

# Canonical Realization of Poincaré Algebra: from Field Theory to Direct-Interaction Theory

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The canonical realization of the Poincaré group for the systems of the pointlike particles coupled with the electromagnetic, massive vector and scalar fields is constructed. The reduction of the canonical field degrees of freedom is done in the linear approximation in the coupling constant. The Poincaré generators in terms of particle variables are found. The relation between covariant and physical particle variables in the Hamiltonian description is written. The approximation up to  $c^{-2}$  is examined.

## 1 Introduction

So many field-theoretical models in the classical relativistic mechanics are based on the Lagrangian formalism due to its conceptual simplicity [1, 2]. However, the transition from Lagrangian description, when the fields are eliminated by means of substitution of the formal solutions of the field equations, to Hamiltonian one is not simple and demands the use of various approximations. For this reason, it is natural to construct the Hamiltonian description of the “particle plus field” systems, and *then* to exclude field degrees of freedom. Such a program is discussed in the series of papers by Lusanna with collaborators (see [3]).

Here at the beginning we apply simpler approach of the use of the geometrical forms of dynamics [2] fixing chronometrical invariance of the action integral. We construct the Hamiltonian description of charged particles with electromagnetic field, and perform the canonical transformation which isolates nonphysical (gauge) degrees of freedom of the electromagnetic field. We also consider the massive scalar and vector interactions and obtain generators of time evolution and Lorentz transformations on the physical phase space. In Section 3 the procedure of the exclusion of the field degrees of freedom is described within the linear approximation in the coupling constant. We obtain the canonical generators of the Poincaré group (the direct-interaction theory) for considered interactions. We demonstrate that the approximation up to  $c^{-2}$  agrees with the well known results of various approaches.

## 2 Hamiltonian formulation of the “field+particle” systems

Let particles be described by their world lines in the Minkowski space-time<sup>1</sup>  $\gamma_a : \tau \mapsto x_a^\mu(\tau)$ . The electromagnetic interaction between charges is mediated by the field  $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$  with the electromagnetic potential  $A_\mu(x)$ ;  $\partial_\nu \equiv \partial/\partial x^\nu$ . An action for the system of  $N$  charges

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<sup>1</sup>The Minkowski space-time is endowed with a metric  $\|\eta_{\mu\nu}\| = \text{diag}(1, -1, -1, -1)$ . The Greek indices  $\mu, \nu, \dots$  run from 0 to 3; the Roman indices from the middle of alphabet,  $i, j, k, \dots$  run from 1 to 3 and both types of indices are subject of the summation convention. The Roman indices from the beginning of alphabet,  $a, b$ , label the particles and run from 1 to  $N$ . The sum over such indices is indicated explicitly.

is

$$S = - \sum_{a=1}^N \int d\tau_a \left\{ m_a \sqrt{u_a^2(\tau_a)} + e_a u_a^\nu(\tau_a) A_\nu[x_a(\tau_a)] \right\} - \frac{1}{4} \int F_{\lambda\sigma}(x) F^{\lambda\sigma}(x) d^4x, \tag{1}$$

where  $m_a$  and  $e_a$  are the mass and the charge of particle  $a$ , respectively, and  $u_a^\mu(\tau_a) = dx_a^\mu(\tau_a)/d\tau_a$ . The action is manifestly invariant under reparametrization of the particle world lines and ordinary gauge transformation of the electromagnetic potential:

$$\tau_a \mapsto \phi(\tau_a), \quad \phi' > 0, \tag{2}$$

$$A_\mu \mapsto A_\mu + \partial_\mu \Lambda. \tag{3}$$

Moreover, action (1) is invariant under (global) transformations of the Poincaré group; this invariance results in the conservation of the symmetric energy-momentum tensor [4]:

$$\theta^{\mu\nu}(x) = \sum_{a=1}^N \int m_a \frac{u_a^\mu(\tau_a) u_a^\nu(\tau_a)}{\sqrt{u_a^2(\tau_a)}} \delta^4(x - x_a(\tau_a)) d\tau_a - F^{\mu\lambda} F^\nu{}_\lambda + \frac{\eta^{\mu\nu}}{4} F_{\lambda\sigma} F^{\lambda\sigma}, \tag{4}$$

$$\theta^{\mu\nu}(x) = \theta^{\nu\mu}(x), \quad \partial_\nu \theta^{\mu\nu}(x) = 0. \tag{5}$$

We fix the freedom in the parametrization of particle world lines by means of gauge condition:

$$x^0 = f(t, \mathbf{x}), \quad \mathbf{x} = (x^1, x^2, x^3), \tag{6}$$

which defines the form of relativistic dynamics. Then, the Minkowski space-time is foliated by the family of space-like or isotropic hypersurfaces  $\Sigma_t$  parametrized by  $t$ . The functions  $x^i = x_a^i(t)$ ,  $i = 1, 2, 3$ , completely determine the parametric equations of the particle world lines in a given form of dynamics:

$$x^0 = f(t, \mathbf{x}_a(t)), \quad x^i = x_a^i(t). \tag{7}$$

The variable  $t$  serves as a common evolution parameter of the system.

Accounting (6), we come to a single-time form of the action [5]

$$S = \int dt L \tag{8}$$

with Lagrangian  $L(t)$  depending on the functions  $\mathbf{x}_a(t)$ ,  $A^\mu(t, \mathbf{x})$  and their first order derivatives with respect to evolution parameter,  $\dot{\mathbf{x}}_a(t) = d\mathbf{x}_a(t)/dt$  and  $\dot{A}^\mu(t, \mathbf{x})$ .

The conservation of the energy-momentum tensor (4) gives us ten conserved quantities in a given form of dynamics:

$$P^\mu = \int_{\Sigma_t} \theta^{\mu\nu} d\sigma_\nu, \quad M^{\mu\nu} = \int_{\Sigma_t} (x^\mu \theta^{\nu\rho} - x^\nu \theta^{\mu\rho}) d\sigma_\rho. \tag{9}$$

However, the Lagrangian  $L$  still remains invariant under gauge transformation (3) and leads to the constrained Hamiltonian description. It is demonstrated in [5] that the form of dynamics determines the structure of the corresponding constraints. In the following we confine ourselves by the most common case of the instant form of dynamics ( $x^0 = t$ ). The Lagrangian function in this form of dynamics is represented by

$$L = - \sum_{a=1}^N \left\{ m_a \sqrt{1 - \dot{\mathbf{x}}_a^2} + e_a [A_0(t, \mathbf{x}_a) + \dot{x}_a^i A_i(t, \mathbf{x}_a)] \right\} - \frac{1}{4} \int (2E_i E^i + F_{ij} F^{ij}) d^3x, \tag{10}$$

where  $F_{ij} = \partial_i A_j - \partial_j A_i$  and  $E_i = \partial_i A_0 - \dot{A}_i$ .

In the Hamiltonian formulation of our system we start with canonical variables  $x_a^i(t)$ ,  $A_\mu(t, \mathbf{x})$  and conjugated momenta  $p_{ai}(t)$ ,  $E^\mu(t, \mathbf{x})$  which are subject of the first class constraints [6]

$$E^0 \approx 0, \quad \Gamma \equiv \varrho - \partial_i E^i \approx 0, \quad (11)$$

where  $\approx$  means “weak equality” in the sense of Dirac and  $\varrho(t, \mathbf{x}) = \sum_{a=1}^N e_a \delta^3(\mathbf{x} - \mathbf{x}_a(t))$  is a charge density.

Now we break the field phase space by means of canonical transformation so that the physical part is described by the gauge invariant variables  $a_\alpha = (\delta_\alpha^i - \delta_3^i \partial_\alpha / \partial_3) A_i$ ,  $E^\alpha$ ;  $\alpha = 1, 2$ , and unphysical part is parametrized by the canonical pairs  $(Q, \Gamma)$  and  $(A_0, E^0)$ .

The time evolution of the physical degrees of freedom is generated by the Hamiltonian

$$H = \sum_{a=1}^N \sqrt{m_a^2 + [\mathbf{p}_a - e_a \mathbf{A}_\perp(\mathbf{x}_a)]^2} - \frac{1}{2} \int \left( A_i^\perp \Delta A_i^\perp - E_\perp^i E_\perp^i + \varrho \Delta^{-1} \varrho \right) d^3 x, \quad (12)$$

where

$$E_\perp^i = (\delta_\alpha^i - \delta_3^i \partial_\alpha / \partial_3) E^\alpha, \quad A_i^\perp = (\delta_i^\alpha + \partial_i \Delta^{-1} \partial^\alpha) a_\alpha. \quad (13)$$

Inverse differential operators are defined so that

$$1/\partial_3 \delta^3(\mathbf{x}) = (1/2) \delta(x^1) \delta(x^2) \operatorname{sgn}(x^3), \quad \Delta^{-1} \delta^3(\mathbf{x}) = -1/(4\pi|\mathbf{x}|). \quad (14)$$

Reexpression of the conserved quantities (9) in the terms of canonical variables leads to the canonical realization of the Poincaré group. On the physical subspace the generator  $P^0$  coincides with the Hamiltonian (12), and the generator of the Lorentz transformation is given by

$$M^{k0} = \sum_{a=1}^N \left\{ x_a^k \sqrt{m_a^2 + [\mathbf{p}_a - e_a \mathbf{A}_\perp(\mathbf{x}_a)]^2} - t p_a^k \right\} - \frac{1}{2} \int x^k \varrho \Delta^{-1} \varrho d^3 x \\ + \int x^k \left( \frac{1}{4} F_{ij}^\perp F_{ij}^\perp + \frac{1}{2} E_\perp^i E_\perp^i + E_\perp^l \partial_l \Delta^{-1} \varrho \right) d^3 x - t \int E_\perp^l \partial^k A_l^\perp d^3 x. \quad (15)$$

where  $F_{ij}^\perp = \partial_i A_j^\perp - \partial_j A_i^\perp$ .

Let us consider in a similar manner the Hamiltonian description of the system of particles with massive vector and scalar interactions. In the first case a system is described by action that differs from (1) by the massive term  $\frac{1}{2} \mu^2 A^\nu A_\nu$ . The instant form Hamiltonian description of the system is based on the canonical variables  $x_a^i(t)$ ,  $A_\mu(t, \mathbf{x})$  and  $p_{ai}(t)$ ,  $E^\mu(t, \mathbf{x})$ . Moreover, there is a pair of the second class constraints:

$$E^0 \approx 0, \quad \Gamma - \mu^2 A_0 \approx 0, \quad (16)$$

which can be excluded by means of the Dirac bracket. The canonical Hamiltonian is

$$H = \sum_{a=1}^N \sqrt{m_a^2 + [\mathbf{p}_a - e_a \mathbf{A}(t, \mathbf{x}_a)]^2} \\ + \int \left[ \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} E^i E^i - \frac{1}{2} \mu^2 A_i A^i + A_0 \left( \Gamma - \frac{1}{2} \mu^2 A_0 \right) \right] d^3 x. \quad (17)$$

After exclusion of the constraints (16) one obtains for the boost generator

$$\begin{aligned}
 M^{k0} = & \sum_{a=1}^N \left\{ x_a^k \sqrt{m_a^2 + [\mathbf{p}_a - e_a \mathbf{A}(t, \mathbf{x}_a)]^2} - t p_a^k \right\} \\
 & + \int x^k \left[ \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} E^i E^i - \frac{1}{2} \mu^2 A_i A^i + \frac{1}{2\mu^2} \Gamma^2 \right] d^3x \\
 & - t \int \left[ E^j \partial^k A_j - \frac{1}{2} \mu^2 A^k \Gamma \right] d^3x.
 \end{aligned} \tag{18}$$

In the case of a system of particles interacting by means of the scalar field  $\varphi(x)$  we construct the standard Hamiltonian formalism without constraints with the Hamiltonian

$$H = \sum_{a=1}^N \sqrt{\mathbf{p}_a^2 + [m_a - e_a \varphi(t, \mathbf{x}_a)]^2} + \frac{1}{2} \int [\pi^2 + (\nabla \varphi)^2 + \mu^2 \varphi^2] d^3x, \tag{19}$$

and the boost generator

$$\begin{aligned}
 M^{k0} = & \sum_{a=1}^N \left\{ x_a^k \sqrt{\mathbf{p}_a^2 + [m_a - e_a \varphi(t, \mathbf{x}_a)]^2} - t p_a^k \right\} \\
 & + \frac{1}{2} \int x^k [\pi^2 + (\nabla \varphi)^2 + \mu^2 \varphi^2] d^3x - t \int \pi \partial^k \varphi d^3x.
 \end{aligned} \tag{20}$$

In the next section we will see that elimination of the field degrees of freedom into the three considered cases gives us the canonical generators of a similar structure.

### 3 Elimination of the field degrees of freedom

In the systems, where the free radiation is not essential, the physical field degrees of freedom can be excluded. As a result, we obtain the description of our systems in the terms of particle variables only.

Let us perform the field reduction by three steps [7]. First, we must find a solution of the field equations of motion. Here, using coupling constant expansion, we solve the linearized equations. However, we touch the problem of choice of Green's function. Fortunately, in the first-order (linear) approximation in the coupling constant the advanced, retarded, or symmetric solutions coincide. We use here the time-symmetric Green's function  $G(x^2) = G(x_0^2 - \mathbf{x}^2)$ . It is well known [1] that the Green's function determines the nonrelativistic potential  $u(r)$ :

$$u(r) = \int d\alpha G(\alpha^2 - r^2). \tag{21}$$

The general solution of the field equations is a sum of the source free field  $A_s^{\text{rad}}$  ( $s$  is the number of the physical field components), which satisfies the homogeneous equation, and the solution of the inhomogeneous equation  $\mathcal{A}_s$  in the terms of canonical particle variables.

Second, we perform a canonical transformation [7]:

$$A_s = A_s^{\text{rad}} + \mathcal{A}_s, \quad E^s = E_{\text{rad}}^s + \mathcal{E}^s, \tag{22}$$

$$x_a^i = q_a^i + \int \left[ \left( A_s^{\text{rad}} + \frac{1}{2} \mathcal{A}_s \right) \frac{\partial \mathcal{E}^s}{\partial k_{ai}} - \left( E_{\text{rad}}^s + \frac{1}{2} \mathcal{E}^s \right) \frac{\partial \mathcal{A}_s}{\partial k_{ai}} \right] d^3x, \tag{23}$$

$$p_{ai} = k_{ai} - \int \left[ \left( A_s^{\text{rad}} + \frac{1}{2} \mathcal{A}_s \right) \frac{\partial \mathcal{E}^s}{\partial q_a^i} - \left( E_{\text{rad}}^s + \frac{1}{2} \mathcal{E}^s \right) \frac{\partial \mathcal{A}_s^\perp}{\partial q_a^i} \right] d^3x, \tag{24}$$

here the free field terms ( $A_s^{\text{rad}}, E_{\text{rad}}^s$ ) are treated as the new canonical variables.

Third step consists in elimination of the field variables by means of constraints

$$A_s^{\text{rad}} \approx 0, \quad E_{\text{rad}}^s \approx 0. \quad (25)$$

The Dirac bracket for the systems with additional canonical constraints (25) coincides with the particle Poisson bracket  $\{q_a^i, k_{bj}\} = -\delta_{ab}\delta_j^i$ .

It is true, in order to simplify the form of the Poincaré generators for the system with vector interaction, we need to canonically transform the particle variables. Finally, the canonical generators of the Poincaré group for the considered interactions in the linear approximation are

$$H = c \sum_{a=1}^N k_a^0 + \frac{c}{2} \sum_{a,b=1}^N e_a e_b \frac{f(\omega_{ab})}{k_a^0} u(\rho_{ab}), \quad k_a^0 = \sqrt{m_a^2 c^2 + \mathbf{k}_a^2}, \quad (26)$$

$$P^k = \sum_{a=1}^N k_a^k, \quad M^{ij} = \sum_{a=1}^N (q_a^i k_a^j - q_a^j k_a^i), \quad (27)$$

$$M^{k0} = \sum_{a=1}^N \left( \frac{q_a^k}{c} k_a^0 - t k_a^k \right) + \frac{1}{2c} \sum_{a,b=1}^N e_a e_b q_b^k \frac{f(\omega_{ab})}{k_a^0} u(\rho_{ab}), \quad (28)$$

where the prime over sum denotes that  $a \neq b$  ( $a = b$  terms is excluded by means of mass renormalization);  $\rho_{ab}^2 = q_{ab}^2 + (\mathbf{k}_a \mathbf{q}_{ab} / k_a^0)^2$ ,  $\mathbf{q}_{ab} = \mathbf{q}_a - \mathbf{q}_b$ ,  $q_{ab} = |\mathbf{q}_{ab}|$ ,  $\omega_{ab} = k_a^\mu k_{b\mu} / m_a m_b c^2$ , and  $f(\omega) = 1$  for the scalar interaction and  $f(\omega) = \omega$  for the vector interaction. It can easy be demonstrated that the expressions (26)–(28) satisfy the commutation relations of the Poincaré group in a given approximation with arbitrary functions  $u(r)$  and  $f(\omega)$ .

According to (23), the covariant particle positions  $x_a^i$  are connected with the canonical variables as

$$x_a^i = q_a^i + \frac{1}{2} \int \left[ \mathcal{A}_s \frac{\partial \mathcal{E}^s}{\partial k_{ai}} - \mathcal{E}^s \frac{\partial \mathcal{A}_s}{\partial k_{ai}} \right] d^3 x. \quad (29)$$

It can be verified directly that in a given approximation the expression (29) satisfies the world line condition

$$\{x_a^i, M^{k0}\} = x_a^k \{x_a^i, H\} - t \delta^{ik}. \quad (30)$$

The Poisson brackets between particle positions do not vanish,

$$\{x_a^i, x_b^j\} = \int \left( \frac{\partial \mathcal{A}_s}{\partial k_{bj}} \frac{\partial \mathcal{E}^s}{\partial k_{ai}} - \frac{\partial \mathcal{E}^s}{\partial k_{bj}} \frac{\partial \mathcal{A}_s}{\partial k_{ai}} \right) d^3 x, \quad (31)$$

in a full agreement with the famous no-interaction theorem [8].

Similarly, the direct-interaction theory can be obtained in the different forms of relativistic dynamics. They are physically equivalent. So, the Poincaré generators in the front form ( $x^0 = t + x^3$ ), which corresponds to foliation of the Minkowski space-time by the isotropic hypersurfaces, are connected with the instant form generators (“in”) by means of the following canonical transformation:

$$q_a^i \rightarrow q_a^i - q_a^3 \frac{k_a^i + \delta_3^i h_a}{h_a}, \quad k_a^i \rightarrow k_a^i + \delta_a^i h_a, \quad (32)$$

$$G_{\text{in}} - G_{\text{fr}} = \{F, G_{\text{in}}\}, \quad (33)$$

$$h_a = \frac{\mathbf{k}_a^2 + m_a^2}{2k_{a3}}, \quad F = \int (\exp(-x^3 \partial_t) - 1) \mathcal{F} d^3 x, \quad (34)$$

where  $\partial_t \mathcal{F}$  is equal to the spatial density of the instant form interaction term.

Now let us examine the generators (26), (28) up to  $c^{-2}$  approximation. We immediately find that

$$u(\rho_{ab}) = u(q_{ab}) + \frac{(\mathbf{q}_{ab}\mathbf{k}_a)^2}{2q_{ab}m_a^2c^2} \frac{du(q_{ab})}{dq_{ab}}, \quad f(\omega_{ab}) = 1 + \frac{f'(0)}{2c^2} \left( \frac{\mathbf{k}_a}{m_a} - \frac{\mathbf{k}_b}{m_b} \right)^2. \quad (35)$$

Performing the canonical transformation generated by the function

$$\Lambda = \frac{1}{4c^2} \sum_{a<b}^N e_a e_b u(q_{ab}) \left[ \mathbf{q}_{ab} \left( \frac{\mathbf{k}_a}{m_a} - \frac{\mathbf{k}_b}{m_b} \right) \right], \quad (36)$$

finally, we obtain the expressions

$$H = H^{(0)} + H^{(1)}, \quad (37)$$

$$M^{k0} = \sum_{a=1}^N (q_a^k m_a - t k_a^k) + \frac{1}{2c^2} \sum_{a,b=1}^N e_a e_b q_b^k u(q_{ab}). \quad (38)$$

where

$$H^{(0)} = \sum_{a=1}^N \left( m_a c^2 + \frac{\mathbf{k}_a^2}{2m_a} \right) + U^{(0)}, \quad U^{(0)} = \sum_{a<b}^N e_a e_b u(q_{ab}), \quad (39)$$

$$H^{(1)} = - \sum_{a=1}^N \frac{\mathbf{k}_a^4}{8m_a^3 c^2} - \sum_{a<b}^N e_a e_b \left\{ \frac{1}{2c^2 m_a m_b} [\mathbf{k}_a \mathbf{k}_b u(q_{ab}) + (\mathbf{k}_a \mathbf{q}_{ab})(\mathbf{k}_b \mathbf{q}_{ab}) \frac{du(q_{ab})}{q_{ab} dq_{ab}}] - \frac{A}{2c^2} \left( \frac{\mathbf{k}_a}{m_a} - \frac{\mathbf{k}_b}{m_b} \right)^2 u(q_{ab}) \right\}, \quad (40)$$

and  $A = f'(0) - 1$ . Specifically,  $A = -1$  for the scalar and  $A = 0$  for the vector interactions. The latter in the massless case produces by the Darwin's Lagrangian for electromagnetic interaction. Expression (40) agrees with the post-Newtonian Hamiltonians obtained within various approaches [1].

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