. The index α represents the $\mathfrak{g}_1$ degrees of freedom, the index i ($i = -1, 1$) the two copies of $\mathfrak{g}_1$ and ε_{ij} the two dimensional antisymmetric tensor.

To conclude, we will give an example of a $3-$Lie (resp. $4-$Lie) algebra, associated to a $1-$Lie algebra (resp. superalgebra).

Example 1. Let $\mathfrak{g}_0$ be a Lie algebra and $\mathfrak{g}_1$ the adjoint representation of $\mathfrak{g}_0$ and $S_3 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We introduce J_a, A_a, $a = 1, \dots, \dim(\mathfrak{g}_0)$ a basis of S_3. We denote $\mathrm{tr}(A_a A_b) = g_{ab}$ the Killing form. The trilinear bracket of the $3-$Lie algebra S_3, associated to the Lie algebra $\mathfrak{g}$, is:

$$\{A_a, A_b, A_c\} = g_{ab}J_c + g_{ac}J_b + g_{bc}J_a. \tag{9}$$

If $\mathfrak{g} = \mathfrak{sl}(2)$, this is the $3-$Lie algebra constructed in [28].

Example 2. As a second example we give the formulae for the quadrilinear bracket of the $4-$Lie algebra constructed from the orthosymplectic superalgebra. Starting from $\mathfrak{osp}(m|2n) = (\mathfrak{so}(m) \oplus \mathfrak{sp}(2n)) \oplus (\mathbf{m}, \mathbf{2n})$, we define $\mathfrak{osp}(m|2n; 4) = (\mathfrak{so}(m) \oplus \mathfrak{sp}(2n) \oplus \mathfrak{u}(1)) \oplus ((\mathbf{m}, \mathbf{2n})^+ \oplus (\mathbf{m}, \mathbf{2n})^-)$.

Let $F_{qi\alpha}$ ($q = -1, +1$, $1 \le i \le m$, $1 \le \alpha \le 2n$) denote the odd part, J_{ij} the $\mathfrak{so}(m)$ generators, $S_{\alpha\beta}$ the $\mathfrak{sp}(2n)$ generators and h the $\mathfrak{u}(1)$ generator (J_{ij} are antisymmetric and $S_{\alpha\beta}$ are symmetric). The invariant tensor on $\mathfrak{so}(m)$ is given by the symmetric tensor δ_{ij} and on $\mathfrak{sp}(2n)$ by the antisymmetric tensor $\Omega_{\alpha\beta}$, hence the invariant tensor for $\mathfrak{osp}(m|2n)$ is given by $\delta_{ij}\Omega_{\alpha\beta}$. Thus, the quadrilinear bracket of the $4-$algebra takes the form

$$\begin{aligned} &\{F_{q_1 i_1 \alpha_1}, F_{q_2 i_2 \alpha_2}, F_{q_3 i_3 \alpha_3}, F_{q_4 i_4 \alpha_4}\} \\ &\quad = \varepsilon_{q_1 q_2}\delta_{i_1 i_2}\Omega_{\alpha_1\alpha_2}\left(\delta_{q_3+q_4}\delta_{i_3+i_4}S_{\alpha_3\alpha_4} + \delta_{q_3+q_4}\Omega_{\alpha_3\alpha_4}J_{i_3 i_4} + a\delta_{q_3+q_4}\varepsilon_{i_3 i_4}\Omega_{\alpha_3\alpha_4}h\right) \\ &\quad + \varepsilon_{q_1 q_3}\delta_{i_1 i_3}\Omega_{\alpha_1\alpha_3}\left(\delta_{q_2+q_4}\delta_{i_2+i_4}S_{\alpha_2\alpha_4} + \delta_{q_2+q_4}\Omega_{\alpha_2\alpha_4}J_{i_2 i_4} + a\delta_{q_2+q_4}\varepsilon_{i_2 i_4}\Omega_{\alpha_2\alpha_4}h\right) \\ &\quad + \varepsilon_{q_1 q_4}\delta_{i_1 i_4}\Omega_{\alpha_1\alpha_4}\left(\delta_{q_2+q_3}\delta_{i_2+i_3}S_{\alpha_2\alpha_3} + \delta_{q_2+q_3}\Omega_{\alpha_2\alpha_3}J_{i_2 i_3} + a\delta_{q_2+q_3}\varepsilon_{i_2 i_3}\Omega_{\alpha_2\alpha_3}h\right) \\ &\quad + \varepsilon_{q_2 q_3}\delta_{i_2 i_3}\Omega_{\alpha_2\alpha_3}\left(\delta_{q_1+q_4}\delta_{i_1+i_4}S_{\alpha_1\alpha_4} + \delta_{q_1+q_4}\Omega_{\alpha_1\alpha_4}J_{i_1 i_4} + a\delta_{q_1+q_4}\varepsilon_{i_1 i_4}\Omega_{\alpha_1\alpha_4}h\right) \\ &\quad + \varepsilon_{q_2 q_4}\delta_{i_2 i_4}\Omega_{\alpha_2\alpha_4}\left(\delta_{q_1+q_3}\delta_{i_1+i_3}S_{\alpha_1\alpha_3} + \delta_{q_1+q_3}\Omega_{\alpha_1\alpha_3}J_{i_1 i_3} + a\delta_{q_1+q_3}\varepsilon_{i_1 i_3}\Omega_{\alpha_1\alpha_3}h\right) \\ &\quad + \varepsilon_{q_3 q_4}\delta_{i_3 i_4}\Omega_{\alpha_3\alpha_4}\left(\delta_{q_1+q_2}\delta_{i_1+i_2}S_{\alpha_1\alpha_2} + \delta_{q_1+q_2}\Omega_{\alpha_1\alpha_2}J_{i_1 i_2} + a\delta_{q_1+q_2}\varepsilon_{i_1 i_2}\Omega_{\alpha_1\alpha_2}h\right), \end{aligned} \tag{10}$$

with $a \in \mathbb{C}$.

Remark 1. It should be noticed that $F-$Lie algebras associated to Lie algebras (resp. to Lie superalgebras) are of odd (resp. even) order.

5 Conclusion

In this paper a sketch of the construction of $F-$Lie algebras associated to Lie (super)algebras was given. More complete results, such as a criteria for simplicity, representation theory, matrix realisations *etc.*, will be given elswhere.

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