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Heat Equation on Riemann Manifolds: Morphisms and Factorization to Smaller Dimension

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In [1, 2, 3, 4, 5] there was proposed a method of a factorization of PDE. The method is based on reduction of complicated systems to more simple ones (for example, due to dimension decrease). This concept is proposed in general case for the arbitrary PDE systems, and its concrete investigation is developing for the heat equation case. The category of second order parabolic equations posed on arbitrary manifolds is considered. In this category, for the given nonlinear heat equation we could find morphisms from it to other parabolic equations with the same or a smaller number of independent variables. This allows to receive some classes of solutions of original equation from the class of all solutions of such a reduced equation. Classification of morphisms (with the selection from every equivalence class of the simplest "canonical" representatives) is carried out. Necessary and sufficient conditions for canonical morphisms of heat equation to the parabolic equation on the other manifold are derived. These conditions are formulated in the differential geometry language. The comparison with invariant solutions classes, obtained by the Lie group methods, is carried out. It is proved that discovered solution classes are richer than invariant solution classes, even if we find any (including discontinuous) symmetry groups of original equation.

1 General equation category

Definition 1. *Task* is a pair $A = (N_A, E_A)$, where N_A is a set, E_A is a system of equations for graph $\Gamma \subset N_A = M_A \times K_A$ of a function $u : M_A \to K_A$.

Let $S(A)$ be a set of all subsets $\Gamma \subset N_A$ satisfying E_A.

Definition 2. We will say that a (ordered) pair of a tasks $A = (N_A, E_A)$, $B = (N_B, E_B)$ *admits a map* $F_{AB} : N_A \to N_B$, if for any $\Gamma \subset N_B$, $\Gamma \in S(B) \Leftrightarrow F_{AB}^{-1}(\Gamma) \in S(A)$.

Of course, these definitions are rather informal, but they will be correct when we define more exactly the notion "system of equations" and the class of assumed subsets $\Gamma \subset N_A$. Let us consider the *general equation category* $\mathcal{E}$, whose objects are tasks (with some refinement of the sense of the notion "system of equations"), and morphisms $\mathrm{Mor}(A, B)$ are admitted by the pair (A, B) maps with natural composition law.

For the given task A we could define the set $\mathrm{Mor}(A, \mathcal{A})$ of all morphisms A in a framework of some fixed subcategory $\mathcal{A}$ of the general equation category (let us call such morphisms and corresponding tasks B "factorization of A"). The tasks, which factorize A, are naturally divided into classes of isomorphic tasks, and morphisms $\mathrm{Mor}(A, \cdot)$ are divided into equivalence classes.

The proposed approach is conceptually close to the developed in [6] approach to investigation of dynamical and controlled systems. In this approach as morphisms of system A to the system B smooth maps of the phase space of system A to the phase space of system B are considered, which transform solutions (phase trajectories) of A to the solutions of B. By contrast, in the approach presented here, for the class of all solutions of reduced system B there is a corresponding class of such solutions of original system A, whose graphs could be projected onto the space of dependent and independent variables of B; when we pass to the reduced system, the number of dependent variables remains the same, and the number of independent variables does not increase. Thus the approach proposed is an analog to the sub-object notion (in terminology of [6]) with respect to information about original system solutions, though it is closed to the factor-object notion with respect to relations between original and reduced systems.

If G is symmetry group of E_A, then natural projection $p : N \to N/G$ is admitted by the pair $(A, A/G)$ in the sense of Definition 2, that is our definition is a generalization of the reduction by the symmetry group. Instead of this the general notion of the group analysis we base on a more wide notion “a map admitted by the task”. We need not require from the group preserving solution of an interesting class (if even such a group should exist) to be continuous admitted by original system. So we could obtain more general classes of solutions and than classes of invariant solutions of Lie group analysis (though our approach is more laborious owing to non-linearity of a system for admissible map). Besides, when we factorize original system, a factorizing map defined here is a more natural object than the group of transformations operating on space of independent and dependent variables of the original task.

2 Category of parabolic equations

Let us consider subcategory $\mathcal{PE}$ of the general equation category, whose objects are second order parabolic equations:

$$E: \quad u_t = Lu, \quad M = T \times X, \quad K = \mathbb{R},$$

where L is differential operator, depending on the time t, defined on the connected manifold X, which has the following form in any local coordinates $\left(x^i\right)$ on X:

$$Lu = b^{ij}\left(t, x, u\right) u_{ij} + c^{ij}\left(t, x, u\right) u_i u_j + b^i\left(t, x, u\right) u_i + q\left(t, x, u\right).$$

Here a lower index i denotes partial derivative by x^i, form $b^{ij} = b^{ji}$ is positively defined, $c^{ij} = c^{ji}$. Morphisms of $\mathcal{PE}$ are all smooth maps admitted by $\mathcal{PE}$ task pairs. Let us describe this morphisms:

Theorem 1. *Any morphism of the category $\mathcal{PE}$ has the form*

$$(t, x, u) \to \left(t'\left(t\right), x'\left(t, x\right), u'\left(t, x, u\right)\right). \tag{1}$$

Set of isomorphisms of the category $\mathcal{PE}$ is the set of all one-to-one maps of kind (1).

Let us consider full subcategory $\mathcal{PE}'$ of the category $\mathcal{PE}$, whose objects are equations $u_t = Lu$, where operator L in local coordinates has the following form:

$$Lu = b^{ij}\left(t, x\right)\left(a\left(t, x, u\right) u_{ij} + c\left(t, x, u\right) u_i u_j\right) + b^i\left(t, x, u\right) u_i + q\left(t, x, u\right),$$

and all morphisms are inherited from $\mathcal{PE}$.

Theorem 2. *If set of morphisms $\mathrm{Mor}_{\mathcal{PE}}\left(A, B\right)$ is nonempty and $A \in \mathcal{PE}'$, then $B \in \mathcal{PE}'$.*

3 Category of autonomous parabolic equations

Let us call the map (1) *autonomous*, if it has the form

$$(t, x, u) \to \left(t, x'\left(x\right), u'\left(x, u\right)\right). \tag{2}$$

Let us call a parabolic equation from the category $\mathcal{PE}'$, defined on a Riemann manifold X, *autonomous*, if it has the form:

$$u_t = Lu = a\left(x, u\right)\Delta u + c\left(x, u\right)\left(\nabla u\right)^2 + \xi\left(x, u\right)\nabla u + q\left(x, u\right), \quad \xi\left(\cdot, u\right) \in T^* X.$$

Theorem 3. *Let $F : A \to B$ be a morphism of the category $\mathcal{PE}$, F be an autonomous map, A be an autonomous equation. Then we could endow with Riemann metric the manifold, on which B is posed, in such a way, that B becomes an autonomous equation.*

Let $\mathcal{APE}$ be the subcategory of $\mathcal{PE}'$, objects of which are autonomous parabolic equations, and morphisms are autonomous morphisms of the category $\mathcal{PE}$.

4 Classification of morphisms of nonlinear heat equation

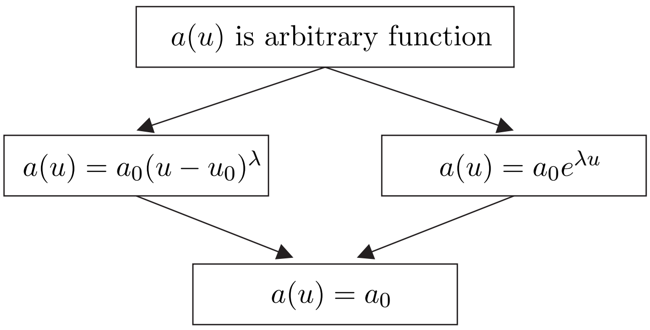
Let us consider a nonlinear heat equation $A \in \mathcal{APE}$, posed on some Riemann manifold X:

$$u_t = a\,(u)\,\Delta u + q(u). \tag{3}$$

(note that any equation $u_t = a(u)\Delta u + c(u)(\nabla u)^2 + q(u))$ is isomorphic to some equation (3) in $\mathcal{APE}$). We will investigate set of morphisms $\mathrm{Mor}\,(A, \mathcal{PE})$ and classes of solutions of equation A, corresponding these morphisms.

Note, that two morphisms $F : A \to B$ and $F' : A \to B'$ are called to be equivalent if there exists such isomorphism $G : B \to B'$ that $F' = G \circ F$. From the point of view of classes of original task solutions obtained from factorization, equivalent morphisms have the same value, that is solution classes are the same for these morphisms. So it is interesting to select from any equivalence class of the simplest (in some sense) morphism, or such morphism for which the factorized equation is the simplest.

When we classify morphisms for the original equation (3), a form of coefficient $a\,(u)$ is important. We will distinguish such option:



The lower the option is situated on this scheme, the richer a collection of morphisms is. Note, that similar relation is observed in the group classification of nonlinear heat equation [7].

Theorem 4. *If $a \neq \mathrm{const}$ then for any morphism of equation (3) into the category $\mathcal{PE}$ there exists an equivalent in $\mathcal{PE}$ autonomous morphism (that is morphism of the category $\mathcal{APE}$).*

Let us give a map $p : X \to X'$ from the manifold X to the manifold X' and a differential operator D on X. We will say that D *is projected on* X', if such a differential operator D' on X' exists that the following diagram is commutative:

$$\begin{array}{ccc} C^\infty(X') & \xrightarrow{p^*} & C^\infty(X) \\ {\scriptstyle D'}\big\downarrow & & \big\downarrow{\scriptstyle D} \\ C^\infty(X') & \xrightarrow[p^*]{} & C^\infty(X) \end{array}$$

Theorem 5. *Let $a \neq \text{const}$. For any morphism of the equation A into the category $\mathcal{PE}$ there exists an equivalent in $\mathcal{PE}$ autonomous morphism $(t, x, u) \to (t, y(x), v(x, u))$ A to $B \in \mathcal{APE}$, for which factorized equation B is $v_t = a(v) Lv + Q(v)$, operator L is projection onto Y at map $x \to y(x)$ of the described below operator D (note that this condition is limitation on the projection $y(x)$), where:*

1) if A is arbitrary (not any of the following special form): $D = \Delta$, $v(x, u) = u$;

2) if A is $u_t = a_0 u^{\lambda} (\Delta u + q_0 u) + q_1 u$ up to shift $u \to u - u_0$, $\lambda \neq 0$, $a_0, q_0, q_1 = \text{const}$: $Df = \beta^{\lambda-1} (\Delta(\beta f) + q_0 \beta f)$ for some function $\beta : X \to \mathbb{R}$, $v(x, u) = \beta^{-1}(x) u$, $Q = q_1 v$;

3) if A is $u_t = a_0 e^{\lambda u} (\Delta u + q_0) + q_1$, $\lambda \neq 0$, $a_0, q_0, q_1 = \text{const}$: $Df = e^{\lambda\beta} (\Delta f + \Delta\beta + q_0)$ for some function $\beta : X \to \mathbb{R}$, $v(x, u) = u - \beta(x)$, $Q = q_1$.

We will call such morphisms "*canonical*". In the category $\mathcal{PE}$ the canonical representative in any class of morphisms is defined uniquely up to diffeomorphism of manifold Y, and in the category $\mathcal{APE}$ it is defined uniquely up to conformal diffeomorphism of Y.

Further we restrict ourselves by the investigation of the canonical maps for the first option, that is will look for such maps p from the given Riemann manifold X onto arbitrary Riemann manifolds Y, for which Laplacian on X is projected to some operator on Y (note that this canonical maps will be canonical for given X in the cases (2), (3) too).

Note that isomorphic autonomous equations B, factorized given A, are distinguished only by arbitrary transformations $v \to v'(y, v)$ and has the same projection $p : x \to y(x)$ up to conformal diffeomorphism of Y. Therefore to find such projection $p : X \to Y$ for canonical morphism is to find all autonomous morphisms from this equivalence class.

5 Factorizing of heat equation in $\mathbb{R}^3$

Let $\mathcal{DAPE}$ be full subcategory of $\mathcal{APE}$, whose objects are autonomous parabolic equations of divergent shape:

$$u_t = c(x, u)^{-1} \operatorname{div}(k(x, u) \nabla u) + q(x, u),$$

and morphisms are autonomous morphisms of the category $\mathcal{APE}$.

Theorem 6. *Let X be a connected region of $\mathbb{R}^3$ with Euclidean metric, Y be a manifold without boundary, A do not have form (2 or 3) from Theorem 5. Then p define canonical morphism of A in $\mathcal{DAPE}$ iff p is restriction on X of factorization $\mathbb{R}^3$ under some (may be discontinuous) group G of isometries.*

6 Factorizing with dimension decrease by 1

Theorem 7. *Let A do not have form (2 or 3) from Theorem 5, and (a) $p : X \to Y$ is a fibering; (b) X and Y are oriented; (c) X is an open domain in complete Riemann space $\tilde{X}$; (d) $\dim Y = \dim X - 1$. Then p define canonical morphism in $\mathcal{DAPE}$ iff the following conditions fulfilled:*

a) p is a superposition of maps $p_1 : X \to Y'$ and $p_0 : Y' \to Y$;

b) $p_1 : X \to Y'$ is a restriction on X of the projection $\tilde{X} \to \tilde{X} \big/ G_1$, where G_1 is some 1-parameter subgroup of group $\operatorname{Isom}\left(\tilde{X}\right)$ of all isometries of $\tilde{X}$;

c) $p_0 : Y' \to \tilde{Y}$ is isomeric covering (for the metric on Y', inherited from X);

d) for the vector field η generating group G_1, the function $\vartheta = \langle \eta, \eta \rangle$, defined on Y', is projectible on Y.

7 Factorizing with dimension decrease by 1: comparison with group analysis

As it was shown in the Section 5, when we were factorizing heat equation in $\mathbb{R}^3$ with Euclidean metric, the class of correspondent (3D) solutions of A coincides with a class of solutions of A, which are invariant under some (maybe discontinuous) group of isometries of $\mathbb{R}^3$.

But these results about coincidence of factorizing maps for the heat equation in $\mathbb{R}^3$ with Euclidean metric with factormaps by symmetry groups (that is isometries groups) are accidental.

At first, projection $p_0 : Y' \to Y$ from previous section is not necessarily generated by some group of transformation of Y'.

At second, let even $Y' = Y/G_0$, where G_0 is some discrete group of the isometries of Y'. The question is: could group G_0 be lifted to some group of the isometries of X, which preserves projection onto Y?

Let the group G_1 be fixed that satisfies conditions of Theorem 7. We consider differential-geometric connection χ on a fibering $p_1 : X \to Y'$ with the structural group G_1, which horizontal planes are orthogonal to G_1 orbits.

Theorem 8. *(necessary condition). If a discrete group G_0, which operates on Y' and satisfies conditions of the Theorem 7, could be lifted to the subgroup of* $\mathrm{Isom}(X)$, *then curvature form $d\chi$, projected on Y', would be invariant respectively G_0.*

Lemma 1. *χ may be decomposed on a sum $\chi = p_{1*}\chi' + dh$, where $\chi' \in T^*Y'$, h is a function from X to $\mathcal{H}$, $\mathcal{H}$ is fiber of p_1 (that is either $\mathbb{R}$, or circle $\mathbb{R} \bmod H$, where $H = \mathrm{const}$ is integral χ on a vertical cycle).*

Theorem 9. *(necessary and sufficient condition). A discrete group G_0, operating on Y' and satisfying conditions of the Theorem 7, could be lifted to the subgroup of* $\mathrm{Isom}\,(X)$, *iff $\forall g \in G_0$ the form $g\chi' - \chi'$ is:*

– exact, if the fiber of p_1 is simply connected;

– closed with periods, multiply H, if the fiber of p_1 is multiply connected.

Particularly, if $X = \mathbb{R}^n$, and G_1 is the rotations group, $\eta = \sum\limits_{i=1}^{m} a_i \partial_{\varphi_i}$, $m \geq 3$, or G_1 is the screw motions group, $\eta = \partial_z + \sum\limits_{i=1}^{m} a_i \partial_{\varphi_i}$, $m \geq 2$, then such groups G_0 exist, which does not lift on X.

8 Factorizing with dimension decrease

Let us equip X with connection generated by planes orthogonal to fibers.

Theorem 10. *Let (a) $p : X \to Y$ be a fibering; (b) $\dim Y < \dim X$. Then p defines canonical morphism to $\mathcal{DAPE}$ iff the following conditions fulfilled:*

1) the fibers of p are parallel;

2) the transformation of a fiber over an initial point to a fiber over a final point changes volumes proportionally when we translate along any curve on Y;

3) the holonomy group preserves volume on a fiber.

Moreover, p define canonical morphism to $\mathcal{APE}$ iff conditions 1)–2) fulfilled.

Example 1. $(\dim X = 4, \dim Y = 2)$. Let $X = \{(x, y, z, w)\}$ with the metric

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\alpha^2+\beta^2 & \alpha & \beta \\ 0 & \alpha & 1 & 0 \\ 0 & \beta & 0 & 1 \end{pmatrix}, \quad \alpha = xe^w, \quad \beta = xe^z,$$

$Y = \{(x, y)\}$ with the Euclidean metric, $p(x, y, z, w) = (x, y)$. Then map p and equation $v_t = v_{xx} + v_{yy}$ are factorization of the equation

$$u_t = u_{xx} + u_{yy} - 2\alpha u_{yz} - 2\beta u_{yw} + \left(1 + \alpha^2\right) u_{zz} + 2\alpha\beta u_{zw} + \left(1 + \beta^2\right) u_{ww} + (\alpha\beta)_w u_z + (\alpha\beta)_z u_w,$$

where $\alpha = xe^w$ and $\beta = xe^z$, by the map $p : (x, y, z, w) \to (x, y)$. (The same is true for the equations $v_t = a(v)\Delta v$ on Y and $u_t = a(u)\Delta u$ on X for arbitrary function a, but for simplicity we will write linear equations in examples.) However the only transformations X, under which both the last equation and all it's solutions projected by p are invariant, are $(x, y, z, w) \to (x, y, w, z)$ and identity. Moreover, another transformation with such properties does not exist even locally (i.e. it could not be defined in any small neighborhood on X), even if we replace the requirement "to keep the equation invariant" by the requirement "to be conformal".

Example 2. ($\dim X = 3$, $\dim Y = 2$). Let $\tilde{X} = \mathbb{R}^3 = \{(x, y, z)\}$ with the metric

$$g_{ij} = \begin{pmatrix} 1 + z^2 & z & -z \\ z & 2 & -1 \\ -z & -1 & 1 \end{pmatrix},$$

$\tilde{Y} = \{(x, y)\}$ with the Euclidean metric. Let us consider group H of isometries $\tilde{X}$, generated by the screw motion $(x, y, z) \to (x + 1, -y, -z)$ (H is projectible on $\tilde{Y}$), $X = \tilde{X}/H$, $Y = \tilde{Y}/H$, $p(x, y, z) = (x, y)$. Y is homeomorphic to the Mobius band without a boundary; X is homeomorphic to the torus without a boundary.

Then map p and equation $e^x v_t = (e^x v_x)_x + (e^x v_y)_y$, or $v_t = v_{xx} + v_{yy} + v_x$ on Y are factorizations of the equation

$$u_t = u_{xx} + u_{yy} + u_x + 2zu_{xz} + 2u_{yz} + \left(\left(2 + z^2\right) u_z\right)_z$$

on X. However the only transformation X, under which both the last equation and all projected by p it's solutions are invariant, is identity map. Moreover, there does not exist a non-identity conformal transformation X, under which all projected by p solutions of the last equation are invariant.

Example 3. ($\dim X = 3$, $\dim Y = 1$). Let $X = S^1 \times \mathbb{R}^2 = \{(x, y, z) : x \in \mathbb{R} \bmod 1, y, z \in \mathbb{R}\}$, equipped with the metric

$$g_{ij} = \begin{pmatrix} \alpha^2 + \beta^2 & \alpha & \beta \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix}, \quad \alpha = -e^z, \quad \beta = 2y,$$

$\tilde{Y} = S^1 = \{x \in \mathbb{R} \bmod 1\}$ equipped with the Euclidean metric, $p(x, y, z) = x$. Then map p and equation $a^{-1}(v)v_t = v_{xx}$ on Y are factorizations of the equation

$$u_t = u_{xx} + \left(1 + \alpha^2\right) u_{yy} + \left(1 + \beta^2\right) u_{zz} + 2\alpha\beta u_{yz} - 2\alpha u_{xy} - 2\beta u_{xz} + (\alpha\beta)_y u_z + (\alpha\beta)_z u_y,$$

on X. However the only transformation X, under which both the last equation and all projected by p it's solutions are invariant, is identity map.

Example 4. ($\dim X = 2$, $\dim Y = 1$). Let $X = \mathbb{R}^2/G$ with the Euclidean metric, when G is the group generated by the sliding symmetry respectively the straight line l. The orthogonal projection of X onto the mean circumference (image of the line l) define equation $v_t = v_{yy}$ on l, factorized the equation $u_t = u_{xx} + u_{yy}$ on X. However the only transformation X, under which both the last equation and all projected by p its solutions are invariant, is reflection with respect to l.

9 Factorization without dimension decrease

If $\dim X = \dim Y$, then $p : X \to \tilde{Y}$ projected Laplacian iff it is isometric projection up to some conformal transformation Y.

Example 5. Let manifold X be a plane without 3 points: $A\,(0,0)$, $B\,(1,0)$ and $C\,(0,2)$. Let's consider heat equation on X with metric $g_{ij} = \lambda^2\,(x)\,\delta_{ij}$:

$$\lambda^2\,(x)\,u_t = u_{11} + u_{22}, \tag{4}$$

where $\lambda\,(x) = \rho\,(x,A)\,\rho\,(x,B)\,\rho\,(x,C)$, ρ is the distance function (in usual plane metric). Let $Y = X$, and map $p : X \to Y$ is given by the formula $y = \frac{1}{4}x^4 - \frac{1+2i}{3}x^3 + ix^2$, where x, y are considered as points at a complex plane.

Because of $|y_x| = |x\,(x-1)\,(x-2i)| = \lambda\,(x)$, heat equation $u_t = u_{11} + u_{22}$ on Y, equipped by Euclidean metric $g_{ij} = \delta_{ij}$, is factorisation of the equation (4) on the manifold $\overset{\circ}{X}$, which is obtained by deleting of pre-images of images of zeroes of λ from X. However, there does not exist a non-identical transformation of $\overset{\circ}{X}$, under which all projected by p solutions of equation (4) are invariant. Moreover, there does not exist a non-identical transformation of any manifold X', under which an equation (4) is invariant, if X' is obtained by deleting an arbitrary discrete set of points from X.

Example 6. Let us consider an equation on $X = \mathbb{R}^2$:

$$u_t = \left(1 + |x|^2\right)^2 (u_{11} + u_{22})\,. \tag{5}$$

Let g be the transformation of $\mathbb{R}^2/\{0\}$ that maps $x \in X$ to the point, obtained from x by inversion under the unit circle with a center in an origin and consequent reflection under this center. Equation (5) is invariant with respect to g, but g is not defined at origin. However the map $p : X \to Y = \mathbb{P}^2$ onto the projective plane, which past together points x and gx at $x \neq 0$, is defined on all X and gives smooth projection. Then inducing on Y heat equation is factorization of original equation on X.

Acknowledgments

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[1] Prokhorova M., Modeling of solutions of heat conduction equation and Stefan problem with dimension decrease, *Russian Acad. Sci. Docl. Math.*, 1998, V.361, N 6, 740–742.

[2] Prokhorova M., Modeling of heat conduction and crystal growth with dimension decrease, IMM UrBr Russian Acad. Sci., Sverdlovsk, 1996, Dep. VINITI N2314-V96.

[3] Prokhorova M., Modeling of the heat conduction equation and Stefan problem, IMM UrBr Russian Acad. Sci., Ekaterinburg, 2000, Dep. VINITI N347-V00.

[4] Prokhorova M., Modeling of nonlinear heat equation with boundary conditions on free interface, *Electronic Journal Differential Equations and Control Processes*, to appear.

[5] Prokhorova M., Dimension decrease for diffusion equation with boundary conditions on the free boundary, in Proceedings of II International Conference "Symmetry and differential equations" (21–25 August, 2000, Krasnoyarsk, Russia), Editors V.K. Andreev and Yu.V. Shan'ko, 2000, 177–180.

[6] Elkin V., Reduction of nonlinear controlled systems. Differential-geometry approach, Moscow, Nauka, 1997.

[7] Ovsyannikov L., Group analysis of differential equations, Moscow, Nauka, 1978.