A Symmetric Treatment of Damped Harmonic Oscillator in Extended Phase Space

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Extended phase space (EPS) formulation of quantum statistical mechanics treats the ordinary phase space coordinates on the same footing and thereby permits the definition of the canonical momenta conjugate to these coordinates. The extended Lagrangian and extended Hamiltonian are defined in EPS by the same procedure as one does for ordinary Lagrangian and Hamiltonian. The combination of ordinary phase space and their conjugate momenta exhibits the evolution of particles and their mirror images in the same manner. As an example the resultant evolution equation in EPS for a damped harmonic oscillator DHO, is such that the energy dissipated by the actual oscillator is absorbed in the same rate by the image oscillator leaving the whole system as a conservative system. We use the EPS formalism to obtain the dual Hamiltonian of a damped harmonic oscillator, first proposed by Bateman, by a simple extended canonical transformations. The extended canonical transformations are capable of converting the damped system of actual and image oscillators to an undamped one, and transform the evolution equation into a simple form. The resultant equation is solved and the eigenvalues and eigenfunctions for damped oscillator and its mirror image are obtained. The results are in agreement with those obtained by Bateman. At last, the uncertainty relation are examined for above system.

1 Introduction

Although the formulation of dissipative systems from the first principles are cumbersome and little transparent, however, it is not so difficult to account for dissipative forces in classical mechanics in a phenomenological manner. Stokes' linear frictional force proportional to the velocity $\boldsymbol{v}$, Coulomb's friction $\sim \boldsymbol{v}/v$, Dirac's radiation damping $\sim \ddot{\boldsymbol{v}}$ and the viscous force $\sim \nabla^2\boldsymbol{v}$ are noteworthy examples in this respect. Unfortunately, the situation is much more complicated in quantum level (see Dekker [1], and the references there in). In his review article on classical and quantum mechanics of the damped harmonic oscillator, Dekker outlines that: "Although completeness is certainly not claimed, it is felt that the present text covers a substantial portion of the relevant work done during the last half century. All models agree on the classical dynamics ... however, the actual quantum mechanics of the various models reveals a considerable variety in fluctuation behavior. ... close inspection further shows that none of them ... are completely satisfactory in all respects". As an example of the dissipative systems, the DHOs is investigated through different approaches by different people. Caldirola [2] and Kanai [3] using the familiar canonical quantization procedure, obtained the Schrödinger equation which gives the eigenvalue and eignfunctions for damped oscillator. However the difficulty with this approach is that it violates the Heisenberg uncertainty relation in the long time limit. Another approach is the Schrödinger–Langevin method, which introduces a nonlinear wave equation for the evolution of the damped oscillator [4]. In this method the superposition principle is obviously violated. Using the Wigner equation, Dodonov and Manko [5] introduced the loss energy state for DHO as consequence of the Bateman dissipation, by introducing a dual Hamiltonian considered the evolution of the DHO in parallel with its mirror image [6]. In this method the energy dissipated by the actual oscillator of interest is absorbed at the same rate by the image oscillator. The image oscillator, in fact, plays the role of the physical reservoir. Therefore, the energy of the total system, as a closed one, is a constant of motion.

Here we use the EPS method [7] to investigate the evolution of the DHO. The method looks like the Bateman approach, however, the uncertainty principle, when looked upon from a different point of view, is not violated. That is, the extended uncertainty relation is satisfied for combination of actual and image oscillators, while reducing into ordinary uncertainty relations for actual and image oscillators, separately, in zero dissipation limit.

This paper is organized as follows. In Section 2, a review of the EPS formulation is given. In Section 3, we investigate the quantization procedure for the DHO. In Section 4, we use the path integral technique directly to calculate the exact propagators, and then the uncertainties of position and momentum for the actual and image oscillator system. Section 5 is devoted to concluding remarks.

2 A review of the EPS formulation

A direct approach to quantum statistical mechanics is proposed by Sobouti and Nasiri [7], by extending the conventional phase space and applying the canonical quantization procedure to extended quantities in this space. Assuming the phase space coordinates q and p to be independent variables on the virtual trajectories allows one to define momenta π_q and π_p, conjugate to q and p, respectively. This is done by introducing the extended Lagrangian

$$\mathcal{L}(q,p,\dot{q},\dot{p}) = -\dot{q}p - \dot{p}q + \mathcal{L}^q(q,\dot{q}) + \mathcal{L}^p(p,\dot{p}), \tag{1}$$

where $\mathcal{L}^q$ and $\mathcal{L}^p$ are the q and p space Lagrangians of the given system. Using equation (1) one may define the momenta, conjugate to q and p, respectively, as follows

$$\pi_q = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}^q}{\partial \dot{q}} - p, \tag{2}$$

$$\pi_p = \frac{\partial \mathcal{L}}{\partial \dot{p}} = \frac{\partial \mathcal{L}^p}{\partial \dot{p}} - q. \tag{3}$$

In the EPS defined by the set of variables $\{q,p,\pi_q,\pi_p\}$, one may define the extended Hamiltonian

$$\begin{aligned}\mathcal{H}(q,p,\pi_q,\pi_p) &= \dot{q}\pi_q + \dot{p}\pi_p - \mathcal{L} = H(p+\pi_q,q) - H(p,q+\pi_p)\\ &= \sum \frac{1}{n}\left\{\frac{\partial^n H}{\partial p^n}\pi_q^n - \frac{\partial^n H}{\partial q^n}\pi_p^n\right\},\end{aligned} \tag{4}$$

where $H(q,p)$ is the Hamiltonian of the system. Using the canonical quantization rule, the following postulates are outlined:

a) Let q, p, π_q and π_p be operators in Hilbert space X, of all square integrable complex functions, satisfying the following commutation relations

$$[\pi_q, q] = -i\hbar, \qquad \pi_q = -i\hbar\frac{\partial}{\partial q}, \tag{5}$$

$$[\pi_p, p] = -i\hbar, \qquad \pi_p = -i\hbar\frac{\partial}{\partial p}, \tag{6}$$

$$[q,p] = [\pi_q,\pi_p] = 0. \tag{7}$$

By virtue of equations (5)–(7), the extended Hamiltonian $\mathcal{H}$, will be an operator in X.

b) A state function $\chi(q,p,t) \in X$ is assumed to satisfy the following dynamical equation

$$i\hbar\frac{\partial\chi}{\partial t} = \mathcal{H}\chi = \left[H\left(p - i\hbar\frac{\partial}{\partial q}, q\right) - H\left(p, q - i\hbar\frac{\partial}{\partial p}\right)\right]\chi$$
$$= \sum\frac{1}{n}\left\{\frac{\partial^n H}{\partial p^n}\pi_q^n - \frac{\partial^n H}{\partial q^n}\pi_p^n\right\}\chi. \tag{8}$$

The general solution for this equation is

$$\chi(q,p,t) = \psi(q)\phi^*(p)e^{-\frac{i}{\hbar}qp}, \tag{9}$$

where $\psi(q)$ and $\phi(p)$ are the solutions of the Schrödinger equation in q and p space, respectively.

c) the averaging rule for an observable $O(q,p)$, a c-number operator in this formalism, is given as

$$\langle O(q,p)\rangle = \int O(q,p)\chi^*(q,p,t)dpdq. \tag{10}$$

For details of selection procedure of the admissible state functions, see Sobouti and Nasiri [7].

3 Damped harmonic oscillator in EPS

Extended Hamiltonian of equation (4) for undamped harmonic oscillator is given by

$$\mathcal{H} = \frac{1}{2}{\pi_q}^2 + p\pi_q - \frac{1}{2}{\pi_p}^2 - q\pi_p. \tag{11}$$

By a canonical transformation of the form

$$q_1 = q, \qquad \pi_{q_1} = -\pi_q - p, \qquad p_1 = p, \qquad \pi_{p_1} = -\pi_p - q,$$

equation (10) yields

$$\mathcal{H} = \frac{1}{2}\pi_{q_1}^2 + q_1^2 - \frac{1}{2}\pi_{p_1}^2 - \frac{1}{2}p_1^2. \tag{12}$$

This extended Hamiltonian evidently represents the subtraction of Hamiltonians of two independent identical oscillators, which is called actual and image oscillators [5]. The position q and momentum π_q denote the actual oscillator, while p and π_p denote the image oscillator. The minus sign has its origin in equation (4) and has an important role in this theory [7]. The following canonical transformation

$$q_2 = q_1, \qquad \pi_{q_2} = \pi_{q_1} - \lambda q_1, \qquad p_2 = p_1, \qquad \pi_{p_2} = \pi_{p_1} + \lambda p_1. \tag{13}$$

changes the extended Hamiltonian of an undamped harmonic oscillator into that of the damped one, i.e.

$$\mathcal{H}_2 = \frac{1}{2}\left\{\pi_{q_2}^2 + 2\lambda q_2\pi_{q_2} + \omega^2 q_2^2\right\} - \frac{1}{2}\left\{\pi_{p_2}^2 - 2\lambda p_2\pi_{p_2} + \omega^2 p_2^2\right\}, \tag{14}$$

where $\omega = 1 + i\lambda$. One further transformation generated by

$$F_2(q_2, p_2, \pi_{q_3}, \pi_{p_3}) = q_2\pi_{q_3}e^{-\lambda, t} + p_2\pi_{p_3}e^{\lambda t}, \tag{15}$$

finally leads to

$$\mathcal{H}_3 = \frac{1}{2}\left\{\pi_{q3}^2 e^{-2\lambda t} + \omega^2 q_3^2 e^{2\lambda t}\right\} - \frac{1}{2}\left\{\pi_{p3}e^{2\lambda t} + \omega^2 p_3^2 e^{-2\lambda t}\right\}. \tag{16}$$

The first part of the extended Hamiltonian in equation (16) is Caldirola–Kanai Hamiltonian, which is widely used to study the dissipation in quantum mechanics [3]. Using equation (16), the extended Hamilton equations [7] gives the following classical evolution equations for actual and image oscillators, respectively

$$\ddot{q_3} + 2\lambda\dot{q_3} + \omega^2 q_3 = 0, \tag{17}$$

and

$$\ddot{p_3} - 2\lambda\dot{p_3} + \omega^2 p_3 = 0. \tag{18}$$

Almost trivially, the energy dissipated by actual oscillator, with phase space coordinates (q_3, π_{q_3}) is completely absorbed at the same pace by the image oscillator with phase space coordinates (p_3, π_{p_3}).

To quantize the above system as usual, the dynamical variables (q_3, π_{q_3}) and (p_3, π_{p_3}) are considered as operators in a linear space. They obey the commutation relations in equations (5)–(7). The dynamical equation (8), now becomes

$$i\hbar\frac{\partial\chi}{\partial t} = \mathcal{H}\chi = \left(\frac{1}{2}\left\{\pi_{q_3}^2 e^{-2\lambda t} + \omega^2 q_3^2 e^{2\lambda t}\right\} - \frac{1}{2}\left\{\pi_{p_3}^2 e^{2\lambda t} + \omega^2 p_3^2 e^{-2\lambda t}\right\}\right)\chi. \tag{19}$$

By an infinitesimal canonical transformation which in quantum level corresponds to the following unitary transformation

$$U = \exp\left(\frac{i\lambda}{2\hbar}\left\{e^{2\lambda t}{q_4}^2 + e^{-2\lambda t}p_4^2\right\} + \frac{i\lambda t}{\hbar}\left\{q_4\pi_{q_4} - p_4\pi_{p_4}\right\}\right), \tag{20}$$

equation (19) may be written as

$$i\hbar\frac{\partial\chi}{\partial t} = \mathcal{H}\chi = \left(\frac{1}{2}\left\{-\hbar^2\frac{\partial^2}{\partial q_4^2} + \omega'^2{q_4}^2\right\} - \frac{1}{2}\left\{-\hbar^2\frac{\partial^2}{\partial p_4^2} + \omega'^2{p_4}^2\right\}\right)\chi. \tag{21}$$

where $\omega' = \omega + i\lambda$. The eigenvalues of equation (21) may be obtained as follows [7],

$$\mathcal{E}_{mn} = E_n - E_m = (n-m)\hbar\omega'. \tag{22}$$

The corresponding eigenfunctions are,

$$\begin{aligned}\chi_{mn}(q_4, p_4, t) &= U\chi_{mn}(q_3, p_3, t)\\ &= \exp\left(\frac{i\lambda}{2\hbar}\left\{e^{2\lambda t}{q_4}^2 + e^{-2\lambda t}{p_4}^2\right\}\right)\psi_m\left(e^{\lambda t}q_4\right)\phi_n^\star\left(e^{-\lambda t}p_4\right)e^{-\frac{i}{\hbar}p_4q_4},\end{aligned} \tag{23}$$

where $\psi_m(q)$ and $\phi_n(p)$ eigenfunctions of the harmonic oscillator in configuration and momentum space (Hermit functions). The result obtained above are in agreement with those obtained by Bateman [6]. However, here in contrast to the Bateman approach, the Heisenberg uncertainty relation is looked upon from a different point of view and is not violated. This is discussed in the next section using the eigenfunctions in equation (23).

4 Uncertainty relations for actual and image oscillators

In this section we calculate the uncertainties in position and momentum for the actual and the image oscillators. We calculate the extended propagator [8] for the combined actual and the image oscillators as follows

$$K(q,p,t,q_i,p_i,t_i) = \left(\frac{1}{2\pi i\hbar}\right)\left[\frac{\omega'}{\sin\omega'(t-t_i)}\right]$$
$$\times \exp\left[\frac{1}{2}\left(\frac{\omega' e^{\lambda(t+t_i)}}{\sin\omega'(t-t_i)}\right)\left\{e^{\lambda(t-t_i)}q^2\left(\cos\omega'(t-t_i) - \frac{\lambda}{\omega'}\sin(\omega'(t-t_i))\right)\right.\right.$$
$$\left.\left. + e^{-\lambda(t-t_i)}q_i^2\left(\cos\omega'(t-t_i) + \frac{\lambda}{\omega'}\sin(\omega'(t-t_i))\right) - 2qq_i\right\}\right]$$
$$\times \exp\left[\frac{1}{2}\left(\frac{\omega' e^{-\lambda(t+t_i)}}{\sin\omega'(t-t_i)}\right)\left\{e^{-\lambda(t-t_i)}p^2\left(\cos\omega'(t-t_i) + \frac{\lambda}{\omega'}\sin(\omega'(t-t_i))\right)\right.\right.$$
$$\left.\left. + e^{\lambda(t-t_i)}p_i^2\left(\cos\omega'(t-t_i) - \frac{\lambda}{\omega'}\sin(\omega'(t-t_i))\right) - 2pp_i\right\}\right]. \tag{24}$$

When $\lambda \to 0$, then equation (24) reduces to the familiar form of the undamped extended harmonic oscillator propagator [8]. We assume that the initial state function for combined system in ground state is $\chi_{00}(q,p,0) = \left(\pi\delta^2\right)^{-\frac{1}{2}}\exp\left(-\frac{q^2+p^2}{2\delta^2}\right)$, where δ is the width of the extended wave packet. Then one gets using equation (9)

$$\chi_{00}(q,p,t) = \int\int dq_i dp_i K(q,p,t,q_i,p_i,0)\chi_{00}(q_i,p_i,0)$$
$$= \left(\frac{\pi}{\delta^2}\right)\left[\frac{1}{2\delta^2} - \frac{i}{2}\frac{\omega'}{\hbar}\left(\frac{\cos\omega' t}{\sin\omega' t} + \frac{\lambda}{\omega'}\right)\right]^{-\frac{1}{2}}$$
$$\times\left(\frac{\omega' e^{\lambda t}}{2\pi i\hbar\sin\omega' t}\right)^{\frac{1}{2}}\exp\left[-\frac{q^2}{2}\left\{\frac{1}{\delta^2}e^{2\lambda t}\left(1+\left[\frac{1}{\delta^4}\left(\frac{\hbar}{\omega'}\right)^2 + 2\left(\frac{\lambda}{\omega'}\right)^2 - 1\right]\sin^2\omega'\right.\right.\right.$$
$$\left. + \frac{\lambda}{\omega'}\sin 2\omega' t\right)^{-1} - i\left\{\frac{\omega'}{\hbar}\frac{e^{2\lambda t}}{\sin\omega' t}\left[\cos\omega' t - \frac{\lambda}{\omega'}\sin\omega' t - \left(\cos\omega' t + \frac{\lambda}{\omega'}\sin\omega' t\right)\right]\right.$$
$$\left.\left.\left.\times\left[1+\frac{1}{\delta^4}\left(\frac{\hbar}{\omega'}\right)^2 + 2\left(\frac{\lambda}{\omega'}\right)^2 - 1\right]\sin 2\omega' t + \frac{\lambda}{\omega'}\sin 2\omega' t\right\}\right\}^{-1}\right]$$
$$\times\left[\frac{1}{2\delta^2} + \frac{i}{2}\frac{\omega'}{\hbar}\left(\frac{\cos\omega' t}{\sin\omega' t} + \frac{-\lambda}{\omega'}\right)\right]^{-\frac{1}{2}}\frac{-\omega' e^{-\lambda t}}{2\pi i\hbar\sin\omega' t}^{\frac{1}{2}}\exp\left[-\frac{p^2}{2}\left\{\frac{1}{\delta^2}e^{-2\lambda t}\right.\right.$$
$$\times\left(1+\left[\frac{1}{\delta^4}\left(\frac{\hbar}{\omega'}\right)^2 + 2\left(\frac{\lambda}{\omega'}\right)^2 - 1\right]\sin^2\omega' - \frac{\lambda}{\omega'}\sin 2\omega' t\right)^{-1}$$
$$+ i\left\{\frac{\omega'}{\hbar}\frac{e^{-2\lambda t}}{\sin\omega' t}\left[\cos\omega' t + \frac{\lambda}{\omega'}\sin\omega' t - \left(\cos\omega' t - \frac{\lambda}{\omega'}\sin\omega' t\right)\right]\right.$$
$$\left.\left.\left.\times\left[1+\frac{1}{\delta^4}\left(\frac{\hbar}{\omega'}\right)^2 + 2\left(\frac{\lambda}{\omega'}\right)^2 - 1\right]\sin 2\omega' t - \frac{\lambda}{\omega'}\sin 2\omega' t\right\}\right\}^{-1}\right]e^{-\frac{ipq}{\hbar}}. \tag{25}$$

Using equations (10) and (25) the uncertainties of positions and momenta we can calculate for the actual and the image oscillators as follows

$$\langle\Delta q\rangle = \frac{\delta}{\sqrt{2}}e^{-\lambda t}\left\{1+\left[\left(\frac{\sqrt{\hbar}}{\delta}\right)^4\left(\frac{1}{\omega'}\right)^2 + \left(\frac{\lambda}{\omega'}\right)^2 - 1\right]\sin^2\omega' t + \frac{\lambda}{\omega'}\sin 2\omega' t\right\}^{\frac{1}{2}}, \tag{26}$$

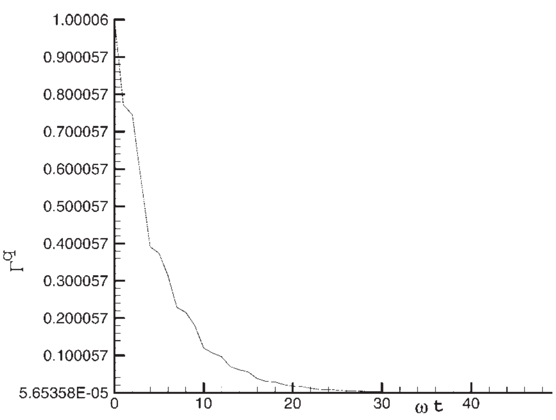

Figure 1. Uncertainty relation for actual oscillator as a function of time, for $\lambda = 0.1\omega$.

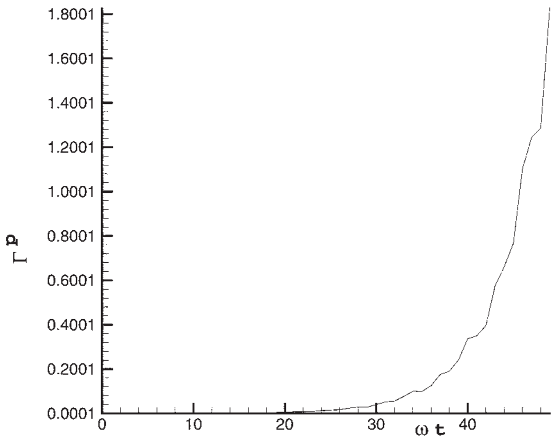

Figure 2. Uncertainty relation for image oscillator as a function of time, for $\lambda = 0.1\omega$.

$$\langle \Delta\pi_q \rangle = \frac{\delta}{\sqrt{2}} e^{-\lambda t} \left\{ 1 + \left[\left(\frac{\sqrt{\hbar}}{\delta} \right)^4 \left(\frac{1}{\omega'} \right)^2 + \left(\frac{\lambda}{\omega'} \right)^2 - 1 \right] \sin^2 \omega' t - \frac{\lambda}{\omega'} \sin 2\omega' t \right\}^{\frac{1}{2}}, \tag{27}$$

$$\langle \Delta p \rangle = \frac{\delta}{\sqrt{2}} e^{\lambda t} \left\{ 1 + \left[\left(\frac{\sqrt{\hbar}}{\delta} \right)^4 \left(\frac{1}{\omega'} \right)^2 + \left(\frac{\lambda}{\omega'} \right)^2 - 1 \right] \sin^2 \omega' t - \frac{\lambda}{\omega'} \sin 2\omega' t \right\}^{\frac{1}{2}}, \tag{28}$$

and

$$\langle \Delta\pi_p \rangle = \frac{\delta}{\sqrt{2}} e^{\lambda t} \left\{ 1 + \left[\left(\frac{\sqrt{\hbar}}{\delta} \right)^4 \left(\frac{1}{\omega'} \right)^2 + \left(\frac{\lambda}{\omega'} \right)^2 - 1 \right] \sin^2 \omega' t + \frac{\lambda}{\omega'} \sin 2\omega' t \right\}^{\frac{1}{2}}. \tag{29}$$

The above results for actual and image oscillators, in separate form, are in agreement with those obtained by Bateman. It is clear that the Heisenberg uncertainty relation is not valid for each oscillator independently. In fact for $\lambda \neq 0$ it is not possible to separate the oscillators, and the Heisenberg uncertainty relations would not hold for them separately, as shown in Figs. 1 and 2. In the presence of dissipation, i.e. $\lambda \neq 0$, the actual and image oscillators are coupled with each other and the area which is preserved during the evolution is $\Gamma(t) = \Delta\pi_q \Delta\pi_p \Delta q \Delta p$ in EPS. In contrast to the case of undamped harmonic oscillator, neither $\Gamma^q(t) = \Delta\pi_q \Delta q$ nor $\Gamma^p(t) = \Delta\pi_p \Delta p$ are preserved for DHO in q and p representation of quantum mechanics. This is shown in Fig. 3, where $\Gamma(t)$ is plotted versus time. It is clear that $\Gamma(t)$ never goes the zero. In other words, $\Gamma^q(t)$ and $\Gamma^p(t)$ which goes to zero and infinity in the long time limit, respectively, behave in such a manner that their product $\Gamma(t)$, always keeps a positive and finite value.

5 Concluding remarks

The EPS formulation of quantum mechanics seems to be a suitable method to handle the dissipative systems. Introducing the notion of mirror image oscillator beside the actual oscillator is a possibility that the extension of the ordinary phase space allows one to consider. This possibility introduces a conservative system of combined actual and image oscillators evolving together in the course of time. The eigenvalues and eigenfunctions obtained in this way is in agreement with those obtained by Bateman by introducing a dual Hamiltonian. However, the uncertainty principle, as one of the major problems on the way of the different approaches to the dissipative systems, including the Bateman approach, is valid in the extended form. This means

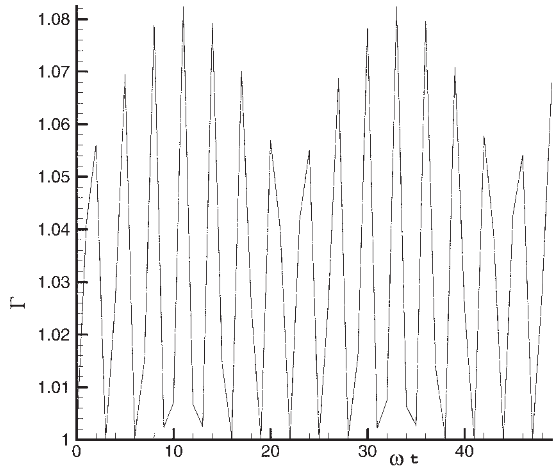

Figure 3. Uncertainty relation for combined system (actual and image oscillator) as a function of time, for $\lambda = 0.1\omega$.

that the dissipative systems can not be considered as isolated systems and it really interacts with its surrounding medium. The effect of the medium must be included as well. The mirror image oscillator plays the role of the interacting medium for the total conservative system, and the uncertainty relation is still valid.

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