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Knot Manifolds of Double-Covariant Systems of Elliptic Equations and Preferred Orthonormal Three-Frames

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We show that the problem of existence of preferred orthonormal frame in general relativity, which is formulated as a problem of solvability for the nonlinear system of elliptic equations, can be reduced to the linear problem. For the obtained system of equations and for more general one we find the necessary and sufficient conditions of existence and uniqueness of the solution for Dirichlet problem and the conditions of zeros absence for the solution, taking into account the availability of double symmetry. This allows in particular to prove existence of wide class of hypersurfaces on which the Sen–Witten equation and the Nester gauge are equivalent (up to the sign).

Let (M, g) be $M = \Sigma \times R$ with spacelike $\Sigma_t \times \{t\}$ and metric g of signature $(+, -, -, -)$[1]. We assume that on Σ_t the constraints of general relativity are satisfied:

$$-R^{(3)} - \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} + \mathcal{K}^2 = 2\mu, \tag{1}$$

$$D_\mu \left(\mathcal{K}^{\mu\nu} - \mathcal{K}h^{\mu\nu}\right) = \mathcal{J}^\nu, \tag{2}$$

where $R^{(3)}$ is scalar curvature of Σ_t, $h = g - n \otimes n$ is induced metric on Σ_t. D_μ is induced by connection ∇_μ on M connection on Σ_t, $\mathcal{K}_{\mu\nu}$ is extrinsic curvature of Σ_t, $\mathcal{K} = \mathcal{K}^\nu_\nu$. μ and $\mathcal{J}^\nu$ are the energy density and momentum density, respectively, of the matter in the frame of reference of an observer, whose one-form of 4-velocity is $\xi = dt$. μ and $\mathcal{J}^\nu$ satisfy the dominant energy condition

$$\mu \geq \mid \mathcal{J}^\nu \mathcal{J}_\nu \mid^{1/2} .$$

There are three globally defined on Σ_t linearly independent one-forms θ^a that may thus be used as a coframe basis. Vector basis will be denoted by e_a. The connection one-forms coefficients $\omega^a{}_{bc}$ are determined as usually: $\omega^a{}_{bc} = \langle \theta^a, \nabla_{e_b} e_c \rangle$.

Definition 1. A set of N $(0 < N \leq 10)$ equations for the components of orthonormal vector basis $e_m{}^\mu$ (tetrad, vierbein)

$$\Phi_N \left(e_{m'}{}^{\mu'},\ \partial_{\nu'} e_{m'}{}^{\nu'},\ \partial^2_{\nu'\rho'} e_{p'}{}^{\pi'} \right) = 0, \tag{3}$$

which are not covariant under the local Lorentz transformations and (or) coordinate basis transformations, is said to be auxiliary conditions.

Definition 2. The auxiliary conditions (3) are said to be gauge fixing conditions in some domain Ω, if in this domain there exists the solution $x^{\mu'}(x^\nu)$, $L^{m'}_n(x)$ of the system of equations

$$\Phi_N \Big(e_n{}^\nu \frac{\partial x^{\mu'}}{\partial x^\nu} L^n{}_{m'}, \ldots, \ldots \Big) = 0 \tag{4}$$

with arbitrary coefficients $e_n{}^\nu$.

[1]Greek indices α to λ run through $1, 2, 3$; indices κ to ω run through $0, 1, 2, 3$. Latin indices are Lorentzian and a to l run through $1, 2, 3$; indices m to z run through $0, 1, 2, 3$.

For construction of tensor method for the proof of the positive energy theorem in general relativity Nester [1] introduced the auxiliary conditions for the choice of special orthonormal frame on three-dimensional Riemannian manifold

$$d\widetilde{q} = 0, \qquad d * q = 0, \tag{5}$$

where

$$\widetilde{q} := i_a d\theta^a, \qquad *q := \theta_a \wedge d\theta^a.$$

For proof the statement that auxiliary conditions (3) are gauge it is necessary to prove the existence of the solution $||R^{a\prime}{}_b|| \in SO(3)$ for the system of equations (5), where

$$q = i_a d\theta^a = q' + \theta_{m'} \wedge R^{m'}{}_b \, dR^b{}_{c'} \wedge \theta^{c'}, \tag{6}$$

$$q = \theta_a \wedge d\theta^a = \widetilde{q}\,' + R^{b'}{}_a R^a{}_{c',b}\theta^b. \tag{7}$$

The system of equations (6)–(7) is a nonlinear second-order elliptic system for the rotation $R^{a'}{}_b$. Nester proved the existence and uniqueness for the solution of the linearization of this system for geometries within a neighborhood of Euclidean space, and therefore the additional conditions (5) are gauge-fixing only asymptotically.

In paper [2] we have proved that conditions (5) are everywhere gauge on maximal hypersurface. Our purpose is to establish the existence of most wide as in [2] class of the conditions, under which auxiliary conditions (5) are gauge everywhere on Σ.

The method for the proof is based on the grounding for substitution of auxiliary conditions (5) by equivalent (up to sign) linear equations for $SU(2)$-spinor field, for which the theorems of existence are known. For these equations the new theorems about uniqueness and zeros of double-covariant system of equations in the bounded closed domain are also proved.

On the spaces, where the forms $\widetilde{q}$ and $*q$ are exact, the conditions (5) are replaced by their first integrals:

$$\widetilde{q} = -4d\ln\rho, \qquad *q = 0.$$

Function ρ is arbitrary and everywhere on Σ_t is positive. Let us consider a case, when the form $\mathcal{K}\theta^3$ is exact, and introduce some function λ, which anywhere on Σ_t does not equal to zero and is defined by the relationship $d\ln\lambda := 4d\ln\rho = \mathcal{K}\theta^3$. Let us complement the triad θ^a to the tetrad defining θ^0 as following: $\theta^0 \equiv n = Ndt$, here n is one form of the normal to Σ_t, and let us introduce the complex one-form $L = \frac{\lambda}{\sqrt{2}}(\theta^1 + i\theta^2)$. Then one-form L satisfies the equation

$$\left\langle \widetilde{L},\, D \otimes L \right\rangle - \mathcal{K}L + 3!\, i * (n \wedge D \wedge L) = 0, \tag{8}$$

where $\widetilde{L} = |L|^{-1} * \left(L \wedge \overline{L}\right)$, and here it is taken into account that λ does not equal to zero. The form L is spatial and, therefore, it defines up to sign the $SU(2)$-spinor λ_A: $L = -\lambda_A\lambda_B$, which is a result of Sen [3] reduction of $SL(2,\mathbb{C})$-spinor on Σ_t according to the definition $\lambda^{A+} = \sqrt{2}n^{A\dot{A}}\lambda_{\dot{A}}$. Equation (8) is the "squared" Sen–Witten equation

$$\mathcal{D}^B{}_C\lambda^C = 0, \tag{9}$$

where an action of the operator $\mathcal{D}_{AB}$ on $SU(2)$ spinor field is

$$\mathcal{D}_{AB}\lambda_C = D_{AB}\lambda_C + \frac{\sqrt{2}}{2}\mathcal{K}_{ABC}{}^D\lambda_D.$$

So, the question about existence of solution for nonlinear elliptic system (5) is reduced to the question about existence of solution for linear elliptic system (9) under the condition that this solution λ_A nowhere on Σ_t equals zero.

Further we will examine the question about conditions of existence and zeros absence for the solution of boundary value problem for general elliptic system of equations

$$\frac{1}{\sqrt{-h}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-h}h^{\alpha\beta}\frac{\partial}{\partial x^\beta}u_A\right)+C_A{}^B u_B=0 \tag{10}$$

on bounded closed spherical-type domain Ω on Σ_t, where $h^{\alpha\beta}$ – metric tensor components, which are arbitrary real functions of independent variables x^α continuous in Ω; the quadratic form $h^{\alpha\beta}\xi_\alpha\xi_\beta$ is negatively defined. The unknown functions u_A are complex twice continuously differentiable functions of independent variables x^α. They are also the elements of vector space $\mathbb{C}^2$, in which the skew symmetric tensor ε^{AB} is defined, and the group $SU(2)$ acts. The matrix $C := ||C_A{}^B||$ is Hermitian, its elements are twice continuously differentiable, and $C_0{}^1 \neq 0$ in Ω. A system (10) is a generalization of a differential result of equation (9) and equations (1), (2) [2].

For strongly elliptic systems of second order equations the sufficient conditions for unique solvability of boundary value problem were obtained in [4]. Let us take into account that the system of equations (10) is covariant under arbitrary transformations of coordinates on Ω, and covariant under the local $SU(2)$ transformations. This allows to obtain the necessary and sufficient conditions for unique solvability of Dirichlet problem.

Let denote by

$$\Delta := C_1{}^1 - C_0{}^0 - \left[\left(C_1{}^1 - C_0{}^0\right)^2 + 4\left|C_0{}^1\right|^2\right]^{1/2}.$$

Theorem 1. *The boundary value problem for equation (10) is uniquely solvable in domain Ω if and only if in this domain there are exist functions of C^2 class which satisfy the inequalities*

$$\det\begin{pmatrix} (-h^{\alpha\beta}) & B_A^\beta \\ B_A^\alpha & \sum\limits_{\gamma=1}^{3}\frac{\partial B_A^\gamma}{\partial x^\gamma}+C_{A'} \end{pmatrix} > 0, \tag{11}$$

here

$$C_{0'} = \frac{4C_0{}^0\left|C_0{}^1\right|^4 + \left(4\Delta\left|C_0{}^1\right|^2 + C_1{}^1\Delta^2\right)\left(4\left|C_0{}^1\right|^2+\Delta^2\right)}{4\left|C_0{}^1\right|^2\left(4\left|C_0{}^1\right|^2+\Delta^2\right)}, \tag{12}$$

$$C_{1'} = \frac{\left(C_0{}^0\Delta^2 - 4\,\Delta\left|C_0{}^1\right|^2\right)\left(4\left|C_0{}^1\right|^2+\Delta^2\right) + 4C_1{}^1\left|C_0{}^1\right|^4}{\left(4\left|C_0{}^1\right|^2+\Delta^2\right)}. \tag{13}$$

Proof. The system of equations (10) is covariant under the arbitrary transformations of coordinates and under the local transformations from spinor group $SU(2)$. This allows us to use them independently. Because the matrix C is Hermitian, there exists a matrix

$$R := ||R_A{}^B|| := \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \qquad \alpha\overline{\alpha}+\beta\overline{\beta}=1,$$

such that $C' = -\varepsilon R\varepsilon C R^{T+} = R^{T+}CR = \operatorname{diag}(C_{0'}, C_{1'})$. The matrix elements satisfy the conditions $\alpha\overline{\alpha}\left(1+\Delta^2/4\left|C_0{}^1\right|^2\right)=1$, $\beta = \alpha\Delta/2C_0{}^1$, and in the new spinor basis

$$u_{0'} = \overline{\alpha}\left(u_0 + \frac{\Delta}{2\overline{C}_0{}^1}\,u_1\right), \qquad u_{1'} = \alpha\left(-\frac{\Delta}{2C_0{}^1}\,u_0 + u_1\right). \tag{14}$$

The eigenvalues of matrix C are real, therefore, the system of equations (10) in the new spinor basis splits into the system of four independent equations. The coefficients before unknown functions are expressed as (12) and (13). Since $C_{0'}$ and $C_{1'}$ are scalars under the transformations of coordinates, we can apply the Skorobohat'ko theorem 1.16 [4] to each of equations. The conditions of the existence for each of equations are (11) with (12) or (13). This proves the Theorem. ■

Corollary 1. *The boundary value problem for equation (10) is uniquely solvable in domain* Ω *if in this domain the matrix* C *is positively defined.*

Theorem 2. *If the matrix* C *is positively defined in* Ω*, then a function* $M = u_0\overline{u}_0 + u_1\overline{u}_1$ *for any solution of class* C^2 *for equation (10) reaches the non-zero maximum only on the boundary of domain* Ω*.*

Proof. In arbitrary domain Ω in Riemannian space V^3 there exist solutions f_γ of class C^2 for a system of differential equations

$$h^{\alpha\beta}\frac{\partial f_\gamma}{\partial x^\alpha}\frac{\partial f_\delta}{\partial x^\beta}, \qquad \gamma \neq \delta.$$

Setting 3-orthogonal hypersurfaces $f_\alpha = \text{const}$ as coordinate hypersurfaces $x^{\alpha'} = \text{const}$ we will obtain the system of coordinates in which $h^{\alpha\beta} = 0$ at $\alpha \neq \delta$ [5].

Let us assume that the function M reaches the non-zero maximum in some intrinsic point of domain Ω. Then in this point $\frac{\partial M}{\partial x^\alpha} = 0$ and $h^{\alpha\alpha}\frac{\partial^2 M}{\partial x^{\alpha^2}} \geq 0$. But, from the other side, in the same point of maximum we have, taking into account that functions u_A and $C_A{}^B$ are scalars under arbitrary transformations of coordinate basis, the equation (10) is covariant with respect to them, and Hermitian matrix C is positively defined:

$$h^{\alpha\alpha}\frac{\partial^2 M}{\partial x^{\alpha^2}} = h^{\alpha\alpha}\frac{\partial u_A}{\partial x^\alpha}\frac{\partial \overline{u}_A}{\partial x^{\alpha^2}} - \sqrt{-h}C_A{}^B u_B \overline{u}^A - \sqrt{-h}\,\overline{C}_A{}^B \overline{u}_B u^A > 0.$$

The contradiction proves the statement of the theorem. ■

Theorem 2 generalizes the Bicadze extremum principle [6] onto the systems of elliptic equations with non-diagonal main parts of operators. It gives effective conditions of the knot points absence.

Corollary 2. *If the matrix* C *is positively defined in domain* Ω*, then in this domain the non-trivial solutions of class* C^2 *for equations (10) do not have the knot points.*

Reula has proved the existence of the C^∞ solution to Sen–Witten equation on Σ_t , if the initial data set $(\Sigma_t, h_{\mu\nu}, \mathcal{K}_{\pi\rho})$ is asymptotically flat [7].

The conditions, when initial data set is asymptotically flat and spinor u_A belongs to a certain Hilbert space, in this case substitute the conditions of Theorem 1. From Theorem 2 we obtain a condition for absence of knot points for Sen–Witten equation.

Theorem 3. *If matrix* G *with the elements*

$$G_A{}^B := \mathcal{D}_A{}^B K + \sqrt{2}\varepsilon_A^B\left(K^2 + \frac{1}{4}K_{\alpha\beta}K^{\alpha\beta} + \frac{1}{2}\mu\right)$$

is positively defined everywhere on Σ_t*, then the solution of equation (10), which tends at infinity to a certain constant non-zero value, differs from zero everywhere on* Σ_t*.*

From existence of the solution for Sen–Witten equation and from absence of knot points for this solution it follows that the Nester additional conditions are gauge on all hypersurfaces, on which the matrix G is positively defined. Therefore on such hypersurfaces there exists the Nester preferred orthonormal frame, and this is also the Sen–Witten frame. In particular, the Nester conditions are gauge, and the Nester frame is the Sen–Witten frame on maximal hypersurface.

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