

Symmetries and Dynamical Symmetry Breaking of General n -Dimensional Self-Consistently Renormalized Spinor Diagrams

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Using the self-consistent renormalization (SCR), a careful study of complicated tangle of problems associated on the one hand with renormalizations and on the other with symmetries conservation, their breaking, the Ward identities (WIs) behavior, the Schwinger terms contributions (STCs), and quantum anomalies is performed for some set of UV-divergent Feynman amplitudes (FAs) connected with general mass-anisotropic spinor diagrams in any space-time dimension $n = 2r + \delta_n$, $\delta_n = 0, 1$. It is shown that the WIs involving SCR FAs do retain (or imitate) the canonical WIs (CWIs). In this context quantum anomalies reveal themselves either as an oversubtraction effect for a non-chiral case and for chiral limits (in these cases the STCs are zero) or as nonzero STCs for the chiral case. Effective formulae for general quantum corrections (QCs) to the CWIs and primitive “daughter reduction identities” (DRIs) are derived for any dimension n . For an anisotropic case ($m_1 \neq m_2$, $m_l \neq 0$), the QCs are the zero degree homogeneous functions of masses and are expressed in terms of hypergeometric functions ${}_2F_1$. For the degenerate nonchiral case ($m_1 = m_2 = m \neq 0$), these QCs either are equal to zero for vector WIs or reduce to mass-independent expressions for axial-vector WIs. The Schwinger-Johnson anomaly for $n = 2$ is a particular case of general formulas obtained. Conditions under which the nonzero STCs exist are obtained and the role of the STCs in the QCs are revealed. The behavior of FAs and QCs in the chiral case ($m = 0$) and in the symmetric chiral limit ($m \rightarrow 0$) is different. In the chiral case only the “left-handed vector” current may be conserved and hence it may be more fundamental than vector or axial-vector currents.

1 Introduction

In the perturbation theory, quantum anomalies manifest themselves as breakdown of the canonical WIs (CWIs) at a level of regular (finite) values of FAs involved in them. Therefore, modes and interpretations of these CWIs violations are extremely important as for the quantum field theory itself and for physical applications [1–8]. Despite the large number of papers which have been written on quantum anomalies, surprisingly many facets of this problem have not been adequately described, if at all. We hope to clarify some obscure points in these violations by employing the SCR [9–14] to general spinor diagram FAs, being very important objects for physical applications [15, 16], and to illustrate possibilities of the SCR. Subjects which will be raised here are: i) mass dependence of quantum anomalies; ii) distinction between chiral and chiral limit anomalies; iii) relation between the Schwinger terms contributions and quantum anomalies. Previously, we have carried out similar investigation for the spinor triangle FAs [17–22] in which new features of quantum anomalies have been exhibited. Recall that the SCR is an effective realization of the Bogoliubov–Parasiuk R-operation [23–26] which is complemented with recurrence, compatibility, and differential relations fixing a renormalization arbitrariness of the R-operation in some universal way based on mathematical properties of FAs only.

2 General spinor diangle amplitudes and their identities

2.1. The main Feynman amplitude corresponding to the spinor diangle graph of the most general kind (different masses, arbitrary Clifford structure of vertices, n -dimensional world with (q, p) -signature of a nondegenerate metric g , where q and p are respectively the number of negative and positive squares in g , i.e. $q + p = n = 2r + \delta_n$, $\delta_n = 0, 1$) looks as follows:

$$\begin{aligned} I^{\gamma_1 \gamma_2}(m, k) &:= \int_{-\infty}^{\infty} (d^n p) \delta(p, k) \frac{\text{tr}[\gamma_1(m_1 + \hat{p}_1) \gamma_2(m_2 + \hat{p}_2)]}{(\mu_1 - p_1^2)(\mu_2 - p_2^2)}, \\ (d^n p) &:= d^n p_1 d^n p_2, \quad \hat{p}_l := \gamma^\sigma p_{l\sigma}, \quad m := (m_1, m_2), \quad k := (k_1, k_2), \\ \delta(p, k) &:= \delta_1(-k_1 + p_2 - p_1) \delta_2(-k_2 + p_1 - p_2), \quad \mu_l := m_l^2 - i\epsilon_l. \end{aligned} \quad (1)$$

The matrices γ_i , γ_σ , I_g , act in the N_g -dimensional space of the faithful representation $\pi(g)$ of the lowest dimension for the Clifford algebra $Cl(g)_{\mathbb{K}}$, $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , with $\gamma^\sigma \in \Lambda^1(g)$, $\sigma = 1, \dots, n$, being the generating elements of the $Cl(g)_{\mathbb{K}}$ -algebra in its matrix representation $\pi(g)$, i.e. $\gamma^\sigma \gamma^\tau + \gamma^\tau \gamma^\sigma = 2g^{\sigma\tau} I_g$; γ_i , $i = 1, 2$, are, as a rule, some k -degree ($k = 0, 1, \dots, n$) homogeneous elements of the $Cl(g)_{\mathbb{K}}$ -algebra in the $\pi(g)$ -representation or some linear combination of such elements; I_g is the N_g -dimensional unit matrix. The n -degree element $\gamma^* \in \Lambda^n(g)$, i.e. the dual conjugation matrix, with the obvious but important properties:

$$\begin{aligned} \gamma^* &:= \gamma^1 \gamma^2 \dots \gamma^n, \quad (\gamma^*)^2 = \varepsilon(g) I_g, \quad \gamma^\sigma \gamma^* = (-1)^{n+1} \gamma^* \gamma^\sigma, \quad \sigma = 1, \dots, n, \\ \varepsilon(g) &:= (-1)^q (-1)^{n(n-1)/2} = (-1)^{\varkappa(\varkappa+1)/2}, \quad \varkappa := (q - p) \pmod{8}, \end{aligned} \quad (2)$$

is the natural analog of the Dirac γ^5 -matrix. For more details on properties of the γ^* -matrix and on the self-consistent version of the dimensional regularization with the γ^* -matrix see [27, 28].

2.2. The UV-divergent FAs (1) satisfy formally the *canonical Ward identities* (CWIs):

$$\begin{aligned} k_{1\mu} I^{(\gamma^\mu \gamma) \gamma_2}(m, k) &= D_1^{\dot{\gamma} \gamma_2}(m, k) \\ &= (-1)^{\pi_1} P_1^{\gamma \gamma_2}(m, k) - P_2^{\gamma \gamma_2}(m, k) + (m_2 - (-1)^{\pi_1} m_1) I^{\gamma \gamma_2}(m, k), \\ k_{2\alpha} I^{\gamma_1 (\gamma^\alpha \gamma)}(m, k) &= D_2^{\gamma_1 \dot{\gamma}}(m, k) \\ &= (-1)^{\pi_2} P_2^{\gamma_1 \gamma}(m, k) - P_1^{\gamma_1 \gamma}(m, k) + (m_1 - (-1)^{\pi_2} m_2) I^{\gamma_1 \gamma}(m, k). \end{aligned} \quad (3)$$

Here the quantities $D_1^{\dot{\gamma} \gamma_2}(m, k)$, $D_2^{\gamma_1 \dot{\gamma}}(m, k)$, $P_l^{\gamma'_1 \gamma'_2}(m, k)$, $l = 1, 2$, $\gamma'_i = \gamma_i$ or γ , $i = 1, 2$, are similar to the main amplitude $I^{\gamma_1 \gamma_2}(m, k)$ and differ from it only in polynomials of the integrand:

$$I^{\gamma_1 \gamma_2}(m, k) \longleftrightarrow \text{tr} [\mathcal{I}^{\gamma_1 \gamma_2}(m, p)] := \text{tr} [\gamma_1 (m_1 + \hat{p}_1) \gamma_2 (m_2 + \hat{p}_2)]; \quad (4)$$

$$D_1^{\dot{\gamma} \gamma_2}(m, k) \longleftrightarrow \text{tr} [\mathcal{D}_1^{\dot{\gamma} \gamma_2}(m, p)] := (p_2 - p_1)_\mu \text{tr} [\mathcal{I}^{(\gamma^\mu \gamma) \gamma_2}(m, p)],$$

$$D_2^{\gamma_1 \dot{\gamma}}(m, k) \longleftrightarrow \text{tr} [\mathcal{D}_2^{\gamma_1 \dot{\gamma}}(m, p)] := (p_1 - p_2)_\alpha \text{tr} [\mathcal{I}^{\gamma_1 (\gamma^\alpha \gamma)}(m, p)]; \quad (5)$$

$$P_1^{\gamma'_1 \gamma'_2}(m, k) \longleftrightarrow \text{tr} [\mathcal{P}_1^{\gamma'_1 \gamma'_2}(m, p)] := \text{tr} [\gamma'_1 (m_1^2 - p_1^2) \gamma'_2 (m_2 + \hat{p}_2)],$$

$$P_2^{\gamma'_1 \gamma'_2}(m, k) \longleftrightarrow \text{tr} [\mathcal{P}_2^{\gamma'_1 \gamma'_2}(m, p)] := \text{tr} [\gamma'_1 (m_1 + \hat{p}_1) \gamma'_2 (m_2^2 - p_2^2)]. \quad (6)$$

In equations (3) the vector CWIs ($\gamma = I_g$) and the axial-vector CWIs ($\gamma = \gamma^*$) are represented in the uniform manner. The factors

$$(-1)^{\pi_i} = \begin{cases} 1, & \text{if } \gamma = I_g, \forall n, \text{ or } \gamma = \gamma^*, n = 2r + 1; \\ -1, & \text{if } \gamma = \gamma^*, n = 2r, \end{cases}$$

stem from the commutation relations $\gamma^\sigma \gamma = (-1)^\pi \gamma \gamma^\sigma$, $\sigma = 1, \dots, n$.

The quantities $D_1^{\dot{\gamma}\gamma_2}(m, k)$, $D_2^{\gamma_1\dot{\gamma}}(m, k)$, correspond to divergencies of current density T-products $\langle 0 | \partial_{1\mu} T(J^{O_1}(x_1) J^{O_2}(x_2)) | 0 \rangle$, $\langle 0 | \partial_{2\alpha} T(J^{O_1}(x_1) J^{O_2}(x_2)) | 0 \rangle$, where $\partial_{i\sigma} \equiv \partial/\partial x_i^\sigma$, $J^{O_i}(x_i) = : \bar{\Psi}(x_i) O_i \Psi(x_i) :$, $O_i := h_i \otimes \gamma_i$, and $h_i = \tau_i \otimes \lambda_i$ are matrices specifying flavor-color structure of current densities. The quantities $P_l^{\dot{\gamma}'\gamma'_2}(m, k)$ are associated with current density commutators and consequently with possible contributions of the Schwinger terms in them.

2.3. Let us consider the obvious identities:

$$P_{l\epsilon}^{\gamma_1\gamma_2}(m, k) = \bar{P}_{l\epsilon}^{\gamma_1\gamma_2}(\bar{m}_{(l)}, k), \quad l = 1, 2, \quad \bar{m}_{(1)} \equiv m_2, \quad \bar{m}_{(2)} \equiv m_1, \quad (7)$$

$$\frac{\partial}{\partial \mu_l} P_{l\epsilon}^{\gamma_1\gamma_2}(m, k) = 0, \quad l = 1, 2, \quad (8)$$

in which the simple idea of cancelling the equal factors in factorized polynomials in numerators and the denominator of integrands is used. Therefore, these identities are named as *reduction identities* (RIs). The *nonreduced* FAs in the l.h.s. of equation (7) are defined as:

$$\left[\begin{array}{c} P_{1\epsilon}^{\gamma_1\gamma_2}(m, k) \\ P_{2\epsilon}^{\gamma_1\gamma_2}(m, k) \end{array} \right] := \int_{-\infty}^{\infty} \frac{(d^n p) \delta(p, k)}{(\mu_1 - p_1^2)(\mu_2 - p_2^2)} \left[\begin{array}{c} \text{tr} [\gamma_1 (\mu_1 - p_1^2) \gamma_2 (m_2 + \hat{p}_2)] \\ \text{tr} [\gamma_1 (m_1 + \hat{p}_1) \gamma_2 (\mu_2 - p_2^2)] \end{array} \right] \quad (9)$$

and the *reduced* FAs in the r.h.s. of equation (7) are:

$$\left[\begin{array}{c} \bar{P}_{1\epsilon}^{\gamma_1\gamma_2}(\bar{m}_{(1)}, k) \\ \bar{P}_{2\epsilon}^{\gamma_1\gamma_2}(\bar{m}_{(2)}, k) \end{array} \right] := \int_{-\infty}^{\infty} (d^n p) \delta(p, k) \left[\begin{array}{c} \text{tr} [\gamma_1 \gamma_2 (m_2 + \hat{p}_2)] / (\mu_2 - p_2^2) \\ \text{tr} [\gamma_1 (m_1 + \hat{p}_1) \gamma_2] / (\mu_1 - p_1^2) \end{array} \right], \quad (10)$$

which are well known as ‘‘tadpole’’ amplitudes.

The RIs (7) are closely related to the CWIs (3). Indeed, due to equation (6) the amplitudes $P_l^{\dot{\gamma}'\gamma'_2}(m, k)$ involving in equations (3) are very similar to the nonreduced FAs in equation (9). The only difference between them is the $i\epsilon_l$ -terms in numerator polynomials of integrands in equation (9). But exactly these terms permit to perform identical cancellations of factors and to obtain independent on $\mu_l = m_l^2 - i\epsilon_l$ expressions that are reflected clearly in equation (8).

The RIs (7) induce primitive *daughter* RIs (DRIs) via decompositions involving: i) the numeric Clifford tensors $\text{tr}[\gamma_1 \gamma_2 m_2]$, $\text{tr}[\gamma_1 \gamma_2 \gamma_\sigma]$, $\text{tr}[\gamma_1 m_1 \gamma_2]$, $\text{tr}[\gamma_1 \gamma_\sigma \gamma_2]$; ii) the irreducible tensor structures constructed by means of independent external momenta (e.g., k_2 , or k_1). Altogether for general spinor diangles, there are 4 primitive DRIs taken two $\forall l = 1, 2$.

3 General spinor diangle amplitudes and identities in the SCR

3.1. The amplitude $I^{\gamma_1\gamma_2}(m, k)$ has the divergence index $\nu = n - 2$ whereas the amplitudes $D_1^{\dot{\gamma}\gamma_2}(m, k)$, $D_2^{\gamma_1\dot{\gamma}}(m, k)$, $P_l^{\dot{\gamma}'\gamma'_2}(m, k)$, $P_{l\epsilon}^{\gamma_1\gamma_2}(m, k)$, $\bar{P}_{l\epsilon}^{\gamma_1\gamma_2}(\bar{m}_{(l)}, k)$, $l = 1, 2$, have the divergence index $\nu + 1 = n - 1$. The regular values for all of them are obtained according to the SCR [9, 10, 11, 13]. Some of them have been given in [29] as the net results. Here once again, we present only some of them, but in such a form permitting to evaluate all unavailable amplitudes.

So, the regular values $(R^\nu T)^{\gamma_1\gamma_2}(m, k)$, $(R^{\nu+1} D_1)^{\dot{\gamma}\gamma_2}(m, k)$, and $(R^{\nu+1} P_{1\epsilon})^{\gamma_1\gamma_2}(m, k)$ of the amplitudes given by equations (1), (5), and (9) have the following α -parametric integral representation, shown here in a form best suitable for general FAs:

$$\left[\begin{array}{c} (R^\nu T)^{\gamma_1\gamma_2}(m, k) \\ (R^{\nu+1} D_1)^{\dot{\gamma}\gamma_2}(m, k) \\ (R^{\nu+1} P_{1\epsilon})^{\gamma_1\gamma_2}(m, k) \end{array} \right] = (2\pi)^n \delta(k) b(g) \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \sum_{s=0}^3 \sum_{j=0}^{[s/2]} \left[\begin{array}{c} \text{tr} [\mathcal{I}_{s_j}^{\gamma_1\gamma_2}](R^\nu \mathcal{F})_{s_j} \\ \text{tr} [\mathcal{D}_{1;s_j}^{\dot{\gamma}\gamma_2}](R^{\nu+1} \mathcal{F})_{s_j} \\ \text{tr} [\mathcal{P}_{1\epsilon;s_j}^{\gamma_1\gamma_2}](R^{\nu+1} \mathcal{F})_{s_j} \end{array} \right]. \quad (11)$$

The integration measure $d\mu(\alpha)$, the integration region Σ^1 , the metric dependent constant $b(g)$, and the overall δ -function $\delta(k)$ are defined as

$$d\mu(\alpha) := \delta(1 - \alpha_1 - \alpha_2) d\alpha_1 d\alpha_2, \quad \Sigma^1 := \{\alpha_l | \alpha_l \geq 0, \forall l, \alpha_1 + \alpha_2 = 1\},$$

$$b(g) := (\pi^{n/2} i^p) / (2\pi)^n, \quad \delta(k) := \delta(-k_1 - k_2), \tag{12}$$

and p is the number of positive squares in a space-time metric g .

The explicit form of the basic functions $(R^\nu \mathcal{F})_{sj}$, $(R^{\nu+1} \mathcal{F})_{sj}$, and the determining numbers ν_{sj} , λ_{sj} , ν_{sj}^1 , λ_{sj}^1 , and ω appearing in them are as follows:

$$(R^\nu \mathcal{F})_{sj} := M_\epsilon^{\omega+j} \Gamma(\lambda_{sj}) / \Gamma(2 + \nu_{sj}) Z_\epsilon^{1+\nu_{sj}} {}_2F_1(1, \lambda_{sj}; 2 + \nu_{sj}; Z_\epsilon),$$

$$\nu_{sj} := [(\nu - s)/2] + j, \quad \lambda_{sj} := -\omega - j + 1 + \nu_{sj}, \quad \omega := n/2 - 2, \tag{13}$$

$$(R^{\nu+1} \mathcal{F})_{sj} := M_\epsilon^{\omega+j} \Gamma(\lambda_{sj}^1) / \Gamma(2 + \nu_{sj}^1) Z_\epsilon^{1+\nu_{sj}^1} {}_2F_1(1, \lambda_{sj}^1; 2 + \nu_{sj}^1; Z_\epsilon),$$

$$\nu_{sj}^1 := [(\nu + 1 - s)/2] + j, \quad \lambda_{sj}^1 := -\omega - j + 1 + \nu_{sj}^1, \quad \omega := n/2 - 2. \tag{14}$$

The $[s/2]$, $[(\nu - s)/2]$, and $[(\nu + 1 - s)/2]$ in equations (11), (13)–(14) are integral parts of the numbers $s/2$, $(\nu - s)/2$, and $(\nu + 1 - s)/2$ respectively. The subscripts (s, j) of the basic functions $(R^\nu \mathcal{F})_{sj}$ and $(R^{\nu+1} \mathcal{F})_{sj}$ just mean that these functions are attached to the homogeneous k -polynomials $\mathcal{I}_{sj}^{\gamma_1 \gamma_2}$, $\mathcal{D}_{1;sj}^{\dot{\gamma}_1 \dot{\gamma}_2}$, and $\mathcal{P}_{1\epsilon;sj}^{\gamma_1 \gamma_2}$ of the degree $s - 2j$ in external momenta. The latter are α -images of the homogeneous p -polynomials $\mathcal{I}_s^{\gamma_1 \gamma_2}(m, p)$, $\mathcal{D}_{1;s}^{\dot{\gamma}_1 \dot{\gamma}_2}(m, p)$, and $\mathcal{P}_{1\epsilon;s}^{\gamma_1 \gamma_2}(m, p)$ of the degree s appearing in $\mathcal{I}^{\gamma_1 \gamma_2}(m, p)$, $\mathcal{D}_1^{\dot{\gamma}_1 \dot{\gamma}_2}(m, p)$, and $\mathcal{P}_{1\epsilon}^{\gamma_1 \gamma_2}(m, p)$ given by equations (4)–(6) and equation (9):

$$\begin{aligned} \mathcal{I}_{00}^{\gamma_1 \gamma_2} &:= \gamma_1 m_1 \gamma_2 m_2, & \mathcal{P}_{1\epsilon;00}^{\gamma_1 \gamma_2} &:= \gamma_1 \mu_1 \gamma_2 m_2, \\ \mathcal{I}_{10}^{\gamma_1 \gamma_2} &:= \gamma_1 \hat{Y}_1 \gamma_2 m_2 + \gamma_1 m_1 \gamma_2 \hat{Y}_2, & \mathcal{P}_{1\epsilon;10}^{\gamma_1 \gamma_2} &:= \gamma_1 \mu_1 \gamma_2 \hat{Y}_2, \\ \mathcal{I}_{20}^{\gamma_1 \gamma_2} &:= \gamma_1 \hat{Y}_1 \gamma_2 \hat{Y}_2, & \mathcal{P}_{1\epsilon;20}^{\gamma_1 \gamma_2} &:= \gamma_1 (-Y_1^2) \gamma_2 m_2, \\ \mathcal{I}_{21}^{\gamma_1 \gamma_2} &:= (-1/2) X_{12} \gamma_1 \gamma^\sigma \gamma_2 \gamma_\sigma, & \mathcal{P}_{1\epsilon;30}^{\gamma_1 \gamma_2} &:= \gamma_1 (-Y_1^2) \gamma_2 \hat{Y}_2, \\ \mathcal{I}_{30}^{\gamma_1 \gamma_2} = \mathcal{I}_{31}^{\gamma_1 \gamma_2} &:= 0; & \mathcal{P}_{1\epsilon;21}^{\gamma_1 \gamma_2} &:= (-1/2) (-n X_{11}) \gamma_1 \gamma_2 m_2, \\ \mathcal{D}_{1;00}^{\dot{\gamma}_1 \dot{\gamma}_2} = \mathcal{D}_{1;21}^{\dot{\gamma}_1 \dot{\gamma}_2} &:= 0, & \mathcal{P}_{1\epsilon;31}^{\gamma_1 \gamma_2} &:= (-1/2) [(-n X_{11}) \gamma_1 \gamma_2 \hat{Y}_2 + (-2 X_{12}) \gamma_1 \gamma_2 \hat{Y}_1], \\ \mathcal{D}_{1;sj}^{\dot{\gamma}_1 \dot{\gamma}_2} &:= (Y_2 - Y_1)_\mu \mathcal{I}_{s-1,j}^{(\gamma^\mu \gamma) \gamma_2}, & \text{if } (s, j) \neq (0, 0), (2, 1); & \mu_1 := m_1^2 - i\epsilon_1. \end{aligned} \tag{15}$$

The α -parametric functions $Z_\epsilon \equiv Z_\epsilon(\alpha, m, k)$, $M_\epsilon \equiv M_\epsilon(\alpha, m)$, $A \equiv A(\alpha, k)$, $\Delta \equiv \Delta(\alpha)$, $Y_l \equiv Y_l(\alpha, k)$ and $X_{ll'} \equiv X_{ll'}(\alpha)$ incoming in equations (11)–(15) have the form:

$$Z_\epsilon := A/M_\epsilon, \quad M_\epsilon := \alpha_1 \mu_1 + \alpha_2 \mu_2, \quad A := \frac{\alpha_1 \alpha_2}{\Delta} k_2^2 = \alpha_1 Y_1^2 + \alpha_2 Y_2^2, \quad \Delta := \alpha_1 + \alpha_2,$$

$$Y_1 := \beta_2 k_2, \quad Y_2 := -\beta_1 k_2, \quad X_{ll'} := \Delta^{-1}, \quad l, l' \in \{1, 2\}, \quad Y_2 - Y_1 = -k_2 = k_1,$$

$$Y_l^2 = -\frac{A}{\Delta} + (1 - \beta_l) k_2^2, \quad Y_1 \cdot Y_2 = -\frac{A}{\Delta}, \quad \alpha_l Y_l^2 = (1 - \beta_l) A, \quad \beta_l := \frac{\alpha_l}{\Delta}, \quad l = 1, 2. \tag{16}$$

Similar considerations for the reduced amplitudes (10) give rise to the zero values:

$$(R^{\nu+1} \overline{\mathcal{P}}_{l\epsilon})^{\gamma_1 \gamma_2}(\overline{m}_{(l)}, k) = 0, \quad \forall n \geq 1, \quad l = 1, 2, \tag{17}$$

confirming once again but in another way the well known result for ‘‘tadpole’’ amplitudes.

3.2. In the SCR, there exist the following compatibility and recurrence relations:

$$(R^\nu \mathcal{F})_{sj} = \mathcal{F}_{sj} := M_\epsilon^{\omega+j} (1 - Z_\epsilon)^{\omega+j} \Gamma(-\omega - j), \quad \text{if } \nu_{sj} \leq -1, \tag{18}$$

$$(R^\nu \mathcal{F})_{sj} = (R^{\nu+1} \mathcal{F})_{s+1,j}, \tag{19}$$

$$M_\epsilon (R^\nu \mathcal{F})_{00} - A (R^\nu \mathcal{F})_{20} + (\omega + 1) (R^\nu \mathcal{F})_{21} = 0,$$

$$M_\epsilon (R^{\nu+1} \mathcal{F})_{10} - A (R^{\nu+1} \mathcal{F})_{30} + (\omega + 1) (R^{\nu+1} \mathcal{F})_{31} = 0, \tag{20}$$

between the basic functions $(R^\nu \mathcal{F})_{sj}$, $(R^{\nu+1} \mathcal{F})_{sj}$. In fact, due to compatibility relations (19) both recurrence relations (20) are different forms of the common one.

3.3. From equations (11), (15), and (16) follow more specific formulae for our quantities. So, the regular value of the main FA defined by equation (1) takes the form:

$$\begin{aligned} (R^\nu I)^{\gamma_1 \gamma_2}(m, k) &= (2\pi)^n \delta(k) b(g) \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \left\{ \text{tr}[\gamma_1 m_1 \gamma_2 m_2] (R^\nu \mathcal{F})_{00} \right. \\ &+ \text{tr}[\gamma_1 \hat{k}_2 \gamma_2 m_2 \beta_2 - \gamma_1 m_1 \gamma_2 \hat{k}_2 \beta_1] (R^\nu \mathcal{F})_{10} \\ &\left. + \text{tr}[\gamma_1 \hat{k}_2 \gamma_2 \hat{k}_2] (-\beta_1 \beta_2) (R^\nu \mathcal{F})_{20} + \text{tr}[\gamma_1 \gamma^\sigma \gamma_2 \gamma_\sigma] (-1/2) \Delta^{-1} (R^\nu \mathcal{F})_{21} \right\}. \end{aligned} \quad (21)$$

The regular values of convolutions and divergence contributions involved in the CWIs (3) and defined by equations (5) and (1) look as follow:

$$\begin{bmatrix} k_{1\mu} (R^\nu I)^{(\gamma^\mu \gamma) \gamma_2}(m, k) \\ k_{2\alpha} (R^\nu I)^{\gamma_1 (\gamma^\alpha \gamma)}(m, k) \end{bmatrix} = \begin{bmatrix} (R^{\nu+1} \mathcal{D}_1)^{\dot{\gamma} \gamma_2}(m, k) \\ (R^{\nu+1} \mathcal{D}_2)^{\gamma_1 \dot{\gamma}}(m, k) \end{bmatrix} = (2\pi)^n \delta(k) b(g) \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \quad (22)$$

$$\times \begin{bmatrix} \text{tr}[\gamma \gamma_2] (R^{\nu+1} \mathcal{D}_1)_{\{0\}}(m, \alpha, k) - \text{tr}[\gamma \gamma_2 \hat{k}_2] (R^{\nu+1} \mathcal{D}_1)_{\{1\}}(m, \alpha, k) \\ \text{tr}[\gamma_1 \gamma] (R^{\nu+1} \mathcal{D}_2)_{\{0\}}(m, \alpha, k) + \text{tr}[\gamma_1 \hat{k}_2 \gamma] (R^{\nu+1} \mathcal{D}_2)_{\{1\}}(m, \alpha, k) \end{bmatrix}, \quad (23)$$

$$(R^{\nu+1} \mathcal{D}_i)_{\{0\}}(m, \alpha, k) := k_2^2 [m_i \beta_i - (-1)^{\pi_i} m_j \beta_j] (R^{\nu+1} \mathcal{F})_{20}, \quad i, j \in \{1, 2\}, \quad j \neq i; \quad (24)$$

$$\begin{aligned} (R^{\nu+1} \mathcal{D}_i)_{\{1\}}(m, \alpha, k) &:= m_1 m_2 (R^{\nu+1} \mathcal{F})_{10} - (-1)^{\pi_i} (A/\Delta) (R^{\nu+1} \mathcal{F})_{30} \\ &+ (-1)^{\pi_i} (n/2 - 1) \Delta^{-1} (R^{\nu+1} \mathcal{F})_{31} \\ &= [m_1 m_2 - (-1)^{\pi_i} (M_\epsilon/\Delta)] (R^{\nu+1} \mathcal{F})_{10}, \quad i = 1, 2. \end{aligned} \quad (25)$$

The regular values of the nonreduced FAs in the RIs (7) defined by equation (9) take the form:

$$\begin{bmatrix} (R^{\nu+1} P_{1\epsilon})^{\gamma_1 \gamma_2}(m, k) \\ (R^{\nu+1} P_{2\epsilon})^{\gamma_1 \gamma_2}(m, k) \end{bmatrix} = \begin{bmatrix} \text{tr}[\gamma_1 \gamma_2] m_2 (R^{\nu+1} P_{1\epsilon})_{\{0\}} - \text{tr}[\gamma_1 \gamma_2 \hat{k}_2] (R^{\nu+1} P_{1\epsilon})_{\{1\}}(m, k) \\ \text{tr}[\gamma_1 \gamma_2] m_1 (R^{\nu+1} P_{2\epsilon})_{\{0\}} + \text{tr}[\gamma_1 \hat{k}_2 \gamma_2] (R^{\nu+1} P_{2\epsilon})_{\{1\}}(m, k) \end{bmatrix}, \quad (26)$$

$$\begin{bmatrix} (R^{\nu+1} P_{l\epsilon})_{\{0\}}(m, k) \\ (R^{\nu+1} P_{l\epsilon})_{\{1\}}(m, k) \end{bmatrix} := (2\pi)^n \delta(k) b(g) \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \begin{bmatrix} (R^{\nu+1} \mathcal{P}_{l\epsilon})_{\{0\}}(m, \alpha, k) \\ (R^{\nu+1} \mathcal{P}_{l\epsilon})_{\{1\}}(m, \alpha, k) \end{bmatrix}, \quad (27)$$

$$\begin{aligned} (R^{\nu+1} \mathcal{P}_{l\epsilon})_{\{0\}}(m, \alpha, k) &:= \mu_l (R^{\nu+1} \mathcal{F})_{00} - Y_l^2 (R^{\nu+1} \mathcal{F})_{20} + (n/2) \Delta^{-1} (R^{\nu+1} \mathcal{F})_{21} \\ &= (R^{\nu+1} \mathcal{P}_l)_{\{0\}}(m, \alpha, k) - i\epsilon_l (R^{\nu+1} \mathcal{F})_{00}; \end{aligned} \quad (28)$$

$$\begin{aligned} (R^{\nu+1} \mathcal{P}_{l\epsilon})_{\{1\}}(m, \alpha, k) &:= \beta_l \mu_l (R^{\nu+1} \mathcal{F})_{10} - \beta_l Y_l^2 (R^{\nu+1} \mathcal{F})_{30} \\ &+ [(n/2) \beta_l - (1 - \beta_l)] \Delta^{-1} (R^{\nu+1} \mathcal{F})_{31} = (R^{\nu+1} \mathcal{P}_l)_{\{1\}}(m, \alpha, k) - i\epsilon_l \beta_l (R^{\nu+1} \mathcal{F})_{10}. \end{aligned} \quad (29)$$

The first equality in equation (22) holds due to compatibility relations (19). The second expression of equation (25) follows from the first one due to the second recurrence relation (20).

3.4. The tensor structure of regular values $(R^{\nu+1} \bar{P}_{l\epsilon})^{\gamma_1 \gamma_2}(\bar{m}_{(l)}, k)$, $l = 1, 2$, is the same as that of $(R^{\nu+1} P_{l\epsilon})^{\gamma_1 \gamma_2}(m, k)$, $l = 1, 2$, given by the r.h.s. of equation (26), and consequently from equations (7), (17), and (26) we obtain four very practical primitive DRIs at the regular level:

$$(R^{\nu+1} P_{l\epsilon})_{\{0\}}(m, k) = 0, \quad (R^{\nu+1} P_{l\epsilon})_{\{1\}}(m, k) = 0, \quad \forall n \geq 1, \quad l = 1, 2, \quad (30)$$

where the l.h.s. are given by equations (27)–(29).

Similarly, the regular values $(R^{\nu+1} P_l)^{\gamma_1 \gamma_2}(m, k)$, $l = 1, 2$, of FAs involved in CWIs (3) and defined by equations (6) and (1) are almost the same as the $(R^{\nu+1} P_{l\epsilon})^{\gamma_1 \gamma_2}(m, k)$, $l = 1, 2$, given by equations (26)–(29). The connection between them is determined completely by the second

expressions of equations (28)–(29). Taking into account equations (30) one obtains very simple representations:

$$\left[\begin{matrix} (R^{\nu+1}P_l)_{\{0\}}(m, k) \\ (R^{\nu+1}P_l)_{\{1\}}(m, k) \end{matrix} \right] := (2\pi)^n \delta(k) b(g) \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \left[\begin{matrix} i\epsilon_l (R^{\nu+1}\mathcal{F})_{00} \\ i\epsilon_l \beta_l (R^{\nu+1}\mathcal{F})_{10} \end{matrix} \right], \quad l = 1, 2, \quad (31)$$

for quantities $(R^{\nu+1}P_l)_{\{\kappa\}}(m, k)$ in terms of which the regular values $(R^{\nu+1}P_l)^{\gamma_1 \gamma_2}(m, k)$ are initially expressed by using of equations (26)–(29) (and replacing μ_l by m_l^2).

By virtue of properties of the hypergeometric function ${}_2F_1$ from equation (31) the important limiting values of these quantities for any space-time dimension $n = 2r + \delta_n$, $\delta_n = 0, 1$, follow:

$$\lim_{\epsilon_1, \epsilon_2 \rightarrow 0, \exists m_s \neq 0; \text{ or } (\epsilon, m) \rightarrow 0} (R^{\nu+1}P_l)_{\{\kappa\}}(m, k) = 0, \quad \forall n \geq 1, \quad l = 1, 2, \quad \kappa = 0, 1, \quad (32)$$

$$\lim_{(m, \epsilon) \rightarrow 0} \begin{cases} m_{l'} (R^{\nu+1}P_l)_{\{0\}}(m, k) = 0, & \forall n \geq 1, \quad l', l \in \{1, 2\}, \quad l' \neq l, \\ (R^{\nu+1}P_l)_{\{1\}}(m, k) = (2\pi)^n \delta(k) b(g) (1 - \delta_n) (k_2^2)^{r-1} \Gamma(r) / (2\Gamma(2r)). \end{cases} \quad (33)$$

Hereafter the (ϵ, m) -limit means first $\epsilon_l \rightarrow 0$ and then $m_l = m \rightarrow 0$, $l = 1, 2$, i.e. it is equivalent to the symmetric chiral limit case ($m_1 = m_2 = m \rightarrow 0$). Analogously, the (m, ϵ) -limit means first $m_l \rightarrow 0$ and then $\epsilon_l = \epsilon \rightarrow 0$, $l = 1, 2$, i.e. it is equivalent to the chiral case ($m_1 = m_2 = 0$).

3.5. It turns out that so calculated regular values of the FAs defined by equations (1) and (4)–(6) satisfy the identities [29]:

$$\begin{aligned} k_{1\mu} (R^\nu T)^{(\gamma^\mu \gamma) \gamma_2}(m, k) &= (R^{\nu+1}D_1)^{\dot{\gamma} \gamma_2}(m, k) \\ &= (-1)^{\pi_1} (R^{\nu+1}P_1)^{\gamma \gamma_2}(m, k) - (R^{\nu+1}P_3)^{\gamma \gamma_2} + (m_2 - (-1)^{\pi_1} m_1) (R^{\nu+1}I)^{\gamma \gamma_2}(m, k), \\ k_{2\alpha} (R^\nu T)^{\gamma_1 (\gamma^\alpha \gamma)}(m, k) &= (R^{\nu+1}D_2)^{\gamma_1 \dot{\gamma}}(m, k) \\ &= (-1)^{\pi_2} (R^{\nu+1}P_2)^{\gamma_1 \gamma}(m, k) - (R^{\nu+1}P_1)^{\gamma_1 \gamma} + (m_1 - (-1)^{\pi_2} m_2) (R^{\nu+1}I)^{\gamma_1 \gamma}(m, k), \end{aligned} \quad (34)$$

which are referred to as *the regular analog of the CWIs* (3) (or the quantum Ward identities (QWIs)). The latter name may be more adequately depict their physical meaning. The first rows of equations (34) are due to the compatibility relations (19). It is important to note also that the last terms in the identities (34) are calculated by the renormalization index $\nu + 1$, although their proper divergence index is ν . It is this peculiarity that permits to the regular analogs of the CWIs (34) both to imitate (or to retain) the CWIs (3) and to differ from them simultaneously. It is this peculiarity that permits to obtain some effective formulae for calculating of *the quantum corrections* (QCs) to the CWIs in the most general nonchiral case [29].

3.6. As a result the regular analogs of the CWIs (34) are equivalent to four scalar equations:

$$\begin{aligned} (R^{\nu+1}D_1)_{\{\kappa\}}(m, k) &= (R^{\nu+1}P_{\bar{1}-2})_{\{\kappa\}}(m, k) + (R^{\nu+1}I_{2-\bar{1}})_{\{\kappa\}}(m, k), \quad \kappa = 0, 1, \\ (R^{\nu+1}D_2)_{\{\kappa\}}(m, k) &= (R^{\nu+1}P_{\bar{2}-1})_{\{\kappa\}}(m, k) + (R^{\nu+1}I_{1-\bar{2}})_{\{\kappa\}}(m, k), \quad \kappa = 0, 1, \end{aligned} \quad (35)$$

$$\left[\begin{matrix} (R^{\nu+1}D_i)_{\{\kappa\}}(m, k) \\ (R^{\nu+1}P_{\bar{i}-j})_{\{\kappa\}}(m, k) \\ (R^{\nu+1}I_{j-\bar{i}})_{\{\kappa\}}(m, k) \end{matrix} \right] := (2\pi)^n \delta(k) b(g) \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \left[\begin{matrix} (R^{\nu+1}\mathcal{D}_i)_{\{\kappa\}}(m, \alpha, k) \\ (R^{\nu+1}\mathcal{P}_{\bar{i}-j})_{\{\kappa\}}(m, \alpha, k) \\ (R^{\nu+1}\mathcal{I}_{j-\bar{i}})_{\{\kappa\}}(m, \alpha, k) \end{matrix} \right], \quad (36)$$

$$\begin{aligned} (R^{\nu+1}\mathcal{D}_i)_{\{0\}}(m, \alpha, k) &= k_2^2 [m_i \beta_i - (-1)^{\pi_i} m_j \beta_j] (R^{\nu+1}\mathcal{F})_{20}, \\ (R^{\nu+1}\mathcal{P}_{\bar{i}-j})_{\{0\}}(m, \alpha, k) &:= (-1)^{\pi_i} m_j (R^{\nu+1}\mathcal{P}_i)_{\{0\}}(m, \alpha) - m_i (R^{\nu+1}\mathcal{P}_j)_{\{0\}}(m, \alpha) \\ &\cong [(-1)^{\pi_i} m_j (i\epsilon_i) - m_i (i\epsilon_j)] (R^{\nu+1}\mathcal{F})_{00}, \end{aligned}$$

$$\begin{aligned}
(R^{\nu+1}\mathcal{I}_{j-\bar{i}})_{\{0\}}(m, \alpha, k) &:= (m_j - (-1)^{\pi_i} m_i) [m_1 m_2 (R^{\nu+1}\mathcal{F})_{00} - \\
&\quad - (-1)^{\pi_i} (A/\Delta) (R^{\nu+1}\mathcal{F})_{20} - (-1)^{\pi_i} (n/2) \Delta^{-1} (R^{\nu+1}\mathcal{F})_{21}] \\
&= m_i (R^{\nu+1}\mathcal{P}_j)_{\{0\}}(m, \alpha) - (-1)^{\pi_i} m_j (R^{\nu+1}\mathcal{P}_i)_{\{0\}}(m, \alpha) \\
&\quad + k_2^2 [m_i \beta_i - (-1)^{\pi_i} m_j \beta_j] (R^{\nu+1}\mathcal{F})_{20} \cong [m_i (i\epsilon_j) - (-1)^{\pi_i} m_j (i\epsilon_i)] (R^{\nu+1}\mathcal{F})_{00} \\
&\quad + k_2^2 [m_i \beta_i - (-1)^{\pi_i} m_j \beta_j] (R^{\nu+1}\mathcal{F})_{20}, \quad i, j \in \{1, 2\}, \quad j \neq i; \tag{37}
\end{aligned}$$

$$\begin{aligned}
(R^{\nu+1}\mathcal{D}_i)_{\{1\}}(m, \alpha, k) &:= [m_1 m_2 - (-1)^{\pi_i} (M_\epsilon/\Delta)] (R^{\nu+1}\mathcal{F})_{10}, \\
(R^{\nu+1}\mathcal{P}_{\bar{i}-j})_{\{1\}}(m, \alpha, k) &:= (-1)^{\pi_i} [(R^{\nu+1}\mathcal{P}_i)_{\{1\}}(m, \alpha, k) + (R^{\nu+1}\mathcal{P}_j)_{\{1\}}(m, \alpha, k)] \\
&= (-1)^{\pi_i} (iE/\Delta) (R^{\nu+1}\mathcal{F})_{10}, \\
(R^{\nu+1}\mathcal{I}_{j-\bar{i}})_{\{1\}}(m, \alpha, k) &:= (m_j - (-1)^{\pi_i} m_i) [m_i \beta_i - (-1)^{\pi_i} m_j \beta_j] (R^{\nu+1}\mathcal{F})_{10} \\
&= [m_1 m_2 - (-1)^{\pi_i} (M/\Delta)] (R^{\nu+1}\mathcal{F})_{10}, \quad i, j \in \{1, 2\}, \quad j \neq i. \tag{38}
\end{aligned}$$

Equations (35) are obeyed for all values of k_2 , m_l , ϵ_l , $l = 1, 2$, and any space-time dimension n . But limiting values of quantities in them depend strongly on the limit employed. Hereafter the quantities M and E are defined as $M := \alpha_1 m_1^2 + \alpha_2 m_2^2$, $E := \alpha_1 \epsilon_1 + \alpha_2 \epsilon_2$, and hence $M_\epsilon = M - iE$, and the congruence relation $\mathcal{A}(m, \alpha, k) \cong \mathcal{B}(m, \alpha, k)$ denotes the equality of the integrals $\int_{\Sigma^1} d\mu(\alpha) \Delta^{-n/2} \mathcal{A}(m, \alpha, k) = \int_{\Sigma^1} d\mu(\alpha) \Delta^{-n/2} \mathcal{B}(m, \alpha, k)$. See also equations (25), (28)–(29), (31), and the relations $Y_l^2 = -A/\Delta + (1 - \beta_l) k_2^2$, $l = 1, 2$, in equations (16).

4 Quantum corrections to the CWIs and STCs in the SCR

4.1. Now we investigate equations (34)–(38) more closely. Let us first consider a general mass-anisotropic nonchiral case. Then, from equations (32) follow $\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} (R^{\nu+1}P_l)_{\{\kappa\}}(m, k) = 0$ if $m_1, m_2 \neq 0, \forall l = 1, 2, \kappa = 0, 1$, and consequently we also obtain $\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} (R^{\nu+1}P_{\bar{i}-j})_{\{\kappa\}}(m, k) = 0, \forall l = 1, 2, \kappa = 0, 1$. The *quantum corrections* (QCs) (or anomalous contributions in usual nomenclature) to the CWIs appear now as an oversubtraction effect and take the form:

$$\begin{aligned}
\begin{bmatrix} a_1^{\hat{\gamma}\gamma_2}(m, k) \\ a_2^{\hat{\gamma}1\gamma}(m, k) \end{bmatrix} &:= \begin{bmatrix} (m_2 - (-1)^{\pi_1} m_1) [(R^{\nu+1}I)^{\gamma\gamma_2}(m, k) - (R^\nu I)^{\gamma\gamma_2}(m, k)] \\ (m_1 - (-1)^{\pi_2} m_2) [(R^{\nu+1}I)^{\gamma_1\gamma}(m, k) - (R^\nu I)^{\gamma_1\gamma}(m, k)] \end{bmatrix} \\
&= \begin{bmatrix} \text{tr}[\gamma\gamma_2] a_{1\{0\}}(m, k) - \text{tr}[\gamma\gamma_2 \hat{k}_2] a_{1\{1\}}(m, k) \\ \text{tr}[\gamma_1\gamma] a_{2\{0\}}(m, k) + \text{tr}[\gamma_1 \hat{k}_2 \gamma] a_{2\{1\}}(m, k) \end{bmatrix}, \tag{39}
\end{aligned}$$

where the scalar functions $a_{i\{\kappa\}}(m, k)$ have the integral representations:

$$a_{i\{\kappa\}}(m, k) := (2\pi)^n \delta(k) b(g) \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} a_{i\{\kappa\}}(m, \alpha, k), \quad i = 1, 2, \quad \kappa = 0, 1, \tag{40}$$

$$\begin{aligned}
a_{i\{0\}}(m, \alpha, k) &:= (m_j - (-1)^{\pi_i} m_i) [m_1 m_2 (\Delta\mathcal{F})_{00} - (-1)^{\pi_i} (A/\Delta) (\Delta\mathcal{F})_{20} \\
&\quad - (-1)^{\pi_i} (n/2) \Delta^{-1} (\Delta\mathcal{F})_{21}] \cong [m_i (i\epsilon_j) - (-1)^{\pi_i} m_j (i\epsilon_i)] (\Delta\mathcal{F})_{00} \\
&\quad + k_2^2 [m_i \beta_i - (-1)^{\pi_i} m_j \beta_j] (\Delta\mathcal{F})_{20}, \quad i, j \in \{1, 2\}, \quad j \neq i; \tag{41}
\end{aligned}$$

$$\begin{aligned}
a_{i\{1\}}(m, \alpha, k) &:= (m_j - (-1)^{\pi_i} m_i) [m_i \beta_i - (-1)^{\pi_i} m_j \beta_j] (\Delta\mathcal{F})_{10} \\
&= [m_1 m_2 - (-1)^{\pi_i} (M/\Delta)] (\Delta\mathcal{F})_{10}, \quad i, j \in \{1, 2\}, \quad j \neq i. \tag{42}
\end{aligned}$$

The quantities $(\Delta\mathcal{F})_{sj}$ appearing in equations (41)–(42) are defined as

$$\begin{aligned}
(\Delta\mathcal{F})_{sj} &:= (R^{\nu+1}\mathcal{F})_{sj} - (R^\nu\mathcal{F})_{sj} = (-1)\Theta_{sj} (\Gamma(\lambda_{sj}) A^{1+\nu_{sj}}) / (\Gamma(2 + \nu_{sj}) M_\epsilon^{\lambda_{sj}}), \\
\Theta_{sj} &:= H(\nu_{sj}^1) \theta_s, \quad \theta_s := \nu_{sj}^1 - \nu_{sj} = (\nu - s) \pmod{2}, \tag{43}
\end{aligned}$$

and $H(x)$ is the Heaviside step function such that $H(x) = 0, x < 0, H(x) = 1, x \geq 0$.

4.2. According to equations (13)–(14) we find that $\lambda_{00} = 2 - \delta_n/2$, $\nu_{00} = r - 1$, $\Theta_{00} = \delta_n H(r - 1 + \delta_n)$; $\lambda_{20} = 1 - \delta_n/2$, $\nu_{20} = r - 2$, $\Theta_{20} = \delta_n H(r - 2 + \delta_n)$; $\lambda_{21} = 1 - \delta_n/2$, $\nu_{21} = r - 1$, $\Theta_{21} = \delta_n H(r - 1 + \delta_n)$; $\lambda_{10} = 1 + \delta_n/2$, $\nu_{10} = r - 2 + \delta_n$, $\Theta_{10} = (1 - \delta_n)H(r - 1)$, and taking into account the representation:

$$\begin{aligned} \Phi\left(\lambda \left| \begin{matrix} a_1, a_2 \\ \mu_1, \mu_2 \end{matrix} \right.\right) &:= \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^b} \frac{\alpha_1^{a_1-1} \alpha_2^{a_2-1}}{M_\epsilon^\lambda} = \begin{cases} B(a_1, a_2)/\mu_2^\lambda {}_2F_1(\lambda, a_1; a_1 + a_2; 1 - \xi_{1/2}), \\ B(a_1, a_2)/\mu_1^\lambda {}_2F_1(\lambda, a_2; a_1 + a_2; 1 - \xi_{2/1}), \end{cases} \\ B(a_1, a_2) &:= \Gamma(a_1)\Gamma(a_2)/\Gamma(a_1 + a_2), \quad b := n/2 + N, \quad N \in \mathbb{Z}_+, \quad \xi_{l/s} := \mu_l/\mu_s, \end{aligned} \tag{44}$$

the scalar functions $a_{i\{\kappa\}}(m, k)$ given by equations (40)–(43) acquire the form:

$$\begin{aligned} a_{i\{0\}}(m, k) &= (2\pi)^n \delta(k) b(g) \delta_n (k_2^2)^r a_{i\{0\}}(m_1, \epsilon_1; m_2, \epsilon_2), \quad i = 1, 2, \\ a_{i\{1\}}(m, k) &= (2\pi)^n \delta(k) b(g) (1 - \delta_n) (k_2^2)^{r-1} a_{i\{1\}}(m_1, \epsilon_1; m_2, \epsilon_2), \quad i = 1, 2, \end{aligned} \tag{45}$$

in which the mass dependent functions $a_{i\{\kappa\}}(m_1, \epsilon_1; m_2, \epsilon_2)$ with ϵ -damping look as follow:

$$\begin{aligned} a_{i\{0\}}(m_1, \epsilon_1; m_2, \epsilon_2) &:= (-1)^i \frac{\Gamma(3/2)}{\Gamma(r+1)} (m_2 - m_1) \{m_1 m_2 \Phi(3/2 \mid \begin{matrix} r+1, r+1 \\ \mu_1, \mu_2 \end{matrix}) \\ &\quad - (4r+1) \Phi(1/2 \mid \begin{matrix} r+1, r+1 \\ \mu_1, \mu_2 \end{matrix})\}, \quad \mu_l := m_l^2 - i\epsilon_l, \quad i = 1, 2, \end{aligned} \tag{46}$$

$$\begin{aligned} a_{i\{1\}}(m_1, \epsilon_1; m_2, \epsilon_2) &:= \frac{(-1)}{\Gamma(r)} \{m_1 m_2 \Phi(1 \mid \begin{matrix} r, r \\ \mu_1, \mu_2 \end{matrix}) \\ &\quad - (-1)^{\pi_i} [m_1^2 \Phi(1 \mid \begin{matrix} r+1, r \\ \mu_1, \mu_2 \end{matrix}) + m_2^2 \Phi(1 \mid \begin{matrix} r, r+1 \\ \mu_1, \mu_2 \end{matrix})]\}, \quad i = 1, 2. \end{aligned} \tag{47}$$

Since in equation (45) the $a_{i\{0\}}(m, k) \neq 0$ only for $\delta_n = 1$, i.e. for $n = 2r + 1$, we put in equation (46) the factors $(-1)^{\pi_i} = 1$, $i = 1, 2$, both for vectors ($\gamma = I_g$) and axial-vectors ($\gamma = \gamma^*$) cases.

The functions $a_{i\{\kappa\}}(m_1, \epsilon_1; m_2, \epsilon_2)$ have the symmetry properties

$$a_{i\{\kappa\}}(m_2, \epsilon_2; m_1, \epsilon_1) = (-1)^{\kappa+1} a_{i\{\kappa\}}(m_1, \epsilon_1; m_2, \epsilon_2), \quad \kappa = 0, 1, \tag{48}$$

and in the limit $\epsilon_l \rightarrow 0$ tend to homogeneous functions $a_{i\{\kappa\}}(x)$ of the zero degree in masses which are named as *mass functions* of the QCs to the CWIs. Using the relation

$$\mu_1 \Phi(\lambda \mid \begin{matrix} a_1+1, a_2 \\ \mu_1, \mu_2 \end{matrix}) + \mu_2 \Phi(\lambda \mid \begin{matrix} a_1, a_2+1 \\ \mu_1, \mu_2 \end{matrix}) = \Phi(\lambda - 1 \mid \begin{matrix} a_1, a_2 \\ \mu_1, \mu_2 \end{matrix}),$$

(which is a consequence of equation (44) and of the identity $M_\epsilon/M_\epsilon^\lambda = M_\epsilon^{\lambda-1}$), and other properties of the function ${}_2F_1$, from equations (46)–(47) follow the explicit form of the mass functions $a_{i\{\kappa\}}(x)$:

$$\begin{aligned} a_{i\{0\}}(x) &:= \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} a_{i\{0\}}(m_1, \epsilon_1; m_2, \epsilon_2) = A_{2r+1} \bar{a}_{i\{0\}}(x), \quad x := m_1/m_2, \\ \bar{a}_{i\{0\}}(x) &:= (-1)^{i-1} (1-x)/C_{2r+1} [(4r+1)\alpha(1/2, r+1; x^2) - x\alpha(3/2, r+1; x^2)], \tag{49} \\ a_{i\{1\}}(x) &:= \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} a_{i\{1\}}(m_1, \epsilon_1; m_2, \epsilon_2) = A_{2r} \bar{a}_{i\{1\}}(x), \quad x := m_1/m_2, \\ \bar{a}_{i\{1\}}(x) &:= [(-1)^{\pi_i} - a_{2r}(x)], \quad a_{2r}(x) := x\alpha(1, r; x^2), \\ \alpha(\lambda, b; x^2) &:= {}_2F_1(\lambda, b; 2b; 1-x^2). \end{aligned} \tag{50}$$

In equations (49)–(50), C_{2r+1} is the normalization constant (that gives $\bar{a}_{i\{0\}}(0) = (-1)^{i-1}$, $\forall r$), and A_{2r+1} , A_{2r} denote the magnitudes of the mass functions of QCs:

$$\begin{aligned} C_{2r+1} &:= (4r+1)\alpha(1/2, r+1; 0) = 2\Gamma(2r+2)\Gamma(r+1/2)/(\Gamma(2r+1/2)\Gamma(r+1)), \\ A_{2r+1} &:= \Gamma(1/2)\Gamma(r+1/2)/\Gamma(2r+1/2), \quad A_{2r} = \Gamma(r)/\Gamma(2r). \end{aligned} \tag{51}$$

There exists the relation $A_{2r+1} = \Gamma(3/2)A_{2r+2}C_{2r+1}$ between them. The magnitudes A_{2r} and A_{2r+1} are monotonically decreasing functions of the variable r , varying from $A_2 = 1$ and $A_1 = (2/3)\sqrt{\pi}$ to $\lim_{r \rightarrow \infty} A_{2r+\delta_n} = 0$. Therefore, when $r \rightarrow \infty$ the QCs go to zero very rapidly.

Similarly, the $\bar{a}_{i\{\kappa\}}(x)$ denote normalized mass functions which determine a shape of the $a_{i\{\kappa\}}(x)$. As far as $(-1)^{\pi_i} = \pm 1$, from equations (49)–(50) follow three primitive mass functions

$$a_{2r+1}(x) := \bar{a}_{1\{0\}}(x); \quad a_{2r}^{(-)}(x) := 1 - a_{2r}(x), \quad a_{2r}^{(+)}(x) := 1 + a_{2r}(x), \quad (52)$$

in term of which all mass functions $a_{i\{\kappa\}}(x)$ of the QCs are finally expressed. The properties of the functions $a_{2r+1}(x)$, $a_{2r}^{(\mp)}(x)$, $a_{2r}(x)$, $\forall r \geq 1$, which may be physical important, are related to the reciprocity relations, the values at $x = 0$, $x = \infty$, and at $x = 1$ (the latter corresponds to the degenerate nonchiral case ($m_1 = m_2 = m \neq 0$)), the range of values for real $0 \leq x \leq \infty$, zeros, extrema, and intervals of monotonicity, are as follows:

$$a_{2r+1}(x) = -a_{2r+1}(1/x); \quad a_{2r}^{(\mp)}(x) = a_{2r}^{(\mp)}(1/x), \quad a_{2r}(x) = a_{2r}(1/x); \quad (53)$$

$$a_{2r+1}(0) = -a_{2r+1}(\infty) = 1; \quad a_{2r}^{(\mp)}(0) = a_{2r}^{(\mp)}(\infty) = 1, \quad a_{2r}(0) = a_{2r}(\infty) = 0; \quad (54)$$

$$a_{2r+1}(1) = 0; \quad a_{2r}^{(-)}(1) = 0, \quad a_{2r}^{(+)}(1) = 2, \quad a_{2r}(1) = 1, \quad (55)$$

$$1 \geq a_{2r+1}(x) \geq -1; \quad 0 \leq a_{2r}^{(-)}(x) \leq 1, \quad 1 \leq a_{2r}^{(+)}(x) \leq 2, \quad 0 \leq a_{2r}(x) \leq 1.$$

The values at $x = 1$ are: the unique zero for $a_{2r+1}(x)$, the unique zero which is the unique minimum for $a_{2r}^{(-)}(x)$, the unique maxima for $a_{2r}^{(+)}(x)$ and $a_{2r}(x)$. The $a_{2r+1}(x)$ are monotonically decreasing on $0 \leq x \leq \infty$; the $a_{2r}^{(-)}(x)$ are monotonically decreasing on $0 \leq x \leq 1$ and are monotonically increasing on $1 \leq x \leq \infty$; the $a_{2r}^{(+)}(x)$ and $a_{2r}(x)$ are monotonically increasing on $0 \leq x \leq 1$ and are monotonically decreasing on $1 \leq x \leq \infty$.

Taking into consideration equations (32), (36)–(43), (45)–(52), one obtains for the regular analogs of the CWIs (35)–(38) the following expressions ($i = 1, 2, j \in \{1, 2\}, j \neq i$):

$$\begin{aligned} (R^{\nu+1}D_i)_{\{0\}}(m, k) &= (R^{\nu+1}I_{j-\bar{i}})_{\{0\}}(m, k) = (R^{\nu}I_{j-\bar{i}})_{\{0\}}(m, k) + a_{i\{0\}}(m, k), \\ a_{i\{0\}}(m, k) &= \tilde{b}(g, k) \delta_n(k_2^2)^r A_{2r+1}(-1)^{i-1} a_{2r+1}(x), \quad \tilde{b}(g, k) := (2\pi)^n \delta(k) b(g), \\ (R^{\nu+1}D_i)_{\{1\}}(m, k) &= (R^{\nu+1}I_{j-\bar{i}})_{\{1\}}(m, k) = (R^{\nu}I_{j-\bar{i}})_{\{1\}}(m, k) + a_{i\{1\}}(m, k), \\ a_{i\{1\}}(m, k) &= \tilde{b}(g, k) (1 - \delta_n)(k_2^2)^{r-1} A_{2r} [(-1)^{\pi_i} - a_{2r}(x)], \quad x := m_1/m_2, \end{aligned} \quad (56)$$

which are valid both for general and degenerate nonchiral cases.

4.3. Now we pass to the chiral behavior. Let us consider some possible ways tending to the chiral state in renormalized amplitudes at hand: i) the symmetric chiral limit ($m_1 = m_2 = m \rightarrow 0$), accomplishing as the $(\epsilon_{1,2}, m)$ –limit, when first $\epsilon_1, \epsilon_2 \rightarrow 0$, and then $m_l = m \rightarrow 0, \forall l$; ii) the nonsymmetric chiral limit ($m_1 \rightarrow 0, m_2 \rightarrow 0$), accomplishing as the $(\epsilon_{1,2}, m_1, m_2)$ –limit, when first $\epsilon_1, \epsilon_2 \rightarrow 0$, then $m_1 \rightarrow 0$, and lastly $m_2 \rightarrow 0$; iii) the nonsymmetric chiral limit ($m_2 \rightarrow 0, m_1 \rightarrow 0$), accomplishing as the $(\epsilon_{1,2}, m_2, m_1)$ –limit, when first $\epsilon_1, \epsilon_2 \rightarrow 0$, then $m_2 \rightarrow 0$, and lastly $m_1 \rightarrow 0$; iv) the chiral case ($m_1 = m_2 = 0$), accomplishing as the $(m_{1,2}, \epsilon)$ –limit, when first $m_1 = m_2 = 0$, and then $\epsilon_l = \epsilon \rightarrow 0, \forall l$.

Equations (53)–(55) imply that the symmetric chiral limit $m_1 = m_2 = m \rightarrow 0$ differs essentially from the nonsymmetric chiral limits $m_1 \rightarrow 0, m_2 \rightarrow 0$ or $m_2 \rightarrow 0, m_1 \rightarrow 0$ for primitive mass functions (52). In addition, for the $a_{2r+1}(x)$ the last two limits are also different.

From equations (36)–(39), (13) and properties of the function ${}_2F_1$ it follows that for all chiral limits, i.e. for i), ii), and iii) cases, $\lim_{m_s \rightarrow 0} (R^{\nu}I_{j-\bar{i}})_{\{\kappa\}}(m, k) = 0, \kappa = 0, 1$, and equations (56) take the form:

$$\lim_{m_s \rightarrow 0} (R^{\nu+1}D_i)_{\{\kappa\}}(m, k) = \tilde{b}(g, k) \begin{cases} \delta_n(k_2^2)^r A_{2r+1}(-1)^{i-1} \{0, 1, -1\}, & \kappa = 0; \\ (1 - \delta_n)(k_2^2)^{r-1} A_{2r} \{[(-1)^{\pi_i} - 1], (-1)^{\pi_i}\}, & \kappa = 1, \end{cases} \quad (57)$$

where the $\{0, 1, -1\}$ in the first row of equation (57) corresponds to the $\{i, ii, iii\}$ -cases, and the $\{[(-1)^{\pi_i} - 1], (-1)^{\pi_i}\}$ in the second row corresponds to the $\{i, ii\}$ or $\{iii\}$ -cases, respectively.

In the chiral case ($m_1 = m_2 = 0$), due to equations (36)–(38), one has:

$$\begin{aligned} (R^{\nu+1}I_{j-\bar{i}})_{\{\kappa\}}(0, k) &= 0, & \kappa &= 0, 1, & \epsilon_1, \epsilon_2 &\neq 0, \\ (R^{\nu+1}D_i)_{\{0\}}(0, k) &= 0, & (R^{\nu+1}P_{\bar{i}-j})_{\{0\}}(0, k) &= 0, & \epsilon_1, \epsilon_2 &\neq 0, \end{aligned} \quad (58)$$

and equations (35),(33) and (51) give rise to the following nontrivial identities

$$\begin{aligned} (R^{\nu+1}D_i)_{\{1\}}(0, k) &= (R^{\nu+1}P_{\bar{i}-j})_{\{1\}}(0, k) \neq 0, & \epsilon_1 \epsilon_2 &\neq 0, \\ \lim_{\epsilon_1=\epsilon_2=\epsilon\rightarrow 0} (R^{\nu+1}D_i)_{\{1\}}(0, k) &= \lim_{\epsilon_1=\epsilon_2=\epsilon\rightarrow 0} (R^{\nu+1}P_{\bar{i}-j})_{\{1\}}(0, k) \\ &= \tilde{b}(g, k) (1 - \delta_n)(k_2^2)^{r-1} A_{2r}(-1)^{\pi_i}, \end{aligned} \quad (59)$$

which are caused by the nonzero Schwinger terms contributions of current density commutators.

From the previous, it follows that for general spinor diangles the STCs may be nonzero only in the chiral case, for even space-time dimension $n = 2r$, for non light-like momenta $k_2^2 \neq 0$, and if $n = 2$ for the light-like momenta $k_2^2 = 0$ also. The dimension $n = 2$ is the unique one for which STCs are nonzero for light-like momenta $k_2^2 = 0$. Clearly, this fact is connected with the well known dynamical mass generation for the two-dimensional vector boson [30, 31].

From equations (57)–(59) also imply that the chiral case and the chiral limit cases in general do not coincide. For example, the expression in equation (59) coincides with that of corresponding to the nonsymmetric chiral limits in equation (57) for ii) and iii) cases and differs from that of corresponding to the symmetric chiral limit in equation (57) for i) case. Similar conclusion follows also from equation (58) and from the first row of equation (57).

5 Conclusions

From the above we have come to the important conclusions:

- There is the technique (SCR) in framework of which the WIs involving regular values of quantities do retain (or imitate) the CWIs. Quantum anomalies reveal themselves either as an oversubtraction effect for a non-chiral case and for the symmetric and nonsymmetric chiral limits (in these cases the STCs are zero) or as nonzero STCs for the chiral case.
- Quantum anomalies are more general phenomena than the well known mass-independent axial-vector and conformal anomalies. The related conclusion has been obtained as early as 1970 by Kummer and Schweda [6, 7, 8]. Our investigations show that canonically non-conserved vector and axial-vector currents can have mass-dependent anomalies. Furthermore, in the chiral case vector and axial-vector currents have the same anomaly (up to the factor $\varepsilon(g) = (-1)^q(-1)^{n(n-1)/2}$). It is the STCs that are responsible for these anomalies. STCs may be nonzero only in the chiral case, for even $n = 2r$, for non light-like momenta $k_2^2 \neq 0$, and if $n = 2$ for the light-like momenta $k_2^2 = 0$ also. The dimension $n = 2$ is the unique one for which STCs and quantum anomalies are nonzero for light-like momenta $k_2^2 = 0$. This fact is connected with the well known dynamical mass generation for the two-dimensional vector boson [30, 31].
- For the complex Clifford algebra $Cl(g)_c$ the matrix dual conjugation γ^* may be always redefined as $\gamma^* := i^{(1-\varepsilon(g))/2} \gamma^1 \gamma^2 \dots \gamma^n$, $(\gamma^*)^2 = I_g$. Then from equations (2) and equations (39)–(40) it follows that in the chiral case the QCs to the vector and axial-vector CWIs are the same exactly. Therefore, in this case “left-handed vector” current can be conserved and hence it can be more fundamental than vector or axial-vector currents. This may give some insight into why just the left-handed neutrino exists in Nature.

- No universal modified operator expressions for divergencies of axial-vector and vector currents exist, even in the framework of some fixed model. Modes of quantum anomalies strongly depend on the type of quantum field quantities under consideration. Moreover, the behavior of FAs and quantum anomalies in the chiral case ($m = 0$) and in the symmetric chiral limit ($m \rightarrow 0$) is different. The same is also true for the Schwinger terms of current commutators.
- A mass spectrum of fermions, appearing in the quantum anomalies, increases the predictive power of formulas widely used in the low energy phenomenological physics, e.g., for describing particle decays [1–5].
- A nontrivial mass dependence of the QCs to the CWIs prevents the standard mechanism of anomaly cancellation and requires a revision of some orthodox ideas of the counter-term renormalization.

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