

# Positive Conjugacy for Simple Dynamical Systems

*Tatiana Yu. MAISTRENKO*

*Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01601 Kyiv-4, Ukraine*  
 E-mail: *tanya@imath.kiev.ua*

In the article the question of topological conjugacy is considered, which provides sufficient condition for the algebras isomorphism. The concept of positive conjugacy for some classes of simple dynamical systems is presented.

## 1 Introduction

It is well-known that dynamical systems play important role in the representation theory of  $C^*$ -algebras. In the book by Yu.S. Samoilenko and V.L. Ostrovskii (see [1]) some results with respect to connection between the theory of representations of  $*$ -algebras given by generators and relations with, generally speaking, non-bijective dynamical systems were presented. In recent paper (see [2]) the issue of description of isomorphism classes of  $C^*$ -algebras associated with  $SU \cap \mathcal{F}_{2^n}$ -mappings has been considered. The question of classification of  $C^*$ -algebras connected with dynamical systems up to isomorphism leads to studying conjugacy of dynamical systems on the set of positive orbits. In the present paper this question is considered for simple dynamical systems.

## 2 Topological conjugacy for simple unimodal dynamical systems

First we recall the necessary material from [4, 5, 6] and the results about conjugacy for  $SU \cap \mathcal{F}_{2^n}$ -mappings (from [3]).

Mapping  $f \in C(I, I)$ ,  $I = [0, 1]$  is called unimodal if there is unique extreme point  $c \in (0, 1)$  and  $f$  is homeomorphism on the intervals  $J_1 = [0, c]$ ,  $J_2 = [c, 1]$ . We consider unimodal mappings  $f \in C(I, I)$  such that  $f(0) = f(1) = 0$  and extreme point  $c$  is point of maximum. Let  $f \in C^3(I, I)$ . Schwarzian derivative is defined by the formula

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2, \quad \text{for } x \text{ such that } f'(x) \neq 0.$$

By  $SU$  we mean a class of unimodal mappings on interval  $I$  with  $Sf(x) < 0$  for all  $x$  different from  $c$ .

Dynamical system is called simple if every its trajectory is periodic or asymptotically periodic. A simple dynamical system can have only cycles of periods  $2^k$ ,  $k = 0, 1, 2, \dots$  [4, p. 64]. By necessary and sufficient condition of a simplicity of dynamical system is  $\text{Per } f = \bigcup_{n=1}^{\infty} \{x : f^n(x) = x\}$  is closed. Denote  $\mathcal{F}_n = \{f \in C(I, I) : \text{Per } f = \text{Fix } f^n\}$ . The class  $\mathcal{F}_n$  consists of mappings for which the period of each cycle is not greater than  $n$  and is a divisor of  $n$ . In fact  $n$  can be only the power of 2. It is known that for  $f \in C^{(1)}(I, I)$ :  $f \in \mathcal{F}_{2^k} \Leftrightarrow (f, I)$  is a simple dynamical system.

Cycle  $B = \{\beta_1, \dots, \beta_m\}$  is called attractive if there is a neighborhood  $U$  of  $B$  such that  $f(U) \subseteq U$  and  $\bigcap_{i>0} f^i(U) = B$ . Cycle  $B$  is called repellent if there is a neighborhood  $U$  such that for every  $x \in U \setminus B$  there is integer  $k > 0$  and  $f^{(k)}(x) \notin U$ .

By the positive orbit of dynamical system  $(f, I)$  we will mean a sequence  $\delta = (x_k)_{k \in \mathbb{Z}}$  such that  $f(x_k) = x_{k+1}$  and  $x_k > 0$  for all integer  $k$ . Unilateral positive orbit is a sequence  $\delta = (x_k)_{k \in \mathbb{N}}$  (Fock orbit) such that  $x_1 = 0$  and  $f(x_k) = x_{k+1}$ ,  $x_k > 0$  for  $k > 1$  or  $\delta = (x_{-k})_{k \in \mathbb{N}}$  (anti-Fock orbit) such that  $x_{-1} = 0$  and  $f(x_k) = x_{k+1}$ ,  $x_k > 0$  for  $k < -1$ . Define  $\text{Orb}_+(f, I)$  be the set of all positive orbits on interval  $I$ .

We say that two maps  $f, g : X \rightarrow X$  are topologically conjugate if there exists a homeomorphism  $h : X \rightarrow X$  such that  $h \circ f = g \circ h$ . This implies that  $h \circ f^n = g^n \circ h$  for every integer  $n$ .

Further we will need some concepts of symbolic dynamics. By the address of a point  $x \in I$  we mean the value  $A(x) = \begin{cases} J_s, & \text{if } x \in J_s \text{ and } x \neq c, \quad s = 1, 2; \\ c, & \text{if } x = c. \end{cases}$

The itinerary of a point  $x$  is the sequence of addresses

$$A_f(x) = (A(x), A(f(x)), A(f^2(x)), \dots) = (A_0, A_1, A_2, \dots).$$

Define the sign of an interval  $J_s$  as  $\varepsilon(J_s) = (-1)^{s+1}$ ,  $\varepsilon(c) = 0$  and put  $\theta_f(x) = (\theta_0, \theta_1, \theta_2, \dots)$  where  $\theta_0 = \varepsilon_0$ ,  $\theta_1 = \varepsilon_0 \varepsilon_1$ ,  $\dots$ ,  $\theta_n = \varepsilon_0 \varepsilon_1 \dots \varepsilon_n$ ,  $\dots$ ,  $\varepsilon_i = \varepsilon(A_i)$ . By the dynamical coordinate of a point  $x \in I$  we mean a formal power series  $\theta(x) = \sum_{i=0}^{\infty} \theta_i(x) t^i$ . Series  $\nu_f = \theta(c^-) = \lim_{x \uparrow c} \theta(x)$  is called kneading invariant of  $f$ .

We will need the following theorem from [7]:

**Theorem 1.** *Let  $f, g \in SU$ ,  $\nu_f = \nu_g$ . Then*

- a) *if the series  $\nu_f$  is nonperiodic then  $f$  and  $g$  are topologically conjugate;*
- b) *if the series  $\nu_f$  is periodic of period  $n$  then  $f$  and  $g$  have an attractive or neutral trajectory of period  $n$  or  $n/2$ ; moreover  $f$  and  $g$  are topologically conjugate when these trajectories are of the same type (i.e. simultaneously either attractive or neutral) and corresponding points of these trajectories have the same dynamical coordinates.*

**Theorem 2.** *Let  $(f_1, I), (f_2, I)$  be dynamical systems such that  $(f_i, I) \in SU \cap \mathcal{F}_{2^n}$ ,  $i = \{1, 2\}$ ,  $c_i$  is the greatest point of local maximum of function  $f_i^{2^n}$ . Then  $\text{sign}(f_1^{2^n}(c_1) - c_1) = \text{sign}(f_2^{2^n}(c_2) - c_2)$  if and only if  $f$  and  $g$  are topologically conjugate.*

**Proof.** We need only to verify that conditions the previous theorem are satisfied. As we can uniquely define a series  $\nu_f$  when  $A_f(c^-)$  is given we consider  $A_f(c^-)$  instead of  $\nu_f$ .

At first let us demonstrate the statement in case  $n = 0$ , i.e.  $f \in SU \cap \mathcal{F}_1$ . In this case the dynamical system has one or two fixed points and does not have cycles of period greater or equal 2. It is easy to see that if  $f$  has only one fixed point ( $s_0 = 0$ ) then  $A_f(c^-) = (+, +, \dots)$ ,  $A_f(0) = (+, +, \dots)$  and  $\text{sign}(f(c) - c) = -1$ . Assume that  $f$  has two fixed points  $s_0 = 0$  and  $0 < s_1 < 1$ . In this case  $s_0$  is repellent,  $s_1$  is attractive and there are three alternatives:

- 1)  $s_1 < c$  (i.e.  $f(c) - c < 0$ );
- 2)  $s_1 = c$  (i.e.  $f(c) - c = 0$ );
- 3)  $s_1 > c$  (i.e.  $f(c) - c > 0$ ).

In the case 1) and 2):  $A_f(c^-) = (+, +, \dots)$ , hence for every  $f_1, f_2 \in SU \cap \mathcal{F}_1$  fulfilling condition 1) or 2):  $\nu_{f_1} = \nu_{f_2} = (+, +, \dots)$ . If  $s_1 > c$  then  $A_f(c^-) = (+, -, -, \dots)$ . Hence for every  $f_1, f_2 \in SU \cap \mathcal{F}_1$  satisfying condition 3):  $\nu_{f_1} = \nu_{f_2} = (+, -, -, \dots)$ . And also if  $f_1$  and  $f_2$  simultaneously satisfy one of the conditions 1), 2), 3) then dynamical coordinates of their attractive periodic trajectories coincide:

- 1)  $A_f(s_1) = (+, +, \dots)$ ,  $\theta_f(s_1) = (+, +, \dots)$ ;
- 2)  $A_f(s_1) = (0, 0, \dots)$ ,  $\theta_f(s_1) = (0, 0, \dots)$ ;
- 3)  $A_f(s_1) = (-, -, \dots)$ ,  $\theta_f(s_1) = (-, +, -, +, \dots)$ .

Thus by previous theorem if  $f_1$  and  $f_2$  simultaneously satisfy one of the conditions 1), 2), 3) then they are topologically conjugate.

We now turn to the case  $n \geq 1$ . Let us consider the mapping  $g(x) = f^2(x)$  and let  $a$  and  $d$  be preimages of  $s_1$  under  $g$  such that  $a < s_1 < d$  and  $[a, d]$  contains no other preimages of  $s_1$  (in other words  $a$  and  $d$  are the closest preimages to  $s_1$ ). Let  $I_1 = [a, s_1]$ ,  $I_2 = [s_1, d]$ . By simple calculations we have  $f(I_1) \subseteq I_2$ ,  $f(I_2) \subseteq I_1$ , i.e. intervals  $I_1, I_2$  are invariant under function  $g$ .

Let us note that dynamical system  $(g, I_2)$  is simple with unimodal  $g$  and  $Sf < 0$  when  $x \neq c'$  ( $c'$  be the point of local maximum of function  $g$  on  $I_2$ ). Moreover  $(g, I_2)$  is conjugate to dynamical system  $(\tilde{f}, [0, 1])$  where  $\tilde{f} = f^2((d - s_1)x + s_1) - s_1$  and satisfying all the conditions of the theorem.

Let us consider the behavior of the trajectory  $f^m(c^-)$ ,  $m \geq 0$ .  $f^0(c^-) = c^- \in J_1$  hence  $\varepsilon(f^0(c^-)) = +1$ . Since  $f(I_1) \subseteq I_2$ ,  $f(I_2) \subseteq I_1$  we have

$$f^m(c^-) \subset I_2, \quad \text{if } m = 2k + 1, \quad f^m(c^-) \subset I_1, \quad \text{if } m = 2k,$$

hence  $\varepsilon(f^{2k+1}(c^-)) = -1$ . In the case  $m = 2k$  we have  $f^{2k}(c^-) \subset I_1$  and since  $c = f(c')$  then

$$f(f^{2k+1}(c^-)) \subset I_1 \cap J_1, \quad \text{if } f^{2k+1}(c^-) > c', \quad f(f^{2k+1}(c^-)) \subset I_1 \cap J_2, \quad \text{if } f^{2k+1}(c^-) < c'.$$

Thus the sign of  $f^{2k}(c^-)$  is calculated by the recursive formula  $\varepsilon(f^{2k}(c^-)) = -\varepsilon(g^k(c'^-))$ . And if  $c_{2^n}$  is the greatest point of local maximum of function  $f^{2^n}$  then the value  $A_f(c^-)$  is uniquely defined by  $A_{f^{2^n}}(c_{2^n}^-)$ , i.e. by value  $\text{sign}(f^{2^n}(c_{2^n}^-) - c_{2^n}^-)$ .

And also it follows from above that the dynamical coordinate of attractive periodical trajectory also uniquely defined by this value. ■

**Corollary 1.** *For every  $n$  there is no more than three isomorphism classes of enveloping  $C^*$ -algebras (see [2]).*

**Corollary 2.** *Let  $(f_1, I), (f_2, I) \in SU \cap \mathcal{F}_{2^n}$  and  $\text{sign}(\mu(B_{2^n}^1)) = \text{sign}(\mu(B_{2^n}^2))$ . Then  $f_1 \sim f_2$ .*

### 3 Positive conjugacy

Define the support of dynamical system  $(f, I)$  to be the union of positive orbits  $X = X(f, I) = \{x \in \delta \mid \delta \in \text{Orb}_+(f, I)\}$ .

**Definition 1.** We will say that two maps  $f_1 : [0, a_1] \rightarrow [0, a_1]$  and  $f_2 : [0, a_2] \rightarrow [0, a_2]$  are positively conjugate if they are topologically conjugate on their supports, i.e.  $X_1 = X(f_1, [0, a_1])$ ,  $X_2 = X(f_2, [0, a_2])$  and there exist a homeomorphism  $\varphi : X_1 \rightarrow X_2$  such that  $\varphi \circ f_1 = f_2 \circ \varphi$ .

**Proposition 1.** *If  $f_1 : [0, a_1] \rightarrow [0, a_1]$  and  $f_2 : [0, a_2] \rightarrow [0, a_2]$  are topologically conjugate then they are positively conjugate.*

**Proof.** Since  $f_1$  and  $f_2$  are conjugate then there exist a homeomorphism  $\varphi : [0, a_1] \rightarrow [0, a_2]$  and  $\varphi \circ f_1(x) = f_2 \circ \varphi(x)$  for all  $x \in [0, a_1]$ . Let us show that  $\varphi(X_1) = X_2$ . Indeed, if  $x \in X_1$  then there exist  $\delta \in \text{Orb}_+(f_1, [0, a_1])$  such that  $x \in \delta$ . Since  $\varphi(\delta) \in \text{Orb}_+(f_2, [0, a_2])$  (where  $\varphi(\delta) = (\varphi(x_k))_{k \in \mathbb{Z}}$ ) we have  $\varphi(x) \in X_2$ , i.e.  $\varphi(X_1) \subseteq X_2$ . Considering  $\varphi^{-1}$  analogously we get  $\varphi^{-1}(X_2) \subseteq X_1$  and consequently  $X_2 \subseteq \varphi(X_1)$ . Obviously  $\varphi|_{X_1}$  is homeomorphism on  $X_2$ . ■

The converse statement to proposition 1 is not true in general. Let us consider the notion of positive conjugacy for the class  $SU \cap \mathcal{F}_{2^n}$ -maps. In the case  $n = 0$  two maps can be positively conjugate but not topologically conjugate.

**Proposition 2.** *Let  $f_1, f_2 \in SU \cap \mathcal{F}_1$ ,  $c_i$  is the point of maximum of function  $f_i$ . Then  $f_1$  and  $f_2$  are positively conjugate iff one of the following statements holds:*

- 1)  $\text{sign}(f_1(c_1) - c_1) = \text{sign}(f_2(c_2) - c_2) \neq 0$ ;
- 2)  $\text{sign}(f_1(c_1) - c_1) \leq 0, \text{sign}(f_2(c_2) - c_2) = 0$ .

**Proof.** If  $f_i \in SU \cap \mathcal{F}_1$  then  $f_i$  has one or two fixed points and does not have any cycles of period more than one (see theorem 1.2 in [3]). Theorem 2 implies 1. In the case 2)  $f_1$  has support  $X_1 = [0, s_1]$ , where  $s_1$  is fixed point of  $f_1$  that is nonequal to 0. Mapping  $f_2$  has support  $X_2 = [0, c_2]$ . Since the functions  $f_1$  on  $X_1$  and  $f_2$  on  $X_2$  are monotonely increasing and fixed points are the ends of the intervals  $X_1$  and  $X_2$  correspondingly hence  $f_1$  and  $f_2$  are topologically conjugate on their supports. Thus  $f_1$  and  $f_2$  are positively conjugate. ■

**Theorem 3.** *Let  $f_1, f_2 \in SU \cap \mathcal{F}_{2^n}$ ,  $n \geq 1$ . If maps  $f_1$  and  $f_2$  are positively conjugate then they are topologically conjugate.*

**Proof.** Let us prove that if  $f_1$  and  $f_2$  are not topologically conjugate then they are not positively conjugate. There are two cases when  $f_1$  and  $f_2$  are not topologically conjugate: 1)  $\text{sign}(f_1^{2^n}(c_1) - c_1) = 0, \text{sign}(f_2^{2^n}(c_2) - c_2) \neq 0$  ( $c_i$  is the greatest point of local maximum of function  $f_i^{2^n}$ ). The support  $X_i$  of the dynamical system  $(f_i, I)$  is interval  $[0, M_i]$ , where  $M_i = \max_{x \in I} f_i(x)$ . Let us note that  $M_1$  is periodic point of period  $2^n$  of function  $f_1$  but  $M_2$  is not periodic. Therefore  $f_1$  and  $f_2$  are not conjugate on their supports. 2)  $\text{sign}(f_1^{2^n}(c_1) - c_1) < 0, \text{sign}(f_2^{2^n}(c_2) - c_2) > 0$ . Let us consider the function  $f_i^{2^n}$  on the interval  $I_i^{2^n}$  that is bounded by the two greatest preimages of fixed point  $s_i$  under  $f_i^{2^n}$ . Mapping  $f_1^{2^n}$  is monotone on the support of  $(f_1^{2^n}, I_1^{2^n})$  but  $f_2^{2^n}$  is not monotone on the support of  $(f_2^{2^n}, I_2^{2^n})$ . Hence  $f_1$  and  $f_2$  are not positively conjugate. ■

Let us consider the notion of positive conjugacy for mappings  $f(x) = 1 + ax - bx^2, a > 0, b > 0$ . Define  $F_+^{-1}, F_-^{-1}$  to be two branches of inverse to  $F$  mapping such that  $F_-^{-1}(0) = 0$ .

**Definition 2.** Let  $(F, I) \in SU$ . By  $p$ -truncated mapping (or truncated by  $p$ ) of  $F$  we mean the mapping  $f_p(x) = F(x + p) - p, p \in (0, c), x \in I_p$ , where  $I_p = [0, F_+^{-1}(p) - p]$ .  $F$  truncated by interval  $P \subset (0, c)$  is the family of mappings  $f_p(x)$ , where  $p \in P$ .

**Remark 1.** In general  $f_p(I_p) \not\subseteq I_p$ , i.e.  $(f_p, I_p)$  is not always a dynamical system.

Obviously, we can consider the mapping  $f(x) = 1 + ax - bx^2 (a > 0, b > 0)$  as truncated mapping of some  $F \in SU$ .

**Proposition 3.** *Let  $(f_1, [0, a_1]), (f_2, [0, a_2])$  are dynamical systems such that  $f_i$  is  $p_i$ -truncated mapping of  $F_i \in SU \cap \mathcal{F}_1$ . Then  $f_1$  and  $f_2$  are positively conjugate.*

**Proof.** Since dynamical system  $(f_i, [0, a_i])$  does not have repellent points then the set  $\text{Orb}_+(f_i)$  consists of Fock-orbit only. Hence the support of such dynamical system is a sequence of points of the interval  $[0, a_i]$  which converge to attractive fixed point  $s_i$ . Let  $\delta_1 = \{x_k \mid x_0 = 0, x_{k+1} = f_1(x_k), k \geq 0\}$  is Fock orbit of  $f_1, \delta_2 = \{y_k \mid y_0 = 0, y_{k+1} = f_2(y_k), k \geq 0\}$  is Fock orbit of  $f_2$ .

The mapping  $\varphi : \delta_1 \rightarrow \delta_2$  defined by the formula  $\varphi(x_k) = y_k$  is homeomorphism since it preserves convergence in both directions and satisfies the condition  $f_2 \circ \varphi(x_k) = \varphi \circ f_1(x_k)$  for all  $k \geq 0$ . Thus  $f_1$  and  $f_2$  are positively conjugate. ■

Further we will consider only dynamical systems truncated by interval  $[0, F^2(M)]$ . This condition guarantees an absence of anti-Fock orbits.

**Theorem 4.** *Let  $(F, I) \in SU \cap \mathcal{F}_2, F^2(c') - c' < 0$  ( $c'$  is the greatest point of local maximum of the fuction  $F^2$ ). Then there exist a countable number of positive conjugacy classes of truncated mappings of  $F$ .*

**Proof.** Define  $B = \{\beta_1, \beta_2\}$  to be the cycle of period 2 of function  $F$ ,  $\beta_1 < \beta_2$ . Let us consider the sequence of intervals  $T_k$ ,  $k \geq 0$ :

$$T_{2k} = F_-^{-k}([\beta_1, \beta_2]), \quad T_{2k+1} = F_-^{-k}((F_-^{-1}(\beta_2), \beta_1)).$$

Interval  $T = [\beta_1, \beta_2]$  has the property:

$$\text{if } x \in T \text{ then } F^n(x) \in T \text{ for all } n \geq 1 \quad (1)$$

(since  $F^2(c') - c' < 0$  then every trajectory of  $F^2$  is attracted to the fixed point  $\beta_2$  monotonely) and

$$\text{for every } x \in T \text{ there is } x' \in T \text{ such that } F(x') = x \quad (2)$$

since  $\max_{x \in I} F(x) > \beta_2$ .

It is evidently that for all  $x \in T_{2k}$  there is  $m \geq 0$  such that  $f^m(x) \in T$ . If  $x \in T_{2k+1}$  then for every  $m \geq 0$ :  $f^m(x) \notin T$ . Any truncated mapping  $f$  of  $F$  has only one repellent cycle (fixed point  $s$ ) and attractive cycle of period 2. Consequently, taking into account properties (1), (2) we can see that for every  $x \in T$  there is  $\delta \in \text{Orb}_+(f)$  such that  $x \in \delta$ . Thus  $T \subset X$ .

Further let us consider Fock-orbits. If  $f(0) \in T_{2k+1}$ ,  $k \geq 0$  then support  $X$  of  $f$  is the set  $T \cup \{x_n \mid x_0 = 0, x_{n+1} = f(x_n), n \geq 0\}$ . Since  $F$  is monotone on  $T$  then all mappings truncated by  $T_{2k+1}$ ,  $k \geq 0$  are positively conjugate. If  $f$  is truncated by  $T_{2k}$ ,  $k \geq 1$  then the support of  $f$  be the set  $T \cup \{x_n \mid x_0 = 0, x_{n+1} = f(x_n), x_n \notin T\}$ . Thus  $k$  defines a class of positive conjugacy of mappings truncated by interval  $T_{2k}$ . ■

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