

First Order Equations of Motion from Breaking of Super Self-Duality

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First order differential equations, which satisfy second order equations of motion for $N = 2$ Super Yang–Mills theory, are obtained with help of breaking of super self-duality.

First order equations of motion, by definition, are differential equations of first order, which satisfy second order equations of motion of the theory. For example, in the $N = 1$ supersymmetric $SU(2)$ Yang–Mills theory

$$L = \text{Tr} \left\{ -\frac{1}{4} F_{mn} F^{mn} - i \bar{\lambda} \bar{\sigma}^m \mathcal{D}_m \lambda + \frac{1}{2} D^2 \right\}, \quad (1)$$

where

$$F_{mn} = \partial_m V_n - \partial_n V_m + ig[V_m, V_n],$$

$$\mathcal{D}_m = \partial_m + ig[V_m, \cdot], \quad \eta_{mn} = \text{diag}(-1, 1, 1, 1),$$

the super self-duality equations in component fields are first order equations of motion. The Yang–Mills strength in spinor indices has the following form

$$F_{\alpha\dot{\alpha},\beta\dot{\beta}} \equiv \sigma^m_{\alpha\dot{\alpha}} \sigma^n_{\beta\dot{\beta}} F_{mn} = \frac{1}{2} \varepsilon_{\alpha\beta} f_{\dot{\alpha}\dot{\beta}} + \frac{1}{2} \varepsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta},$$

where

$$f_{\dot{\alpha}\dot{\beta}} \equiv \varepsilon^{\alpha\gamma} F_{\gamma\dot{\alpha},\alpha\dot{\beta}}, \quad f_{\alpha\beta} \equiv \varepsilon^{\dot{\alpha}\dot{\gamma}} F_{\alpha\dot{\gamma},\beta\dot{\alpha}}.$$

The super self-duality equations of the theory (1) look as follows [1]

$$f_{\alpha\beta} = 0, \quad D = 0, \quad \lambda_\alpha = 0, \quad \mathcal{D}_{\alpha\dot{\beta}} \bar{\lambda}^{\dot{\beta}} = 0. \quad (2)$$

The system (2) is invariant under the following $N = 1$ supersymmetric transformations

$$\delta_\xi V_{\alpha\dot{\alpha}} = -2i(\xi_\alpha \bar{\lambda}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}} \lambda_\alpha), \quad \delta_\xi D = -\xi^\alpha \mathcal{D}_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}} \mathcal{D}^{\alpha\dot{\alpha}} \lambda_\alpha,$$

$$\delta_\xi \lambda_\alpha = \frac{1}{2} \xi^\beta f_{\alpha\beta} + i \xi_\alpha D, \quad \delta_\xi \bar{\lambda}_{\dot{\alpha}} = \frac{1}{2} \bar{\xi}^{\dot{\beta}} f_{\dot{\alpha}\dot{\beta}} - i \bar{\xi}_{\dot{\alpha}} D, \quad (3)$$

where $\xi_\alpha, \bar{\xi}_{\dot{\alpha}}$ are the parameters of $N = 1$ supersymmetric transformations. Invariance of (2) under transformations (3) means that the system of transformed equations

$$\delta_\xi f_{\alpha\beta} = 0, \quad \delta_\xi D = 0, \quad \delta_\xi \lambda_\alpha = 0, \quad \delta_\xi \left(\mathcal{D}_{\alpha\dot{\beta}} \bar{\lambda}^{\dot{\beta}} \right) = 0$$

is satisfied on the system (2).

It easy to verify that the system of super self-duality equations (2) can be derived from the only equation

$$\lambda_\alpha = 0 \quad (4)$$

by applying supersymmetric transformations to this equation twice. In other words, the system (2) can be written in the following form

$$\lambda_\alpha = 0, \quad \delta_\xi \lambda_\alpha = 0, \quad \delta_\eta \delta_\xi \lambda_\alpha = 0.$$

Adding to (4) one more equation (which breaks super self-duality)

$$\lambda_\alpha = 0, \quad \bar{\lambda}_1 = k\bar{\lambda}_2, \tag{5}$$

where k is complex number, and by applying twice the transformations (3) to (5), we obtain another system of first order equations of motion

$$f_{\alpha\beta} = 0, \quad f_{1\dot{\beta}} = kf_{2\dot{\beta}}, \quad D = 0, \quad \lambda_\alpha = 0, \quad \bar{\lambda}_1 = k\bar{\lambda}_2, \quad \mathcal{D}_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}} = 0. \tag{6}$$

Though this system is overdetermined, it is invariant under supersymmetric transformations.

This example prompts the procedure for obtaining of first order equations of motion in supersymmetric theories.

In this paper we present some systems of first order equations in the $N = 2$ supersymmetric Yang–Mills theory, which are obtained by breaking super self-duality. By definition, the system of super-self-duality equations has the following properties: i) it includes the self-duality equation for pure Yang–Mills theory $f_{\alpha\beta} = 0$; ii) it satisfies the equations of motion of the corresponding supersymmetric theory; iii) it is invariant under supersymmetric transformations.

The $SU(2)$ Yang–Mills theory with extended $N = 2$ supersymmetry, given by the Lagrangian [2]

$$L = \text{Tr} \left(-\frac{1}{4} F_{mn} F^{mn} - i\bar{\lambda}_{\dot{\alpha}i} \bar{\sigma}^{m\dot{\alpha}\beta} \mathcal{D}_m \lambda_\beta^i - 2\mathcal{D}_m C \mathcal{D}^m C^* - \frac{1}{2} \vec{C}^2 + igC\{\bar{\lambda}_{\dot{\alpha}i}, \bar{\lambda}^{\dot{\alpha}i}\} + igC^*\{\lambda_\alpha^i, \lambda_i^\alpha\} + 4g^2 C[C, C^*]C^* \right), \tag{7}$$

is invariant under $N = 2$ supersymmetric transformations [3]:

$$\begin{aligned} \delta_\xi C &= -\xi_i^\alpha \lambda_\alpha^i, \\ \delta_\xi C^* &= -\bar{\xi}_{\dot{\alpha}i} \bar{\lambda}^{\dot{\alpha}i}, \\ \delta_\xi V_{\alpha\dot{\alpha}} &= 2i(\xi_\alpha^i \bar{\lambda}_{\dot{\alpha}i} + \bar{\xi}_{\dot{\alpha}i} \lambda_\alpha^i), \\ \delta_\xi \lambda_\alpha^i &= -\frac{1}{2} \xi^{\beta i} f_{\alpha\beta} + 2ig\xi_\alpha^i [C, C^*] - \xi_{\alpha j} \vec{C} \vec{\tau}^{ij} + 2i\xi^{\dot{\alpha}i} \mathcal{D}_{\alpha\dot{\alpha}} C, \\ \delta_\xi \bar{\lambda}_{\dot{\alpha}i} &= -\frac{1}{2} \bar{\xi}_{\dot{\alpha}i} f_{\dot{\alpha}\dot{\beta}} - 2ig\bar{\xi}_{\dot{\alpha}i} [C, C^*] + \bar{\xi}_{\dot{\alpha}i} \vec{C} \vec{\tau}_{ij} + 2i\xi_\alpha^i \mathcal{D}_{\alpha\dot{\alpha}} C^*, \\ \delta_\xi \vec{C} &= -i\xi^{\alpha i} (\mathcal{D}_{\alpha\dot{\beta}} \bar{\lambda}^{\dot{\beta}j} + 2g[\lambda_\alpha^j, C^*]) \vec{\tau}_{ij} + i\bar{\xi}_{\dot{\alpha}i} (\mathcal{D}^{\alpha\dot{\beta}} \lambda_\alpha^j - 2g[\bar{\lambda}^{\dot{\beta}j}, C]) \vec{\tau}_{ij}, \end{aligned} \tag{8}$$

where $\xi_\alpha^i, \bar{\xi}_{\dot{\alpha}i}$ are the parameters of $N = 2$ supersymmetric transformations, and $\vec{\tau}_i^j$ are Pauli matrices.

The $N = 2$ super self-dual system

$$\begin{aligned} f_{\alpha\beta} = 0, \quad C = 0, \quad \mathcal{D}_{\alpha\dot{\beta}} \mathcal{D}^{\alpha\dot{\beta}} C^* - ig\{\bar{\lambda}_{\dot{\alpha}i}, \bar{\lambda}^{\dot{\alpha}i}\} = 0, \\ \vec{C} = 0, \quad \lambda_\alpha^i = 0, \quad \mathcal{D}_{\alpha\dot{\beta}} \bar{\lambda}^{\dot{\beta}i} = 0 \end{aligned} \tag{9}$$

includes one second order equation. The system (9) can be written as

$$\lambda_\alpha^i = 0, \quad \delta_\xi \lambda_\alpha^i = 0, \quad \delta_\eta \delta_\xi \lambda_\alpha^i = 0.$$

In order to obtain the systems of first order equations, we can break super self-duality in two ways: i) adding to the equation $\lambda_\alpha^i = 0$ other conditions for spinor fields; ii) imposing some conditions on supersymmetric parameters. In the case of proper choice of the above-mentioned conditions, after applying twice the supersymmetric transformations (8) to the equations for spinor fields we will obtain the system of first order equations.

The first example is (we have two systems, which correspond to $i = 1$ and $i = 2$)

$$\begin{aligned} \lambda_\alpha^i &= 0, & \delta_\xi \lambda_\alpha^i &= 0, & \delta_\eta \delta_\xi \lambda_\alpha^i &= 0, \\ \bar{\lambda}_{1i} - k\bar{\lambda}_{2i} &= 0, & \delta_\xi (\bar{\lambda}_{1i} - k\bar{\lambda}_{2i}) &= 0, & \delta_\eta \delta_\xi (\bar{\lambda}_{1i} - k\bar{\lambda}_{2i}) &= 0. \end{aligned} \quad (10)$$

After transformations, we obtain from (10)

$$\begin{aligned} f_{\alpha\beta} &= 0, & f_{1\dot{\beta}} &= kf_{2\dot{\beta}}, & C &= 0, & \mathcal{D}_{\alpha 1} C^* &= k\mathcal{D}_{\alpha 2} C^*, \\ \vec{C} &= 0, & \lambda_\alpha^i &= 0, & \mathcal{D}_{\alpha\dot{\beta}} \bar{\lambda}^{\dot{\beta}i} &= 0, & \bar{\lambda}_{1i} &= \bar{\lambda}_{2i}. \end{aligned} \quad (11)$$

The system (11) is invariant under $N = 2$ supersymmetric transformations and satisfies the equations of motion of the theory (7). It is not overdetermined.

The following system of first order equations of motion looks as follows

$$\begin{aligned} \bar{\xi}_{\dot{\alpha}i} &= 0, \\ \lambda_{\alpha i} &= 0, & \delta_\xi \lambda_{\alpha i} &= 0, & \delta_\eta \delta_\xi \lambda_{\alpha i} &= 0, \\ \bar{\lambda}_{\dot{\alpha}i} &= 0, & \delta_\xi \bar{\lambda}_{\dot{\alpha}i} &= 0, & \delta_\eta \delta_\xi \bar{\lambda}_{\dot{\alpha}i} &= 0, \\ \lambda_1^i - k\lambda_2^i &= 0, & \delta_\xi (\lambda_1^i - k\lambda_2^i) &= 0, & \delta_\eta \delta_\xi (\lambda_1^i - k\lambda_2^i) &= 0, \end{aligned} \quad (12)$$

or, in the equivalent form,

$$\begin{aligned} f_{\alpha\beta} &= 0, & \mathcal{D}_{1\dot{\alpha}} C &= k\mathcal{D}_{2\dot{\alpha}} C, & C^* &= \vec{C} = 0, \\ \mathcal{D}^{\alpha\dot{\beta}} \lambda_\alpha^i - 2g[\bar{\lambda}^{\dot{\beta}i}, C] &= 0, & \mathcal{D}_{\alpha\dot{\beta}} \bar{\lambda}^{\dot{\beta}i} &= 0, \\ \lambda_{\alpha i} &= 0, & \bar{\lambda}_{\dot{\alpha}i} &= 0, & \lambda_1^i &= k\lambda_2^i, & \bar{\xi}_{\dot{\alpha}i} &= 0. \end{aligned} \quad (13)$$

In this case we have put a constraint on supersymmetric parameters and included it into the system of first order equations of motion. This underlines that the system (13) is invariant under $N = 2$ supersymmetric transformations on the condition that $\bar{\xi}_{\dot{\alpha}1} = 0$ or $\bar{\xi}_{\dot{\alpha}2} = 0$ correspondingly.

Another system of first order equations of motion

$$\begin{aligned} \xi_{\alpha i} &= 0, & \bar{\xi}_{\dot{\alpha}i} &= 0, \\ \lambda_{\alpha i} &= 0, & \delta_\xi \lambda_{\alpha i} &= 0, & \delta_\eta \delta_\xi \lambda_{\alpha i} &= 0, \\ \bar{\lambda}_{\dot{\alpha}i} &= 0, & \delta_\xi \bar{\lambda}_{\dot{\alpha}i} &= 0, & \delta_\eta \delta_\xi \bar{\lambda}_{\dot{\alpha}i} &= 0, \end{aligned} \quad (14)$$

or, in the explicit form,

$$\begin{aligned} f_{\alpha\beta} &= 0, & C &= 0, & \mathcal{D}_{1\dot{\alpha}} C^* &= k\mathcal{D}_{2\dot{\alpha}} C^*, & \vec{C} &= 0, \\ \mathcal{D}^{\alpha\dot{\beta}} \lambda_\alpha^i &= 0, & \mathcal{D}_{\alpha\dot{\beta}} \bar{\lambda}^{\dot{\beta}i} + 2g[\lambda_\alpha^i, C^*] &= 0, \\ \lambda_{\alpha i} &= 0, & \bar{\lambda}_{\dot{\alpha}i} &= 0, & \xi_{\alpha i} &= 0, & \bar{\xi}_{\dot{\alpha}i} &= 0. \end{aligned} \quad (15)$$

In such way one can find some more systems of first order equations of motion.

- [1] Volovich I.V., Super-selfduality for supersymmetric Yang–Mills theory, *Phys. Lett. B*, 1983, V.123, N 5, 329–331.
- [2] Grimm R., Sohnius M. and Wess J., Extended supersymmetry and gauge theories, *Nucl. Phys. B*, 1978, V.133, N 2, 275–284.
- [3] Sohnius M.F., Introducing supersymmetry, *Phys. Rep.*, 1985, V.128, N 2–3, 39–204.