Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 2, 508–519

Nonlinear Supersymmetry

Mikhail PLYUSHCHAY and Sergey KLISHEVICH

Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile & *Institute for High Energy Physics, Protvino, Russia*
E-mail: *mplyushc@lauca.usach.cl*

After a short discussion of the intimate relation between the generalized statistics and supersymmetry, we review the recent results on the nonlinear supersymmetry obtained in the context of the quantum anomaly problem and of the universal algebraic construction associated with the holomorphic nonlinear supersymmetry.

Introduction

Nonlinear supersymmetry is a natural generalization of the usual linear supersymmetry [1, 2]. It is realized variously in such different systems as the parabosonic [3] and parafermionic [4] oscillator models, and the P,T-invariant models of planar fermions [5] and Chern–Simons fields [6]. It is also the symmetry of the fermion-monopole system [7, 8]. The algebraic structure of the nonlinear supersymmetry resembles the structure of the finite W-algebras [9] for which the commutator of generating elements is proportional to a finite order polynomial in them. In the simplest case the nonlinear supersymmetry is characterized by the superalgebra of the form

$$[Q^{\pm}, H] = 0, \qquad (Q^{\pm})^2 = 0, \qquad \{Q^+, Q^-\} = P_n(H), \tag{1}$$

where $P_n(\cdot)$ is a polynomial of the n-th degree. The nonlinear supersymmetry with such a superalgebra was investigated for the first time by Andrianov, Ioffe and Spiridonov [10].

The pseudoclassical construction underlies the supersymmetric quantum mechanics of Witten [1, 2] corresponding to the linear ($n = 1$) case of the superalgebra (1). Though the nonlinear supersymmetry can also be realized classically, there is an essential difference from the linear case: the attempt to quantize the nonlinear supersymmetry immediately faces the problem of the quantum anomaly [3, 11]. It was shown [12] that the universal algebraic structure with associated "integrability conditions" in the form of the Dolan–Grady relations [13] underlies the so called holomorphic nonlinear supersymmetry [11]. This structure allows one to find a broad class of anomaly-free quantum mechanical systems related to the exactly and quasi-exactly solvable systems [14, 15, 16, 17, 18, 19], and gives a nontrivial centrally extended generalization of the superalgebra (1) [12].

In this talk, after a short discussion of the intimate relation between the generalized statistics and supersymmetry [3], we shall review the recent results on the nonlinear supersymmetry obtained in the context of the quantum anomaly problem and of the universal algebraic construction associated with the holomorphic nonlinear supersymmetry [11, 12].

Nonlinear supersymmetry in purely parabosonic systems

Some time ago it was shown that the linear supersymmetry can be realized without fermions [20, 21, 22]. The nonlinear supersymmetry admits a similar realization revealing the close relationship between the generalized statistics and supersymmetry [3]. The relationship can be observed in the following way. Let us consider a single-mode paraboson system defined by the relations

$$[\{a^+, a^-\}, a^\pm] = \pm a^\pm, \qquad a^- a^+|0\rangle = p|0\rangle, \qquad a^-|0\rangle = 0,$$

where $p \in \mathbb{N}$ is the order of a paraboson [23]. Then the direct calculation shows that the pure parabosonic system of the even order $p = 2(k+1)$, $k \in \mathbb{Z}_+$, with the Hamiltonian of the simplest quadratic form $H = a^+a^-$ reveals a spectrum typical for the nonlinear supersymmetry: all its states are paired in doublets except the $k+1$ singlet states $|2l\rangle \propto (a^+)^{2l}|0\rangle$, $l = 0, \ldots, k$. In correspondence with this property, the system has two integrals of motion

$$Q^+ = (a^+)^{2k+1} \sin^2 \frac{\pi}{4}\{a^+, a^-\}, \qquad Q^- = (a^-)^{2k+1} \cos^2 \frac{\pi}{4}\{a^+, a^-\}, \tag{2}$$

which together with the Hamiltonian form the nonlinear superalgebra (1) of the order $n = 2k+1$ with $P_{2k+1}(H) = H \cdot \prod_{m=1}^{k} \left(H^2 - 4m^2\right)$. This simplest system reflects the peculiar feature of the parabosonic realization of supersymmetry: the supercharges are realized in the form of the infinite series in $a^\pm$, and the role of the grading operator is played here by $R = (-1)^N = \cos \pi N$, where $N = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}p$ is the parabosonic number operator.

It is known that the deformed Heisenberg algebra with reflection

$$[a^-, a^+] = 1 + \nu R, \qquad \{R, a^\pm\} = 0, \qquad R^2 = 1, \tag{3}$$

underlies the parabosons [24, 25]. This algebra possesses unitary infinite-dimensional representations for $\nu > -1$, and at the integer values of the deformation parameter, $\nu = p - 1$, $p \in \mathbb{N}$, is directly related to parabosons of order p [23, 24, 25]. On the other hand, at $\nu = -(2p+1)$ the R-deformed Heisenberg algebra has finite-dimensional representations corresponding to the deformed parafermions of order $2p$ [25]. In the coordinate representation the operator R is the parity operator and the operators $a^\pm$ can be realized in the form $a^\pm = \frac{1}{\sqrt{2}}(x \mp iD_\nu)$ with $D_\nu = -i\left(\frac{d}{dx} - \frac{\nu}{2x}R\right)$. In the context of the Calogero-like models, the operator D_ν is known as the Yang–Dunkl operator [26, 27], where R is treated as the exchange operator. In the coordinate representation the Hamiltonian $H = a^+a^-$ and supercharges (2) read as [3]

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + \frac{\nu^2}{4x^2} - 1 + \nu\left(\frac{1}{2x^2} - 1\right)R\right), \tag{4}$$

$$Q^+ = (Q^-)^\dagger = \frac{1}{2^{3\left(k+\frac{1}{2}\right)}}\left(\left(-\frac{d}{dx} + x + \frac{\nu}{2x}\right)(1-R)\right)^{2k+1} \tag{5}$$

with $\nu = 2k+1$. The system given by the Hamiltonian (4) can be treated as a 2-particle Calogero-like model with exchange interaction, where x has a sense of a relative coordinate and R has to be understood as the exchange operator [28, 29]. Therefore, at odd values of the parameter ν, the class of Calogero-like systems (4) possesses a hidden supersymmetry, which at $\nu = 1$ is the linear ($n = 1$) supersymmetry in the unbroken phase, whereas at $\nu = 2k+1$, the supersymmetry is characterized by the supercharges being differential operators of order $2k+1$ satisfying the nonlinear superalgebra (1). Recently the realization of the nonlinear supersymmetry was extended within the standard approach with fermion degrees of freedom to the case of multi-particle Calogero and related models [30].

Classical supersymmetry

Let us turn now to the classical formulation of the supersymmetry (1). For the purpose, we consider a non-relativistic particle in one dimension described by the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - V(x) - L(x)N + i\theta^+\dot{\theta}^-, \tag{6}$$

where $\theta^{\pm}$ are the Grassman variables, $(\theta^+)^* = \theta^-$, $N = \theta^+\theta^-$, and $V(x)$ and $L(x)$ are two real functions. The nontrivial Poisson–Dirac brackets for the system are $\{x, p\}_* = 1$ and $\{\theta^+, \theta^-\}_* = -i$, and the Hamiltonian is

$$H = \frac{1}{2}p^2 + V(x) + L(x)N. \tag{7}$$

The Hamiltonian H and the nilpotent quantity N are the even integrals of motion for any choice of the functions $V(x)$ and $L(x)$, and one can put the question: when the system (6) has also local in time odd integrals of motion of the form $Q^{\pm} = B^{\mp}(x, p)\theta^{\pm}$, where $(B^+)^* = B^-$? It is obvious that such odd integrals can exist only for a special choice of the functions $V(x)$ and $L(x)$. Restricting ourselves to the physically interesting class of the systems given by the potential $V(x)$ bounded from below, we can generally represent it in terms of a superpotential $W(x)$: $V(x) = \frac{1}{2}W^2(x) + v$, $v \in \mathbb{R}$. Then all the supersymmetric systems are separated into the three classes defined by the behaviour of the superpotential and the results can be summarized as follows [11].

i) When the physical domain given by $z = W(x)+ip$ includes the origin $z = 0$ ($a < W(x) < b$, $a < 0$, $b > 0$), the corresponding supersymmetric system is characterized by the Hamiltonian and the supercharges of the form

$$\begin{aligned} &H = \frac{p^2}{2} + \frac{1}{2}W^2(x) + v + W'(x)\left[n + W(x)M\left(W^2(x)\right)\right]N, \\ &Q^+ = (Q^-)^* = z^n e^{i\int_0^p M\left(p^2-y^2+W^2(x)\right)dy}\theta^+, \qquad n \in \mathbb{Z}, \end{aligned} \tag{8}$$

where $M(W^2)$ is an arbitrary regular function, $|M(0)| < \infty$. The appearance of the integer parameter illustrates in this case the known classical "quantization phenomenon" [31]. The appropriate canonical transformation reduces the system with these Hamiltonian and supercharges to the form of the supersymmetric system with the holomorphic supercharges [11]:

$$H = \frac{1}{2}p^2 + \frac{1}{2}W^2(x) + v + nW'(x)\theta^+\theta^-, \qquad Q^+ = (Q^-)^* = z^n\theta^+, \qquad n \in \mathbb{Z}_+. \tag{9}$$

The integrals (9) obey the classical nonlinear superalgebra:

$$\left\{Q^-, Q^+\right\}_* = -iH^n, \qquad \{Q^{\pm}, H\}_* = 0. \tag{10}$$

The presence of the integer number n in the Hamiltonian means that the instant frequencies of the oscillator-like odd, $\theta^{\pm}$, and even, z, $\bar{z}$, variables are commensurable. Only in this case the regular odd integrals of motion can be constructed, and the factor z^n in the supercharges $Q^{\pm}$ corresponds to the n-fold conformal mapping of the complex plane (or the strip $a < \mathrm{Re}\, z < b$) on itself (or on the corresponding region in $\mathbb{C}$).

ii) The physical domain is defined by the condition $\mathrm{Re}\, z \geq 0$ (or $\mathrm{Re}\, z \leq 0$) and also includes the origin of the complex plane. But unlike the previous case, there are no closed contours around $z = 0$. In this case the most general form of the Hamiltonian and the supercharge is

$$H = \frac{p^2}{2} + \frac{1}{2}W^2(x) + v + W'(x)\left[\alpha + R(W(x))\right]N, \qquad Q^+ = z^{\alpha} e^{i\int_{\varphi_0}^{\varphi} R(\rho\cos\lambda)\, d\lambda}\theta^+, \tag{11}$$

where $\alpha \in \mathbb{R}$, and we assume that the function $R(W)$ is analytical at $W = 0$ and $R(0) = 0$. By the canonical transformation [11], the Hamiltonian and the supercharges can be reduced to

$$H = \frac{1}{2}p^2 + \frac{1}{2}W^2(x) + v + \alpha W'(x)\theta^+\theta^-, \qquad Q^+ = (Q^-)^* = z^{\alpha}\theta^+, \qquad \alpha \in \mathbb{R}_+.$$

iii) The physical domain is defined by the condition $\mathsf{Re}\, z > 0$ (or $\mathsf{Re}\, z < 0$), i.e. the origin of the complex plane is not included. Though in this case the Hamiltonian and the supercharges have a general form

$$H = \frac{p^2}{2} + \frac{1}{2}W^2 + v + W'(x)\phi(W(x))N, \qquad Q^+ = (Q^-)^* = f(H)e^{i\int_{\varphi_0}^{\varphi}\phi(\rho\cos\lambda)\,d\lambda}\theta^+, \quad (12)$$

where ϕ is some function, the appropriate canonical transformation reduces it to [11]

$$H = \frac{1}{2}p^2 + \frac{1}{2}W^2(x) + v, \qquad Q^{\pm} = \theta^{\pm}.$$

This means that *classically* the supersymmetry of any system with bounded non-vanishing superpotential has a "fictive" nature.

In what follows we will refer to the nonlinear supersymmetry generated by the holomorphic supercharges with the Poisson bracket (anticommutator) being proportional to the n-th order polynomial in the Hamiltonian as to the *holomorphic n-supersymmetry*.

Though the form of the Hamiltonians (8), (11), and (12) can be simplified by applying in every case the appropriate canonical transformation reducing the associated supercharges to the holomorphic or antiholomorphic form, the quantization breaks the equivalence between the corresponding classical systems (even in the linear case $n = 1$) [11]. Moreover, alternative classical forms for the Hamiltonians and associated supercharges are important because of the quantum anomaly problem to be discussed below. Having in mind the importance of alternative classical formulations of the nonlinear supersymmetry from the viewpoint of subsequent quantization, one can look for the classical formulation characterized by the supercharges of the n-th degree polynomial form in p [11]. The problem of finding such a formulation can be solved completely in the simplest case $n = 2$, for which the supercharges are given by

$$Q^{\pm} = \frac{1}{2}\left[(\pm ip + W(x))^2 + \frac{c}{W^2(x)}\right]\theta^{\pm}, \qquad c \in \mathbb{R}, \quad (13)$$

while the Hamiltonian is

$$H = \frac{1}{2}\left[p^2 + W^2(x) - \frac{c}{W^2(x)}\right] + 2W'(x)N + v. \quad (14)$$

Note that the Hamiltonian (14) has the Calogero-like form: at $W(x) = x$ its projection to the unit of Grassman algebra takes the form of the Hamiltonian of the two-particle Calogero system. Depending on the value of the parameter c, classically the Calogero-like $n = 2$ supersymmetric system (14) is symplectomorphic to the holomorphic n-supersymmetry with $n = 0$ ($c > 0$), $n = 1$ ($c < 0$) or $n = 2$ ($c = 0$) [11].

Quantum anomaly and quasi-exactly solvable (QES) systems

According to the results on the supersymmetry in pure parabosonic systems, a priori one cannot exclude the situation characterized by the supercharges to be the nonlocal operators represented in the form of some infinite series in the operator $\frac{d}{dx}$. Since such nonlocal supercharges have to anticommute for some function of the Hamiltonian being a usual local differential operator of the second order, they have to possess a very peculiar structure. We restrict ourselves by the discussion of the supersymmetric systems with the supercharges being the differential operators of order n. Classically this corresponds to the system (9) with the holomorphic supercharges or to the system (14).

In the simplest case of the superoscillator possessing the nonlinear n-supersymmetry and characterized by the holomorphic supercharges of the form (9) with $W(x) = x$, the form of the classical superalgebra $\{Q_n^+, Q_n^-\} = H^n$ is changed for $\{Q_n^+, Q_n^-\} = H(H-\hbar)(H-2\hbar)\cdots(H-\hbar(n-1))$. Moreover, it was pointed out in [3] that for $W(x) \neq ax+b$ a global quantum anomaly arises in a generic case: the direct quantum analogues of the superoscillators and the Hamiltonian do not commute, $[Q_n^\pm, H_n] \neq 0$. Therefore, we arrive at the problem of looking for the classes of superpotentials and corresponding quantization prescriptions leading to the anomaly-free quantum n-supersymmetric systems.

Let us begin with the quantum supercharges in the holomorphic form corresponding to the classical n-supersymmetry,

$$Q^\pm = (A^\mp)^n \theta^\pm, \qquad \text{where} \qquad A^\pm = \mp\hbar\frac{d}{dx} + W(x). \tag{15}$$

Choosing the quantum Hamiltonian in the form (7), from the requirement of conservation of the supercharges, $[Q^\pm,\, H] = 0$, one arrives at the supersymmetric quantum system given by the Hamiltonian [11]

$$H = \frac{1}{2}\left(-\hbar^2\frac{d^2}{dx^2} + W^2(x) + 2v + n\hbar\sigma_3 W'\right), \qquad W(x) = w_2x^2 + w_1x + w_0. \tag{16}$$

For any other form of the superpotential, the nilpotent operators (15) are not conserved. The family of supersymmetric systems (16) is reduced to the superoscillator at $w_2 = 0$ with the associated exact n-supersymmetry [3]. For $w_2 \neq 0$, the n-supersymmetry is realized always in the spontaneously broken phase since in this case the supercharges (15) have no zero modes (normalized eigenfunctions of zero eigenvalue).

One can also look for the supercharges in the form of polynomial of order n in the oscillator-like operators $A^\pm$ defined in (15):

$$Q^\pm = \left(A^\mp\right)^n \theta^\pm + \sum_{k=0}^{n-1} q_{n-k}\left(A^\mp\right)^k \theta^\pm, \tag{17}$$

where q_k are real parameters which have to be fixed. As in the case of the supercharges (15), the requirement of conservation of (17) results in the Hamiltonian (16) but with the exponential superpotential [11]:

$$W(x) = w_+e^{\omega x} + w_-e^{-\omega x} + w_0, \qquad \omega^2 = -\frac{24q}{n\left(n^2-1\right)}, \tag{18}$$

where all the parameters $w_{\pm,0}$ are real, while the parameter ω is real or pure imaginary depending on the sign of the real parameter q, and for the sake of simplicity we put $\hbar = 1$. In the limit $\omega \to 0$ this superpotential is reduced to the quadratic form (16) via the appropriate rescaling of the parameters $w_{\pm,0}$.

The family of n-supersymmetric systems given by the superpotential (18) is tightly related to the so called quasi-exactly solvable problems [15, 16, 17, 18, 19]. Indeed, both of the Hamiltonians constituting the supersymmetric Hamiltonian of the form (16) with the exponential superpotential belong to the $sl(2,\mathbb{R})$ scheme of one-dimensional QES systems [15, 16, 17]. Besides, the QES family given by the superpotential (18) is related to the exactly solvable Morse potential for some choice of the parameters [11].

The $n = 2$ non-holomorphic supersymmetry corresponding to equations (13), (14) occupies an especial position. Like the linear supersymmetry, it admits the anomaly-free quantum formulation in terms of an arbitrary superpotential. Indeed, the quantization of the supersymmetric system (14) leads to [11]

$$H = \frac{1}{2}\left[-\hbar^2\frac{d^2}{dx^2} + W^2 - \frac{c}{W^2} + v + 2\hbar W'\sigma_3 + \Delta(W)\right], \tag{19}$$

$$Q^+ = (Q^-)^\dagger = \frac{1}{2}\left[\left(\hbar\frac{d}{dx}+W\right)^2 + \frac{c}{W^2} - \Delta(W)\right]\theta^+, \tag{20}$$

$$\Delta = \frac{\hbar^2}{4W^2}\left(2W''W - W'^2\right). \tag{21}$$

Looking at the quantum Hamiltonian (19) and supercharges (20), we see that the presence of the quadratic in $\hbar^2$ term (21) in the operators H and Q^+ is crucial for preserving the supersymmetry at the quantum level. Therefore, one can say that the quantum correction (21) cures the problem of the quantum anomaly since without it the operators $Q^\pm$ would not be the integrals of motion. The supercharges (20) satisfy the relation $\{Q^+, Q^-\} = (H-v)^2 + c$, and the structure of the lowest bounded states in the cases $c>0$, $c<0$ and $c=0$ for $v=0$ is reflected in the table and on the figure (for the details see Ref. [11]).

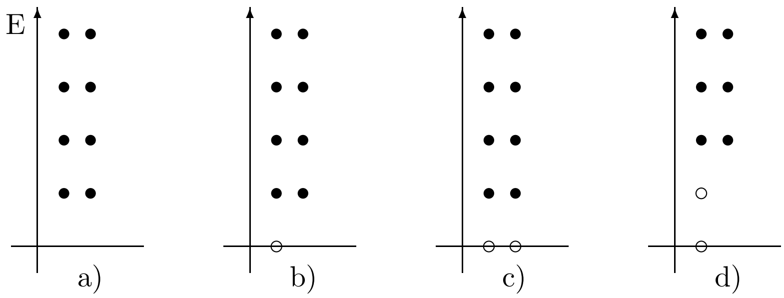

Figure. The four types of the spectra for the $n=2$ supersymmetry for bounded states.

Table. The structure of the lowest states for the $n=2$ supersymmetry.

		$c>0$	$c=0$	$c<0$
a)	Completely broken phase, there are no singlet states	+	+	+
b)	One singlet state in either bosonic or fermionic sector		+	+
c)	Two singlet states with $E=0$, one is in fermionic sector, another is in bosonic sector		+	+
d)	Two singlet states in one of two sectors			+

From this structure one can see, in particular, that the quantum theory "remembers" its classical origin: the case $c>0$ corresponding classically to the holomorphic $n=0$ supersymmetry gives the systems in the completely broken phase for any superpotential providing the existence of bounded states.

In conclusion of the discussion of the nonlinear supersymmetry for the 1D quantum systems, we note that for the first time the close relationship between the nonlinear supersymmetry and QES systems was observed in Ref. [11]. Recently, it was demonstrated [32] that the so called type A $\mathcal{N}$-fold supersymmetry [33] being a generalization of the one-dimensional holomorphic supersymmetry is, in essence, equivalent to the one-dimensional QES systems associated with the $sl(2,\mathbb{R})$ algebra.

Nonlinear supersymmetry on plane in magnetic field

The nonlinear holomorphic supersymmetry we have discussed has a universal nature due to the algebraic construction underlying it and revealed in Ref. [12]. This universality allows us, in particular, to generalize the above analysis to the case of the two-dimensional systems.

The classical Hamiltonian of a charged spin-1/2 particle ($-e = m = 1$) with gyromagnetic ratio g moving on a plane and subjected to a magnetic field $B(\boldsymbol{x})$ is given by

$$H = \frac{1}{2}\boldsymbol{\mathcal{P}}^2 + gB(\boldsymbol{x})\theta^+\theta^-, \tag{22}$$

where $\boldsymbol{\mathcal{P}} = \boldsymbol{p} + \boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{A}(\boldsymbol{x})$ is a 2D gauge potential, $B(\boldsymbol{x}) = \partial_1 A_2 - \partial_2 A_1$. The variables x_i, p_i, $i = 1, 2$, and complex Grassman variables $\theta^\pm$, $(\theta^+)^* = \theta^-$, are canonically conjugate with respect to the Poisson–Dirac brackets, $\{x_i, p_j\}_* = \delta_{ij}$, $\{\theta^-, \theta^+\}_* = -i$. For even values of the gyromagnetic ratio $g = 2n$, $n \in \mathbb{N}$, the system (22) is endowed with the nonlinear n-supersymmetry. In this case the Hamiltonian (22) takes the form

$$H_n = \frac{1}{2}Z^+Z^- + \frac{i}{2}n\left\{Z^-, Z^+\right\}_*\theta^+\theta^-, \qquad Z^\pm = \mathcal{P}_2 \mp i\mathcal{P}_1, \tag{23}$$

which admits the existence of the odd integrals of motion

$$Q^\pm = 2^{-\frac{n}{2}}\left(Z^\mp\right)^n\theta^\pm \tag{24}$$

generating the nonlinear n-superalgebra (10). The n-superalgebra does not depend on the explicit form of the even complex conjugate variables $Z^\pm$. Therefore, in principle, $Z^\pm$ can be arbitrary functions of the bosonic dynamical variables of the system.

The nilpotent quantity $N = \theta^+\theta^-$ is, as in the 1D case, the even integral of motion. When the gauge potential $\boldsymbol{A}(\boldsymbol{x})$ is a 2D vector, the system (23) possesses the additional even integral of motion $L = \varepsilon_{ij}x_i p_j$. The integrals N and L generate the $U(1)$ rotations of the odd, $\theta^\pm$, and even, $Z^\pm$, variables, respectively. Their linear combination $J = L + nN$ is in involution with the supercharges, $\{J, Q^\pm\}_*$=0, and plays the role of the central charge of the classical n-superalgebra. As we shall see, at the quantum level the form of the nonlinear n-superalgebra (10) is modified generically by the appearance of the nontrivial central charge in the anticommutator of the supercharges.

A spin-1/2 particle moving on a plane in a constant magnetic field represents the simplest case of a quantum 2D system admitting the nonlinear supersymmetry. Such a system corresponds to the n-supersymmetric quantum oscillator [12]. As in the case of the one-dimensional theory, the attempt to generalize the n-supersymmetry of the system to the case of the magnetic field of general form faces the problem of quantum anomaly. The generalization is nevertheless possible for the magnetic field of special form [12].

To analyse the nonlinear n-supersymmetry for arbitrary $n \in \mathbb{N}$, it is convenient to introduce the complex oscillator-like operators

$$Z = \partial + W(z, \bar{z}), \qquad \bar{Z} = -\bar{\partial} + \bar{W}(z, \bar{z}), \tag{25}$$

where the complex superpotential is defined by $\mathsf{Re}\, W = A_2(\boldsymbol{x})$, $\mathsf{Im}\, W = A_1(\boldsymbol{x})$, and the notations $z = \frac{1}{2}(x_1 + ix_2)$, $\bar{z} = \frac{1}{2}(x_1 - ix_2)$, $\partial = \partial_z$, $\bar{\partial} = \partial_{\bar{z}}$ are introduced.

The magnetic field is defined by the relation $[Z, \bar{Z}] = 2B(z, \bar{z})$. The n-supersymmetric Hamiltonian has the form

$$H_n = \frac{1}{4}\left\{\bar{Z}, Z\right\} + \frac{n}{4}\left[Z, \bar{Z}\right]\sigma_3. \tag{26}$$

For $n = 1$ we reproduce the usual supersymmetric Hamiltonian. Unlike the linear supersymmetry, the nonlinear holomorphic supersymmetry exists only when the operators (25) obey the relations

$$\left[Z, \left[Z, \left[Z, \bar{Z}\right]\right]\right] = \omega^2\left[Z, \bar{Z}\right], \qquad \left[\bar{Z}, \left[\bar{Z}, \left[Z, \bar{Z}\right]\right]\right] = \bar{\omega}^2\left[Z, \bar{Z}\right]. \tag{27}$$

Here $\omega \in \mathbb{C}$ and $\bar{\omega} = \omega^*$. Using equation (27), one can prove algebraically by the mathematical induction that for the system (26) the odd operators defined by the recurrent relations

$$Q^+_{n+2} = \frac{1}{2}\left(Z^2 - \left(\frac{n+1}{2}\right)^2 \omega^2\right) Q^+_n, \qquad Q^+_0 = \theta^+, \qquad Q^+_1 = 2^{-\frac{1}{2}} Z\theta^+, \tag{28}$$

are the integrals of motions, i.e. they are supercharges. One can make sure [11] that in the 1D case these operators generate the nonlinear supersymmetry with the polynomial superalgebra (1).

In the representation (25) the conditions (27) acquire the form of the differential equations for magnetic field:

$$\left(\partial^2 - \omega^2\right) B(z, \bar{z}) = 0, \qquad \left(\bar{\partial}^2 - \bar{\omega}^2\right) B(z, \bar{z}) = 0. \tag{29}$$

The general solution to these equations is

$$B(z, \bar{z}) = w_+ e^{\omega z + \bar{\omega}\bar{z}} + w_- e^{-(\omega z + \bar{\omega}\bar{z})} + w e^{\omega z - \bar{\omega}\bar{z}} + \bar{w} e^{-(\omega z - \bar{\omega}\bar{z})}, \tag{30}$$

where $w_\pm \in \mathbb{R}$, $w \in \mathbb{C}$, $\bar{w} = w^*$. On the other hand, for $\omega = 0$ the solution to equation (29) is the polynomial,

$$B(\boldsymbol{x}) = c\left((x_1 - x_{10})^2 + (x_2 - x_{20})^2\right) + c_0, \tag{31}$$

with c, c_0, x_{10}, x_{20} being some real constants. Though the latter solution can be obtained formally from (30) in the limit $\omega \to 0$ by rescaling appropriately the parameters $w_\pm$, w, the corresponding limit procedure is singular and the cases (30) and (31) have to be treated separately.

Since the conservation of the supercharges is proved algebraically, the operators Z, $\bar{Z}$ can have any nature (the action of Z, $\bar{Z}$ is supposed to be associative). For example, they can have a matrix structure. With this observation the nonlinear supersymmetry can be applied to the case of matrix Hamiltonians [34, 35, 36].

Thus, the introduction of the operators Z, $\bar{Z}$ allows us to reduce the two-dimensional holomorphic n-supersymmetry to the pure algebraic construction. It is worth noting that in the literature the algebraic relations (27) are known as Dolan–Grady relations. The relations of such a form appeared for the first time in the context of integrable models [13].

The essential difference of the n-supersymmetric 2D system (26) from the corresponding 1D supersymmetric system is the appearance of the central charge

$$J_n = -\frac{1}{4}\left(\omega^2 \bar{Z}^2 + \bar{\omega}^2 Z^2\right) + \partial B \bar{Z} + \bar{\partial} B Z - B^2 + \frac{n}{2}\bar{\partial}\partial B \sigma_3, \tag{32}$$

$[H_n, J_n] = [Q^\pm_n, J_n] = 0$. The anticommutator of the supercharges contains it for any $n > 1$. For example, the $n = 2$ nonlinear superalgebra is

$$\left\{Q^-_2, Q^+_2\right\} = H^2_2 + \frac{1}{4} J_2 + \frac{|\omega|^4}{64}. \tag{33}$$

The systems (26) with the magnetic field (30) of the pure hyperbolic ($w = 0$) or pure trigonometric ($w_\pm = 0$) form can be reduced to the one-dimensional problems with the nonlinear holomorphic supersymmetry [12].

Let us turn now to the polynomial magnetic field (31). One can see that this case reveals a nontrivial relation of the holomorphic n-supersymmetry of the 2D system to the nonholomorphic 1D $\mathcal{N}$-fold supersymmetry of Aoyama et al [33].

In the system (26) with the polynomial magnetic field (31) the central charge has the form

$$J_n = \frac{1}{4c}\left(\partial B(z, \bar{z})\bar{Z} + \bar{\partial} B(z, \bar{z}) Z - B^2(z, \bar{z}) + \frac{n}{2}\bar{\partial}\partial B(z, \bar{z})\sigma_3\right). \tag{34}$$

It can be obtained from the operator (32) in the limit $\omega \to 0$ via the same rescaling of the parameters of the exponential magnetic field which transforms (30) into (31). The essential feature of this integral is its linearity in derivatives.

The polynomial magnetic field (31) is invariant under rotations about the point (x_{10}, x_{20}). Therefore, one can expect that the operator (34) should be related to a generator of the axial symmetry. To use the benefit of this symmetry, one can pass over to the polar coordinate system with the origin at the point (x_{10}, x_{20}). Then the magnetic field is radial, $B(r) = cr^2 + c_0$. The supercharges have the simple structure: $Q_n^+ = 2^{-\frac{n}{2}} Z^n \theta^+ = (Q_n^-)^\dagger$. As in the case $\omega \neq 0$, the anticommutator of the supercharges is a polynomial of the n-th degree in H_n, $\{Q_n^-, Q_n^+\} = H_n^n + P(H_n, J_n)$, where $P(H_n, J_n)$ denotes a polynomial of the $(n-1)$-th degree. For example, for $n = 2$ one has

$$\{Q_2^-, Q_2^+\} = H_2^2 + cJ_2.$$

For the radial magnetic field it is convenient to use the gauge

$$A_\varphi = \frac{1}{4}cr^4 + \frac{1}{2}c_0 r^2, \qquad A_r = 0. \tag{35}$$

In this gauge the Hamiltonian (26) reads

$$H_n = -\frac{1}{2}\left(\partial_r^2 + r^{-1}\partial_r - r^{-2}\left(A_\varphi^2(r) - 2iA_\varphi(r)\partial_\varphi - \partial_\varphi^2\right)\right) + \frac{n}{2}B(r)\sigma_3, \tag{36}$$

while the central charge (34) takes the form $J_n = -i\partial_\varphi - \frac{c_0^2}{4c} + \frac{n}{2}\sigma_3$. Thus, the integral J_n is associated with the axial symmetry of the system under consideration. The simultaneous eigenstates of the operators H_n and J_n have the structure

$$\Psi_m(r,\varphi) = \begin{pmatrix} e^{i(m-n)\varphi}\chi_m(r) \\ e^{im\varphi}\psi_m(r) \end{pmatrix}. \tag{37}$$

Since the angular variable φ is cyclic, the 2D Hamiltonian (36) can be reduced to the 1D Hamiltonian. The kinetic term of the Hamiltonian (36) is Hermitian with respect to the measure $d\mu = rdrd\varphi$. In order to obtain a one-dimensional system with the usual scalar product defined by the measure $d\mu = dr$, one has to perform the similarity transformation $H_n \to UH_nU^{-1}$, $\Psi \to U\Psi$ with $U = \sqrt{r}$. Since the system obtained after such a transformation is originated from the two-dimensional system, one should keep in mind that the variable r belongs to the half-line, $r \in [0, \infty)$. After the transformation, the reduced one-dimensional Hamiltonian acting on the lower (Bose) component of the state (37) reads as

$$\mathcal{H}_n^{(2)} = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{c^2}{32}r^6 + \frac{c_0 c}{8}r^4 + \frac{1}{8}\left(c_0^2 - 2c(2n-m)\right)r^2 + \frac{m^2 - \frac{1}{4}}{2r^2} - \frac{1}{2}(n-m)c_0. \tag{38}$$

This Hamiltonian gives the well-known family of the quasi-exactly solvable systems [15, 16, 34, 19]. The superpartner $\mathcal{H}_n^{(1)}$ acting on the upper (Fermi) component of the state (37) can be obtained from $\mathcal{H}_n^{(2)}$ by the substitution $n \to -n$, $m \to m - n$.

The reduced supercharges have the form

$$\mathcal{Q}_n^+ = 2^{-\frac{n}{2}}\mathcal{Z}_n\theta^+ = (\mathcal{Q}_n^-)^\dagger, \qquad \mathcal{Z}_n = \left(A - \frac{n-1}{r}\right)\left(A - \frac{n-2}{r}\right)\cdots A,$$

where $A = \frac{d}{dr} + W(r)$ and the superpotential is

$$W(r) = \frac{1}{4}cr^3 + \frac{1}{2}c_0 r + \frac{m - \frac{1}{2}}{r}.$$

The operators $\mathcal{Q}_n^{\pm}$, $\mathcal{H}_n^{(i)}$, $i = 1, 2$, generate the non-holomorphic type A $\mathcal{N}$-fold supersymmetry discussed in [33]. The supersymmetry is exact for $c > 0$ ($c < 0$) and corresponding zero modes of the supercharge $\mathcal{Q}_n^+$ ($\mathcal{Q}_n^-$) can be found. The relation of the $\mathcal{N}$-fold supersymmetry with the cubic superpotential to the family of QES system (38) with the sextic potential was also noted in Ref. [37].

Resume

To conclude, let us summarize the main results of our consideration of the nonlinear supersymmetry.

- Generalized statistics and supersymmetry are intimately related.
- Linear supersymmetry at the classical level is a particular case of a classical supersymmetry characterized by the Poisson algebra being nonlinear in Hamiltonian.
- Any classical 1D supersymmetric system is symplectomorphic to the supersymmetric system of the canonical form characterized by the holomorphic supercharges. There are three different classes of the classical canonical supersymmetric systems defined by the behaviour of the superpotential.
- The anomaly-free quantization of the classical 1D holomorphic n-supersymmetry is possible for the quadratic and exponential superpotentials.
- The nonlinear supersymmetry is closely related to the quasy-exactly solvable systems.
- The $n = 2$ supersymmetric Calogero-like systems (14) admit the anomaly-free quantization for any superpotential; the specific quantum term ($\sim \hbar^2$) "cures" the quantum anomaly problem.
- The anomaly-free quantization of the classical 2D holomorphic n-supersymmetry fixes the form of the magnetic field to be the quadratic or exponential one.
- Realization of the holomorphic n-supersymmetry in 2D systems leads to the appearance of the central charge entering nontrivially into the superalgebra.
- The holomorphic nonlinear supersymmetry can be related to other known forms of nonlinear supersymmetry via the dimensional reduction procedure.
- There is the universal algebraic foundation associated with the Dolan–Grady relations which underlies the holomorphic n-supersymmetry.

The universal algebraic structure underlying the holomorphic nonlinear supersymmetry opens the possibility to apply the latter for investigation of the wide class of the quantum mechanical systems including the models described by the matrix Hamiltonians, the models on the noncommutative space, and integrable models [38].

Acknowledgements

M.P. thanks the organizers for hospitality and A. Zhedanov for useful discussions. The work was supported by the grants 1010073 and 3000006 from FONDECYT (Chile) and by DICYT (USACH).

[1] Witten E., Dynamical breaking of supersymmetry, *Nucl. Phys. B*, 1981, V.188, 513–554.

[2] Cooper F., Khare A. and Sukhatme U., Supersymmetry and quantum mechanics, *Phys. Rep.*, 1995, V.251, 267–385; hep-th/9405029.

[3] Plyushchay M., Hidden nonlinear supersymmetries in pure parabosonic systems, *Int. J. Mod. Phys. A*, 2000, V.23, 3679–3698; hep-th/9903130.

[4] Klishevich S. and Plyushchay M., Supersymmetry of parafermions, *Mod. Phys. Lett. A*, 1999, V.14, 2739–2752; hep-th/9905149.

[5] Grignani G., Plyushchay M. and Sodano P., A pseudoclassical model for P, T invariant planar fermions, *Nucl. Phys. B*, 1996, V.464, 189–212; hep-th/9511072.

[6] Nirov Kh.S. and Plyushchay M.S., P, T invariant system of Chern–Simons fields: pseudoclassical model and hidden symmetries, *Nucl. Phys. B*, 1998, V.512, 295–319; hep-th/9803221.

[7] De Jonghe F., Macfarlane A.J., Peeters K. and van Holten J.W., New supersymmetry of the monopole, *Phys. Lett. B*, 1995, V.359, 114–117; hep-th/9507046.

[8] Plyushchay M.S., On the nature of fermion monopole supersymmetry, *Phys. Lett. B*, 2000, V.485, 187–192; hep-th/0005122.

[9] De Boer J., Harmsze F. and Tjin T., Nonlinear finite W symmetries and applications in elementary systems, *Phys. Rept.*, 1996, V.272, 139–214; hep-th/9503161.

[10] Andrianov A.A., Ioffe M.V. and Spiridonov V.P., Higher derivative supersymmetry and the Witten index, *Phys. Lett. A*, 1993, V.174, 273–279; hep-th/9303005.

[11] Klishevich S. and Plyushchay M., Nonlinear supersymmetry, quantum anomaly and quasi-exactly solvable systems, *Nucl. Phys. B*, 2001, V.606, 583–612; hep-th/0012023.

[12] Klishevich S. and Plyushchay M., Nonlinear supersymmetry on the plane in magnetic field and quasi-exactly solvable systems, *Nucl. Phys. B*, 2001, V.616, 403–418; hep-th/0105135.

[13] Dolan L. and Grady M., Conserved charges from self-duality, *Phys. Rev. D*, 1982, V.25, 1587–1604.

[14] Ulyanov V.V. and Zaslavskii O.B., New classes of exact solutions of the Schrödinger equation and a description of spin systems by means of potential field, *Zh. Eksp. Teor. Fiz.*, 1984, V.87, 1724–1733.

[15] Turbiner A., Quasiexactly solvable problems and SL(2) group, *Comm. Math. Phys.*, 1988, V.118, 467–474.

[16] Shifman M.A., New findings in quantum mechanics (partial algebraization of the spectral problem), *Int. J. Mod. Phys. A*, 1989, V.4, 2897–2952.

[17] Ushveridze A., Quasi-exactly solvable models in quantum mechanics, Bristol, IOP Publishing, 1994.

[18] Finkel F., Gonzalez-Lopez A., Kamran N., Olver P.J. and Rodriguez M.A., Lie algebras of differential operators and partial integrability, hep-th/9603139.

[19] Bender C.M. and Dunne G.V., Quasi-exactly solvable systems and orthogonal polynomials, *J. Math. Phys.*, 1996, V.37, 6–11; hep-th/9511138.

[20] Plyushchay M.S., Deformed Heisenberg algebra, fractional spin fields and supersymmetry without fermions, *Annals Phys.*, 1996, V.245, 339–360; hep-th/9601116.

[21] Plyushchay M.S., Minimal bosonization of supersymmetry, *Mod. Phys. Lett. A*, 1996, V.11, 397–408; hep-th/9601141.

[22] Gamboa J., Plyushchay M. and Zanelli J., Three aspects of bosonized supersymmetry and linear differential field equation with reflection, *Nucl. Phys. B*, 1999, V.543, 447–465; hep-th/9808062.

[23] Ohnuki Y. and Kamefuchi S., Quantum field theory and parastatistics, University Press of Tokyo, 1982.

[24] Macfarlane A.J., Generalized oscillator systems and their parabosonic interpretation, in Proc. Inter. Workshop on Symmetry Methods in Physics, Editors A.N. Sissakian, G.S. Pogosyan and S.I. Vinitsky, Dubna, JINR, 1994, 319.

[25] Plyushchay M.S., Deformed Heisenberg algebra with reflection, *Nucl. Phys. B*, 1997, V.491, 619–634; hep-th/9701091.

[26] Yang L.M., A note on the quantum rule of the harmonic oscillator, *Phys. Rev.*, 1951, V.84, 788–790.

[27] Dunkl C.F., Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.*, 1989, V.311, 167–183.

[28] Polychronakos A.P., Exchange operator formalism for integrable systems of particles, *Phys. Rev. Lett.*, 1992, V.69, 703–705; hep-th/9202057.

[29] Polychronakos A.P., Exact spectrum of $SU(N)$ spin chain with inverse square exchange, *Nucl. Phys. B*, 1994, V.419, 553–566; hep-th/9310095.

[30] Sasaki R. and Takasaki K., Quantum Inozemtsev model, quasi-exact solvability and N-fold supersymmetry, *J. Phys. A*, 2001, V.34, 9533–9554; hep-th/0109008.

[31] Nirov K.S. and Plyushchay M., Symmetries and classical quantization, *Phys. Lett. B*, 1997, V.405, 114–120; hep-th/9707070.

[32] Aoyama H., Nakayama N., Sato M. and Tanaka T., $sl(2)$ construction of type-A N-fold supersymmetry, *Phys. Lett. B*, 2001, V.519, 260–268; hep-th/0107048.

[33] Aoyama H., Sato M. and Tanaka T., General forms of a N-fold supersymmetric family, *Phys. Lett. B*, 2001, V.503, 423–429; quant-ph/0012065.

[34] Turbiner A., Quasiexactly solvable differential equations, hep-th/9409068.

[35] Brihaye Y. and Kosinski P., Quasi-exactly solvable 2×2 matrix equations. *J. Math. Phys.*, 1994, V.35, 3089–3098; hep-th/9307099.

[36] Finkel F., Gonzalez-Lopez A. and Rodriguez M.A., Quasi-exactly solvable spin 1/2 Schrödinger operators, *J. Math. Phys.*, 1997, V.38 2795–2811; hep-th/9509057.

[37] Dorey P., Dunning C. and Tateo R., Spectral equivalences from Bethe ansatz equations, *J. Phys. A*, 2001, V.34, 5679–5704; hep-th/0103051.

[38] Klishevich S. and Plyushchay M., Nonlinear holomorphic supersymmetry, Dolan–Grady relations and Onsager algebra, *Nucl. Phys. B*, 2002, to appear; hep-th/0112158.