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Realizations of Real 4-Dimensional Solvable Decomposable Lie Algebras

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We construct a complete set of inequivalent realizations of real 4-dimensional solvable decomposable Lie algebras in vector fields on a space of an arbitrary (finite) number of variables.

Realizations of Lie algebras in vector fields are applied, in particular, for integrating of ordinary differential equations, group classification of partial differential equations, classification of gravity fields of a general form with respect to motion groups or groups of conformal transformations. In spite of importance for applications, the problem of complete description of realizations have not been solved even for cases when either the dimension of algebras or the dimension of realization space is a fixed small integer. An exception is Lie's classification of all possible Lie groups of point and contact transformations acting on a two-dimensional complex space without fixed points [1], which is equivalent to classification of all possible realizations of Lie algebras in vector fields on a two-dimensional complex space.

The necessary step to classify realizations of low-dimensional Lie algebras is classification of these algebras, i.e. classification of possible commutative relations between basis elements. All the possible complex Lie algebras of dimension less than 4 were already obtained by S. Lie [2]. L. Bianchi investigated three-dimensional real Lie algebras [3]. Complete and correct classification of four-dimensional real Lie algebras was firstly obtained by G.M. Mubarakzyanov [4].

C. Wafo Soh and F.M. Mahomed [5] used Mubarakzyanov's results to classify realizations of three- and four-dimensional real Lie algebras in the space of three variables and to describe systems of two second-order ODEs admitting real four-dimensional real symmetry Lie algebras, but unfortunately their paper contains a number of misprints and incorrect statements. Therefore, this classification cannot be considered as complete.

Preliminary classification of realizations of solvable three-dimensional Lie algebras in the space of any (finite) number of variables was given in [6]. In this paper we present a complete set of inequivalent realizations for real 4-dimensional solvable decomposable Lie algebras in vector fields on a space of an arbitrary (finite) number n of variables $x = (x_1, x_2, \ldots, x_n)$. Analogous results for indecomposable algebras in the case $n = 4$ have been obtained in [7].

The technics of classification is the following. For each algebra A from Mubarakzyanov's classification of abstract four-dimensional Lie algebras [4] we find the automorphism group $G(A)$ and the set of *megaideals* of A, i.e. the ideals invariant with respect to $G(A)$. Knowledge of the megaideals is important to construct realizations and to prove their inequivalence in a simpler way. Then, we take four linearly independent vector fields of the general form $e_i = \xi^{ia}(x)\partial_a$, where $\operatorname{rank}(\xi^{ia}) = 4$, $\partial_a = \partial/\partial x_a$, and demand from them to satisfy commutative relations of A. (Our notions of low-dimensional algebras, choice of their basis elements, and, consequently, the form of commutative relations coincide with Mubarakzyanov's ones.) As a result, we obtain a system of first-order PDEs for the coefficients ξ^{ia} and integrate it, considering all the possible cases. For each case we transform the found solution to the simplest form, using local diffeomorphisms of the space of x and automorphisms of A.

Consideration is essentially simplified if it is taken into account that any four-dimensional algebra contains a three-dimensional ideal. We can use classification of realizations of three-dimensional algebras with respect to local diffeomorphisms of the space of x, extending them to realizations of four-dimensional algebras by means of joining the fourth vector field. Then we obtain a system of first-order PDEs only for the coefficients ξ^{4a}.

Table 1. Realizations of real decomposable solvable four-dimensional Lie algebras.

Algebra	N	Realization
$A_{3.1}\oplus A_1$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2$
	2	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_3,\ \partial_2$
$[e_2,e_3]=e_1$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+\varphi(x_4)\partial_2+\psi(x_4)\partial_3,\ \partial_2$
	4	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1$
	5	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1$
	6	$\partial_1,\ \partial_3,\ x_3\partial_1+\varphi(x_2)\partial_3,\ x_2\partial_1$
$A_{3.2}\oplus A_1$	1	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+\partial_3,\ \partial_4$
	2	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+\partial_3,\ x_4\partial_3$
$[e_1,e_3]=e_1,$	3	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2,\ \partial_3$
$[e_2,e_3]=e_1+e_2$	4	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+\partial_3,\ x_4e^{x_3}(x_3\partial_1+\partial_2)$
	5	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+\partial_3,\ e^{x_3}(x_3\partial_1+\partial_2)$
	6	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+\partial_3,\ e^{x_3}\partial_1$
	7	$\partial_1,\ x_2\partial_1,\ x_1\partial_1-\partial_2,\ \partial_3$
	8	$\partial_1,\ x_2\partial_1,\ x_1\partial_1-\partial_2,\ x_3e^{-x_2}\partial_1$
	9	$\partial_1,\ x_2\partial_1,\ x_1\partial_1-\partial_2,\ e^{-x_2}\partial_1$
$A_{3.3}\oplus A_1$	1	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2+\partial_3,\ \partial_4$
	2	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2+\partial_3,\ x_4\partial_3$
$[e_1,e_3]=e_1,$	3	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2,\ \partial_3$
$[e_2,e_3]=e_2$	4	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2+\partial_3,\ e^{x_3}(\partial_1+x_4\partial_2)$
	5	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2+\partial_3,\ e^{x_3}\partial_1$
	6	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+\partial_3,\ \partial_4$
	7	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+\partial_3,\ x_4\partial_3$
	8	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+\partial_3,\ \varphi(x_2)\partial_3$
	9	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+\partial_3,\ e^{x_3}\partial_1$
$A_{3.4}\oplus A_1$	1	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ \partial_4$
	2	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ x_4\partial_3$
$[e_1,e_3]=e_1,$	3	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2,\ \partial_3$
$[e_2,e_3]=ae_2,$	4	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ e^{x_3}\partial_1+x_4e^{ax_3}\partial_2$
$-1\le a<1,\ a\ne 0$	5	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ e^{x_3}\partial_1+e^{ax_3}\partial_2$
	6	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ e^{x_3}\partial_1$
	7	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ e^{ax_3}\partial_1,\quad 0<\lvert a\rvert<1$
	8	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+(1-a)x_2\partial_2,\ \partial_3$
	9	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+(1-a)x_2\partial_2,\ x_3\lvert x_2\rvert^{\frac{1}{1-a}}\partial_1$
	10	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+(1-a)x_2\partial_2,\ \lvert x_2\rvert^{\frac{1}{1-a}}\partial_1$
$A_{3.5}\oplus A_1$	1	$\partial_1,\ \partial_2,\ (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\ \partial_4$
	2	$\partial_1,\ \partial_2,\ (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\ x_4\partial_3$
$[e_1,e_3]=be_1-e_2,$	3	$\partial_1,\ \partial_2,\ (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2,\ \partial_3$
$[e_2,e_3]=e_1+be_2,$	4	$\partial_1,\ \partial_2,\ (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\ x_4e^{bx_3}(\cos x_3\partial_1-\sin x_3\partial_2)$
$b\ge 0$	5	$\partial_1,\ \partial_2,\ (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\ e^{bx_3}(\cos x_3\partial_1-\sin x_3\partial_2)$
	6	$\partial_1,\ x_2\partial_1,\ (b-x_2)x_1\partial_1-(1+x_2^2)\partial_2,\ \partial_3$
	7	$\partial_1,\ x_2\partial_1,\ (b-x_2)x_1\partial_1-(1+x_2^2)\partial_2,\ x_3\sqrt{1+x_2^2}\,e^{-b\arctan x_2}\partial_1$
	8	$\partial_1,\ x_2\partial_1,\ (b-x_2)x_1\partial_1-(1+x_2^2)\partial_2,\ \sqrt{1+x_2^2}\,e^{-b\arctan x_2}\partial_1$

Continuation of of Table 1.

Algebra	N	Realization
$A_{2.2} \oplus 2A_1$	1	$\partial_1,\ x_1\partial_1+\partial_4,\ \partial_2,\ \partial_3$
	2	$\partial_1,\ x_1\partial_1+x_4\partial_2+x_5\partial_3,\ \partial_2,\ \partial_3$
$[e_1,e_2]=e_1$	3	$\partial_1,\ x_1\partial_1+x_4\partial_2+\psi(x_4)\partial_3,\ \partial_2,\ \partial_3$
	4	$\partial_1,\ x_1\partial_1,\ \partial_2,\ \partial_3$
	5	$\partial_1,\ x_1\partial_1+x_3\partial_3,\ \partial_2,\ x_3\partial_1+x_4\partial_2$
	6	$\partial_1,\ x_1\partial_1+x_3\partial_3,\ \partial_2,\ x_3\partial_1$
	7	$\partial_1,\ x_1\partial_1+\partial_4,\ \partial_2,\ x_3\partial_2$
	8	$\partial_1,\ x_1\partial_1+x_4\partial_2,\ \partial_2,\ x_3\partial_2$
	9	$\partial_1,\ x_1\partial_1+\theta(x_3)\partial_2,\ \partial_2,\ x_3\partial_2$
	10	$\partial_1,\ x_1\partial_1+x_2\partial_2+x_3\partial_3,\ x_2\partial_1,\ x_3\partial_1$
$A_{2.1} \oplus A_{2.1}$	1	$\partial_1,\ x_1\partial_1+\partial_3,\ \partial_2,\ x_2\partial_2+\partial_4$
	2	$\partial_1,\ x_1\partial_1+\partial_3,\ \partial_2,\ x_2\partial_2+x_4\partial_3$
$[e_1,e_2]=e_1$,	3	$\partial_1,\ x_1\partial_1+\partial_3,\ \partial_2,\ x_2\partial_2+C\partial_3,\ \lvert C\rvert\le 1$
$[e_3,e_4]=e_3$	4	$\partial_1,\ x_1\partial_1+x_3\partial_2,\ \partial_2,\ x_2\partial_2+x_3\partial_3$
	5	$\partial_1,\ x_1\partial_1,\ \partial_2,\ x_2\partial_2$
	6	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_2\partial_1,\ -x_2\partial_2+\partial_3$
	7	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_2\partial_1,\ -x_2\partial_2$
$4A_1$	1	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4$
	2	$\partial_1,\ \partial_2,\ \partial_3,\ x_4\partial_1+x_5\partial_2+x_6\partial_3$
	3	$\partial_1,\ \partial_2,\ \partial_3,\ x_4\partial_1+x_5\partial_2+\lambda(x_4,x_5)\partial_3$
	4	$\partial_1,\ \partial_2,\ \partial_3,\ x_4\partial_1+\varphi(x_4)\partial_2+\psi(x_4)\partial_3$
	5	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ x_5\partial_1+x_6\partial_2$
	6	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ x_5\partial_1+\theta(x_3,x_4,x_5)\partial_2$
	7	$\partial_1,\ \partial_2,\ x_3\partial_1+\varphi(x_3,x_4)\partial_2,\ x_4\partial_1+\psi(x_3,x_4)\partial_2$
	8	$\partial_1,\ \partial_2,\ x_3\partial_1+\varphi(x_3)\partial_2,\ \lambda(x_3)\partial_1+\psi(x_3)\partial_2$
	9	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ x_4\partial_1$
	10	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ \lambda(x_2,x_3)\partial_1$
	11	$\partial_1,\ x_2\partial_1,\ \varphi(x_2)\partial_1,\ \psi(x_2)\partial_1$

We plan to publish our results on complete classification of realizations for real Lie algebras of dimensions less than 5 in vector fields on a space of an arbitrary (finite) number of variables in the near future, giving detailed description of the technics of classification and a number of applications of obtained realizations to theory of differential invariants, integrating of ODEs and group classification of PDEs.

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