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A Covering Second-Order Lagrangian for the Relativistic Top without Forces

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A parameter-homogeneous manifestly covariant Lagrangian of second order is considered, which covers the case of the free relativistic top at constraint manifold of constant acceleration. Relation to other models is discussed in brief.

1 Introduction

The interest to the description of quasi-classical physical particle by the means of some higher-order equations of motion and the methods of generalized Ostrohrads'kyj mechanics arose some 60 years ago and since then has been continuous [1–7]. Recently renewed attention was paid to such the models, which basically involve the notions of the first and higher curvatures of the particle's world line [8–12]. In most cases, people start with an *a priori* given higher order Lagrangian, and then try to interpret the dynamical system thus obtained as one describing the motion of quasi-classical spin (the relativistic top). Technical misunderstanding of two kinds happens to arise. First, certain nonholonomic constraints sometimes are imposed from the very beginning. These constraints are chosen in such a way as to ensure that the Lagrangian is in fact written in terms of the moving frame components [13]. But, as shown in [14], non-holonomic constraints require a more subtle approach. In particular, the constraint system does not retain the property of variationality any more. Second, sometimes the very tempting assumption of unit four-velocity vector is imposed *after* the variation procedure has already been carried out (cf. [15]). Such approach was quite justifiably criticized by several authors [16, 17]. On the other hand, there exist the established equations of Mathisson & Papapetrou [18] and of Dixon [19], which are believed to be well based from the point of view of physics. In 1945 Weyssenhoff [2] asserted, referring to one paper of Mathisson [20], "Even for a free particle in Galileian domains the equations of motion of a material particle endowed with spin do not coincide with the Newtonian laws of motion; there remains an additional term depending on the internal angular momentum or spin of the particle, which *raises the order of these differential equations to three*". We add to this that the procedure of *complete elimination* of spin variables in fact raises the order of the differential equations to *four*. In the present note this fourth order differential equation will be shown to follow from Dixon's form of the relativistic top equation of motion and in case of flat space-time a Lagrange function will be proposed which produces the world lines of thus governed spinning particle without any preliminary constraints being imposed *before* the variation procedure in undertaken. A constraint of *constant curvature* must be imposed *after* the variation, and this is why we call the corresponding Lagrange function a *covering* Lagrangian.

2 Relativistic top

To start from the lowest possible order let us recall the Dixon equations of the quasi-classical spinning particle in the gravitational field:

$$\dot{\mathcal{P}}^\alpha = \frac{1}{2} R^\alpha{}_{\beta\gamma\delta} u^\beta \mathcal{S}^{\gamma\delta}, \qquad \dot{\mathcal{S}}^{\alpha\beta} = \mathcal{P}^\alpha u^\beta - \mathcal{P}^\beta u^\alpha. \tag{1}$$

This system (1) does not prescribe any preferable way of parametrization along the world line of the particle.

It was proved in [21] and announced in [22] that under the so-called auxiliary condition of Pirani

$$u_\beta \mathcal{S}^{\alpha\beta} = 0, \tag{2}$$

equations (1), (2) are equivalent to the following system of equations (3), (4), and (5)

$$\varepsilon_{\alpha\beta\gamma\delta}\ddot{u}^\beta u^\gamma s^\delta - 3\frac{\dot{u}_\beta u^\beta}{\|\boldsymbol{u}\|^2}\varepsilon_{\alpha\beta\gamma\delta}\dot{u}^\beta u^\gamma s^\delta - m\left(\|\boldsymbol{u}\|^2\dot{u}_\alpha - \dot{u}_\beta u^\beta u_\alpha\right) = \frac{\|\boldsymbol{u}\|^2}{2}\varepsilon_{\mu\nu\gamma\delta}R_{\alpha\beta}{}^{\mu\nu}u^\beta u^\gamma s^\delta, \tag{3}$$

$$\|\boldsymbol{u}\|^2\dot{s}^\alpha + s_\beta\dot{u}^\beta\, u^\alpha = 0, \tag{4}$$

$$s_\alpha u^\alpha = 0. \tag{5}$$

The correspondence between the skewsymmetric spin tensor $\mathcal{S}^{\alpha\beta}$ and spin four-vector s^α under the assumption that we recognize Pirani's condition is given by

$$s_\alpha = \frac{1}{2\|\boldsymbol{u}\|}\,\varepsilon_{\alpha\beta\gamma\delta}u^\beta S^{\gamma\delta}, \qquad S_{\alpha\beta} = \frac{1}{\|\boldsymbol{u}\|}\,\varepsilon_{\alpha\beta\gamma\delta}u^\gamma s^\delta.$$

Equation (3) in flat space-time was considered from variational point of view in [21] and some Lagrange functions for it were offered in [23].

As promised, from now on we put $R_{\alpha\beta}{}^{\mu\nu} = 0$ and proceed to eliminate the variable s^α (in fact, a four-vector constant quantity). To facilitate the calculations, it is appropriate to chose the would line parametrization in the usual way: $\|\boldsymbol{u}\| = 1$. Then we get immediately that (3) takes on the shape ("$*$" denotes the dual tensor)

$$*\,\ddot{\boldsymbol{u}}\wedge\boldsymbol{u}\wedge\boldsymbol{s} + m\dot{\boldsymbol{u}} = \boldsymbol{0} \tag{6}$$

and possesses the first integral $k^2 = \dot{\boldsymbol{u}}^2$, which is nothing but the squared first curvature of the world line.

Now contract the above vector equation with the tensor $*\,\boldsymbol{u}\wedge\boldsymbol{s}$ and remember of (5) to obtain after some algebraic manipulations

$$\boldsymbol{s}^2\left(\ddot{\boldsymbol{u}} + k^2\boldsymbol{u}\right) = -m\,*\,\dot{\boldsymbol{u}}\wedge\boldsymbol{u}\wedge\boldsymbol{s}.$$

Differentiating and then substituting the right hand side from (6), we finally obtain

$$\dddot{\boldsymbol{u}} + \left(k^2 - \frac{m^2}{\boldsymbol{s}^2}\right)\dot{\boldsymbol{u}} = 0. \tag{7}$$

Now let us return to equations (1) and recall the standard fact that under Pirani's condition (2) the particle's momentum $\boldsymbol{\mathcal{P}}$ may be expressed in terms of spin tensor $\mathcal{S}^{\alpha\beta}$, or, equivalently, in terms of spin for-vector $\boldsymbol{s}$

$$\boldsymbol{\mathcal{P}} = \frac{m}{\|\boldsymbol{u}\|}\,\boldsymbol{u} + \frac{1}{\|\boldsymbol{u}\|^3}\,*\,\dot{\boldsymbol{u}}\wedge\boldsymbol{u}\wedge\boldsymbol{s},$$

where $m = \frac{\boldsymbol{\mathcal{P}}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|}$ is a constant of motion, and that the square momentum

$$\boldsymbol{\mathcal{P}}^2 = m^2 - k^2\boldsymbol{s}^2 + \frac{1}{\|\boldsymbol{u}\|^6}\left[(\dot{\boldsymbol{u}}\cdot\boldsymbol{s})\,\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{s})\,\dot{\boldsymbol{u}}\right]^2 = m^2 - k^2\boldsymbol{s}^2$$

by virtue of (2) is a constant of motion too. Thus denoting $\omega^2 = -\frac{\boldsymbol{\mathcal{P}}^2}{\boldsymbol{s}^2}$, we finally obtain the desired fourth-order equation for the free relativistic top:

$$\dddot{\boldsymbol{u}} + \omega^2\dot{\boldsymbol{u}} = 0 \tag{8}$$

3 Hamilton–Ostrohrads'kyj approach

Let us again notify that we tend to set a parameter-invariant variational problem in order to get the world lines without any additional parametrization. Recall the general formula for the first curvature of the world line in arbitrary parametrization

$$k = \frac{\|\boldsymbol{u} \wedge \dot{\boldsymbol{u}}\|}{\|\boldsymbol{u}\|^3} \tag{9}$$

and consider the following Lagrange function:

$$\mathcal{L} = \frac{1}{2}\|\boldsymbol{u}\| \left(k^2 + A\right). \tag{10}$$

This Lagrange function (10) constitutes a parameter-homogeneous variational problem because it satisfies the Zermelo conditions:

$$\left(\boldsymbol{u}\boldsymbol{.}\frac{\partial}{\partial \boldsymbol{u}} + 2\,\dot{\boldsymbol{u}}\boldsymbol{.}\frac{\partial}{\partial \dot{\boldsymbol{u}}}\right)\mathcal{L} = \mathcal{L}, \qquad \boldsymbol{u}\boldsymbol{.}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}} = 0. \tag{11}$$

Variational equations are given by

$$-\dot{\boldsymbol{\wp}} = \boldsymbol{0}, \tag{12}$$

where

$$\boldsymbol{\wp} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{u}} - \left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}}\right)^{\boldsymbol{\cdot}}.$$

Now, one can calculate the quantity $\dot{\boldsymbol{\wp}}$ and *afterwards* set $\|\boldsymbol{u}\| = 1$, thus benefiting from the parameter homogeneity of equation (12). We get for (12):

$$\dddot{\boldsymbol{u}} + \left(\frac{3}{2}\,\dot{\boldsymbol{u}}^2 - A\right)\dot{\boldsymbol{u}} + 3\,(\ddot{\boldsymbol{u}}\boldsymbol{\cdot}\dot{\boldsymbol{u}})\,\boldsymbol{u} = \boldsymbol{0}. \tag{13}$$

Now, on the surface $k = k_0$ equation (13) will coincide with (8) if we put

$$A = \frac{3}{2}\,{k_0}^2 - \omega^2.$$

This completes the proof, as asserted in [24].

To pass to the canonical formalism, it is necessary to introduce the parametrization by time, setting $x^0 = t$, $u^0 = 1$, and denoting $\frac{dx^i}{dt} = v^i$. In this coordinates formula (10) suggests the following expression for the Lagrange function:

$$L = \frac{1}{2}\sqrt{1 + \boldsymbol{v}^2}\left(k^2 + A\right), \qquad k^2 = \frac{\boldsymbol{v}'^2 + (\boldsymbol{v}' \times \boldsymbol{v})^2}{(1 + \boldsymbol{v}^2)^3}. \tag{14}$$

Generalized Hamilton function H is expressed in terms of $\boldsymbol{v}$ and the couple of momenta

$$\boldsymbol{p}' = \frac{\partial L}{\partial \boldsymbol{v}'}, \qquad \boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} - \frac{d}{dt}\,\boldsymbol{p}', \tag{15}$$

namely,

$$H = \boldsymbol{p}\boldsymbol{.}\boldsymbol{v} + \boldsymbol{p}'\boldsymbol{.}\boldsymbol{v}' - L.$$

It is possible to find the inverse of the generalized Legendre transformation (15) and after some laborious calculating efforts the generalized Hamiltonian reads:

$$H = \boldsymbol{p}\boldsymbol{.}\boldsymbol{v} + \frac{1}{2}\left(1 + \boldsymbol{v}^2\right)^{3/2}\left(\boldsymbol{p}'^{\boldsymbol{2}} + (\boldsymbol{p}'\boldsymbol{.}\boldsymbol{v})^2\right) - \frac{A}{\sqrt{1 + \boldsymbol{v}^2}}.$$

4 Concluding notes

1. Equation (8) was known to Riewe [15], but its deduction directly from (1) or from the Mathisson–Papapetrou equations [18] apparently was not obvious.

2. By means of the formula $kk_2k_3 = \|\boldsymbol{u}\wedge\dot{\boldsymbol{u}}\wedge\ddot{\boldsymbol{u}}\wedge\dddot{\boldsymbol{u}}\|$, which presents the relationship between the successive curvatures of a curve (in natural parametrization), we see immediately, that all the extremals of (10) have zero third curvature, and in terms of the space-like world line it means that the particle evolves in a plane.

3. In [21] we proved by means of generalized Ostrohrads'kyj momenta approach, that every one of the successive curvatures of a curve, taken as the Lagrange function, produces the extremals with this same curvature being the constant of motion. This was also observed by Arodź for the first curvature [9]. But the problem of the simultaneous conservation of all the curvatures, i.e. the variational description of helices, remains open (cf. [25]).

4. Surprisingly enough, the Lagrange function (10) in fact coincides with one, considered by Bopp in [1] for the motion of a charged particle in electromagnetic field (in part, not including the external four-potential itself). That equations (1) in their differential prolongation cover both the Mathisson–Papapetrou equations of spinning particle and the Lorentz–Dirac equations of self-radiating particle, was already noted in [23] in relation to the prediction of Barut [26]. This gives still more grounds to call (10) the *covering* Lagrangian.

5. Following the ideas of [6] we considered in [27] some non-local transformations which leave invariant the exact form of the action integral

$$\int \sqrt{\epsilon^2 d\tau^2 - d\alpha^2} = \int \mathcal{L}_\epsilon d\tau, \tag{16}$$

where $d\alpha$ measures the rotation of the tangent to the world line during the increment $d\tau$ of the proper time along it, so the curvature $k = \frac{d\alpha}{d\tau}$. There was an attempt to interpret these non-local transformations (linear in α and τ) as such that explain the transition between the uniformly accelerated frames of reference in special relativity. Treating in quite formal way the variables α and τ as independent, one may stay hoping that the variation of (16) will produce the world lines of constant curvature (i.e. constant acceleration). On the other hand, looking more closely at the Lagrange function

$$\mathcal{L}_\epsilon = \sqrt{\epsilon^2 - k^2}, \tag{17}$$

immediately leads to the concept of maximal acceleration $\epsilon = c^{7/2}G^{(-1/2)}\hbar^{-1/2} = 3/5{\cdot}10^{52}\mathrm{m/sec}^2$ [28].

Two shortcomings spring up. First, the Lagrange function (17), viewed as a higher-order Lagrangian, does not correspond to constant curvature world lines. Second, the variational problem is not parameter-independent, at least because $\mathcal{L}_\epsilon$, with k given by (9), does not satisfy the Zermelo conditions (11). The Lagrangian (10) is free of these shortcomings.

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