

# Method of Replacing the Variables for Generalized Symmetry of D'Alembert Equation

Gennadii KOTEL'NIKOV

RRC Kurchatov Institute, Kurchatov Sq. 1, Moscow 123182, Russia

E-mail: kga@electronics.kiae.ru

It is shown that by generalized understanding of symmetry the d'Alembert equation for one component field is invariant with respect to arbitrary reversible coordinate transformations.

## 1 Introduction

Symmetries play an important role in particle physics and quantum field theory [1], nuclear physics [2], mathematical physics [3]. It is proposed some receptions for finding the symmetries, for example, the method of replacing the variables [4], the Lie algorithm [3], the theoretical-algebraic approach [5]. The purpose of the present work is the generalization of the method of replacing the variables. We start from the following Definition of symmetry.

## 2 Definition of symmetry. Examples

**Definition 1.** Let some partial differential equation  $\hat{L}'\phi'(x') = 0$  be given. By symmetry of this equation with respect to the variables replacement  $x' = x'(x)$ ,  $\phi' = \phi'(\Phi\phi)$  we shall understand the compatibility of the engaging equations system  $\hat{A}\phi'(\Phi\phi) = 0$ ,  $\hat{L}\phi(x) = 0$ , where  $\hat{A}\phi'(\Phi\phi) = 0$  is obtained from the initial equation by replacing the variables,  $\hat{L}' = \hat{L}$ ,  $\Phi(x)$  is some weight function. If the equation  $\hat{A}\phi'(\Phi\phi) = 0$  can be transformed into the form  $\hat{L}(\Psi\phi) = 0$ , the symmetry will be named the standard Lie symmetry, otherwise the generalized symmetry.

Elements of this Definition were used to study the Maxwell equations symmetries [6, 7, 8]. In the present work we shall apply Definition 1 for investigation of symmetries of the one-component d'Alembert equation:

$$\square'\phi'(x') = \frac{\partial^2\phi'}{\partial x_1'^2} + \frac{\partial^2\phi'}{\partial x_2'^2} + \frac{\partial^2\phi'}{\partial x_3'^2} + \frac{\partial^2\phi'}{\partial x_4'^2} = 0.$$

Let us introduce arbitrary reversible coordinate transformations  $x' = x'(x)$  and a transformation of the field variable  $\phi' = \phi(\Phi\phi)$ , where  $\Phi(x)$  is some unknown function, as well as take into account

$$\begin{aligned} \frac{\partial\phi'}{\partial x_i'} &= \sum_j \frac{\partial\phi'}{\partial\xi} \frac{\partial\Phi\phi}{\partial x_j} \frac{\partial x_j}{\partial x_i'}, \\ \frac{\partial^2\phi'}{\partial x_i'^2} &= \sum_j \frac{\partial^2 x_j}{\partial x_i'^2} \frac{\partial\phi'}{\partial\xi} \frac{\partial\Phi\phi}{\partial x_j} + \sum_{jk} \frac{\partial^2\Phi\phi}{\partial x_j\partial x_k} \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'} \frac{\partial\phi'}{\partial\xi} + \sum_{jk} \frac{\partial^2\phi'}{\partial\xi^2} \frac{\partial\Phi\phi}{\partial x_j} \frac{\partial\Phi\phi}{\partial x_k} \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'}, \end{aligned}$$

where  $\xi = \Phi\phi$ . After replacing the variables we find that the equation  $\square'\phi' = 0$  transforms into

itself, if the system of the engaging equations is fulfilled:

$$\begin{aligned} & \sum_i \sum_j \frac{\partial^2 x_j}{\partial x_i'^2} \frac{\partial \phi'}{\partial \xi} \frac{\partial \Phi \phi}{\partial x_j} + \sum_i \sum_{j=k} \left( \frac{\partial x_j}{\partial x_i'} \right)^2 \frac{\partial \phi'}{\partial \xi} \frac{\partial^2 \Phi \phi}{\partial x_j^2} + \sum_i \sum_{j<k} \sum_k 2 \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'} \frac{\partial \phi'}{\partial \xi} \frac{\partial^2 \Phi \phi}{\partial x_j \partial x_k} \\ & + \sum_i \sum_{j=k} \left( \frac{\partial x_j}{\partial x_i'} \right)^2 \frac{\partial^2 \phi'}{\partial \xi^2} \left( \frac{\partial \Phi \phi}{\partial x_j} \right)^2 + \sum_i \sum_{j<k} \sum_k 2 \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'} \frac{\partial^2 \phi'}{\partial \xi^2} \frac{\partial \Phi \phi}{\partial x_j} \frac{\partial \Phi \phi}{\partial x_k} = 0, \\ \square \phi &= 0. \end{aligned} \quad (1)$$

Here  $x = (x_1, x_2, x_3, x_4)$ ,  $x_4 = ict$ ,  $c$  is the speed of light,  $t$  is the time. Let us put the solution of d'Alembert equation  $\phi$  into the first equation of the set (1). If the obtained equation has a solution, then the set (1) will be compatible. According to Definition 1 it will mean that the arbitrary reversible transformations  $x' = x'(x)$  are the symmetry transformations of the initial equation  $\square \phi = 0$ . Owing to presence of the expressions  $(\partial \Phi \phi / \partial x_j)^2$  and  $(\partial \Phi \phi / \partial x_j)(\partial \Phi \phi / \partial x_k)$  in the first equation from the set (1), the latter has non-linear character. Since the analysis of non-linear systems is difficult we suppose that

$$\frac{\partial^2 \phi'}{\partial \xi^2} = 0. \quad (2)$$

In this case the non-linear components in the set (1) turn to zero and the system will be linear. As result we find the field transformation law by integrating the equation (2)

$$\phi' = C_1 \Phi \phi + C_2 \rightarrow \phi' = \Phi \phi. \quad (3)$$

Here we suppose for simplicity that the constants of integration are  $C_1 = 1$ ,  $C_2 = 0$ . It is this law of field transformation that was used within the algorithm [7, 8]. It marks the position of the algorithm in the generalized variables replacement method. Taking into account the formulae (2) and (3), we find the following form for the system (1):

$$\begin{aligned} & \frac{\partial^2 \phi'}{\partial \xi^2} = 0, \quad \phi' = \Phi \phi, \\ & \sum_j \square' x_j \frac{\partial \Phi \phi}{\partial x_j} + \sum_i \sum_j \left( \frac{\partial x_j}{\partial x_i'} \right)^2 \frac{\partial^2 \Phi \phi}{\partial x_j^2} + \sum_i \sum_{j<k} \sum_k 2 \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'} \frac{\partial^2 \Phi \phi}{\partial x_j \partial x_k} = 0, \\ \square \phi &= 0. \end{aligned} \quad (4)$$

Since here  $\Phi(x) = \phi'(x' \rightarrow x) / \phi(x)$ , where  $\phi'(x')$  and  $\phi(x)$  are the solutions of d'Alembert equation, the system (4) has a common solution and consequently is compatible. This means that the arbitrary reversible transformations of coordinates  $x' = x'(x)$  are symmetry transformations for the one-component d'Alembert equation if the field is transformed with the help of weight function  $\Phi(x)$  according to the law (3). The form of this function depends on d'Alembert equation solutions and the law of the coordinate transformations  $x' = x'(x)$ .

Next we shall consider the following examples.

Let the coordinate transformations belong to the *Poincaré group*  $P_{10}$ :

$$x'_j = L_{jk} x_k + a_j,$$

where  $L_{jk}$  is the matrix of the Lorentz group  $L_6$ ,  $a_j$  are the parameters of the translation group  $T_4$ . In this case we have

$$\square' x_j = \sum_k L'_{jk} \square' x'_k = 0, \quad \sum_i \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'} = \sum_i L'_{ji} L'_{ki} = \delta_{jk}.$$

The last term in the second equation (4) turns to zero. The set reduces to the form

$$\square\Phi\phi = 0, \quad \square\phi = 0. \quad (5)$$

According to Definition 1 this is a sign of the Lie symmetry. The weight function belongs to the set in [8]:

$$\Phi_{P_{10}}(x) = \frac{\phi'(x)}{\phi(x)} \in \left\{ 1; \frac{1}{\phi(x)}; \frac{P_j\phi(x)}{\phi(x)}; \frac{M_{jk}\phi(x)}{\phi(x)}; \frac{P_jP_k\phi(x)}{\phi(x)}; \frac{P_jM_{kl}\phi(x)}{\phi(x)}; \dots \right\},$$

where  $P_j, M_{jk}$  are the generators of Poincaré group,  $j, k, l = 1, 2, 3, 4$ . In the space of d'Alembert equation solutions the set defines a rule of the change from a solution to solution. The weight function  $\Phi(x) = 1 \in \Phi_{P_{10}}(x)$  determines transformational properties of the solutions  $\phi' = \phi$ , which means the well-known relativistic symmetry of d'Alembert equation [9, 10].

Let the transformations of coordinates belong to the *Weyl group*  $W_{11}$ :

$$x'_j = \rho L_{jk}x_k + a_j,$$

where  $\rho = \text{const}$  is the parameter of the scale transformations of the group  $\Delta_1$ . In this case we have

$$\square'x_j = \rho' \sum_k L'_{jk} \square'x'_k = 0, \quad \sum_i \frac{\partial x_j}{\partial x'_i} \frac{\partial x_k}{\partial x'_i} = \sum_i \rho'^2 L'_{ji} L'_{ki} = \rho'^2 \delta_{jk} = \rho^{-2} \delta_{jk}.$$

The set (4) reduces to the set (5) and has the solution  $\Phi_{W_{11}} = C\Phi_{P_{10}}$ , where  $C = \text{const}$ . The weight function  $\Phi(x) = C$  and the law  $\phi' = C\phi$  means the well-known Weyl symmetry of d'Alembert equation [9, 10]. Let here  $C$  be equal  $\rho^l$ , where  $l$  is the conformal dimension<sup>1</sup> of the field  $\phi(x)$ . Consequently, d'Alembert equation is  $W_{11}$ -invariant for the field  $\phi$  with arbitrary conformal dimension  $l$ . This property is essential for the Voigt [4] and Umov [12] works as will be shown just below.

Let the coordinate transformations belong to the *Inversion group*  $I$ :

$$x'_j = -\frac{x_j}{x^2}, \quad x^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2, \quad x^2 x'^2 = 1.$$

In this case we have

$$\square'x_j = \frac{4x'_j}{x'^4} = -4x_j x^2, \quad \sum_i \frac{\partial x_j}{\partial x'_i} \frac{\partial x_k}{\partial x'_i} = \rho'^2 (x') \delta_{jk} = \frac{1}{x'^4} \delta_{jk} = x^4 \delta_{jk}.$$

The set (4) reduces to the set:

$$-4x_j \frac{\partial \Phi\phi}{\partial x_j} + x^2 \square\Phi\phi = 0, \quad \square\phi = 0. \quad (6)$$

The substitution of  $\Phi(x) = x^2\Psi(x)$  transforms the equation (6) for  $\Phi(x)$  into the equation  $\square\Psi\phi = 0$  for  $\Psi(x)$ . It is a sign of the Lie symmetry. The equation has the solution  $\Psi = 1$ . The result is  $\Phi(x) = x^2$ . Consequently, the field transforms according to the law  $\phi' = x^2\phi(x) = \rho^{-1}(x)\phi(x)$ . This means the conformal dimension  $l = -1$  of the field  $\phi(x)$  in the case of d'Alembert equation symmetry with respect to the Inversion group  $I$  in agreement with [5, 10]. In a general case the weight function belongs to the set:

$$\Phi_I(x) = x^2\Psi(x) \in \left\{ x^2; \frac{x^2}{\phi(x)}; x^2 \frac{P_j\phi(x)}{\phi(x)}; x^2 \frac{M_{jk}\phi(x)}{\phi(x)}; x^2 \frac{P_jP_k\phi(x)}{\phi(x)}; \dots \right\}. \quad (7)$$

<sup>1</sup>The conformal dimension is the number  $l$  characterizing the behavior of the field under scale transformations  $x' = \rho x, \phi'(x') = \rho^l \phi(x)$  [11].

Let the coordinate transformations belong to the *Special Conformal Group*  $C_4$ :

$$x'_j = \frac{x_j - a_j x^2}{\sigma(x)}, \quad \sigma(x) = 1 - 2a \cdot x + a^2 x^2, \quad \sigma\sigma' = 1.$$

In this case we have

$$\square' x_j = 4(a_j - a^2 x_j) \sigma(x), \quad \sum_i \frac{\partial x_j}{\partial x'_i} \frac{\partial x_k}{\partial x'_i} = \rho'^2(x') \delta_{jk} = \sigma^2(x) \delta_{jk}.$$

The set (4) reduces to the set:

$$4\sigma(x) (a_j - a^2 x_j) \frac{\partial \Phi \phi}{\partial x_j} + \sigma^2(x) \square \Phi \phi = 0, \quad \square \phi = 0. \quad (8)$$

The substitution of  $\Phi(x) = \sigma(x)\Psi(x)$  transforms the equation (8) into the equation  $\square \Psi \phi = 0$  which corresponds to the Lie symmetry. From this equation we have  $\Psi = 1$ ,  $\Phi(x) = \sigma(x)$ . Therefore  $\phi' = \sigma(x)\phi(x)$  and the conformal dimension of the field is  $l = -1$  as above. Analogously to (7), the weight function belongs to the set:

$$\Phi_{C_4}(x) = \sigma(x)\Psi(x) \in \left\{ \sigma(x); \frac{\sigma(x)}{\phi(x)}; \sigma(x) \frac{P_j \phi(x)}{\phi(x)}; \sigma(x) \frac{M_{jk} \phi(x)}{\phi(x)}; \dots \right\}.$$

From here we can see that  $\phi(x) = 1/\sigma(x)$  is the solution of d'Alembert equation. Combination of  $W_{11}$ ,  $I$  and  $C_4$  symmetries means the well-known d'Alembert equation conformal  $C_{15}$ -symmetry [5, 9, 10].

Let the coordinate transformations belong to the *Galilei group*  $G_1$ :

$$x'_1 = x_1 + i\beta x_4, \quad x'_2 = x_2, \quad x'_3 = x_3, \quad x'_4 = \gamma x_4, \quad c' = \gamma c,$$

where  $\beta' = -\beta/\gamma$ ,  $\gamma' = 1/\gamma$ ,  $\beta = V/c$ ,  $\gamma = (1 - 2\beta n_x + \beta^2)^{1/2}$ . In this case we have

$$\begin{aligned} \square' x_j &= 0, \quad \sum_i \left( \frac{\partial x_1}{\partial x'_i} \right)^2 = 1 - \beta'^2, \quad \sum_i \left( \frac{\partial x_2}{\partial x'_i} \right)^2 = \sum_i \left( \frac{\partial x_3}{\partial x'_i} \right)^2 = 1, \\ \sum_i \left( \frac{\partial x_4}{\partial x'_i} \right)^2 &= \gamma'^2, \quad \sum_i \frac{\partial x_1}{\partial x'_i} \frac{\partial x_2}{\partial x'_i} = \sum_i \frac{\partial x_1}{\partial x'_i} \frac{\partial x_3}{\partial x'_i} = \sum_i \frac{\partial x_2}{\partial x'_i} \frac{\partial x_3}{\partial x'_i} = \sum_i \frac{\partial x_2}{\partial x'_i} \frac{\partial x_4}{\partial x'_i} = 0, \\ \sum_i \frac{\partial x_1}{\partial x'_i} \frac{\partial x_4}{\partial x'_i} &= i\beta'\gamma' = -i\beta/\gamma^2. \end{aligned}$$

After putting these expressions into the set (4) we find [8]:

$$\square \Phi \phi - \frac{\partial^2 \Phi \phi}{\partial x_4^2} - \left( i \frac{\partial}{\partial x_4} + \beta \frac{\partial}{\partial x_1} \right)^2 \frac{\Phi \phi}{\gamma^2} = \left[ \frac{(i\partial_4 + \beta\partial_1)^2}{\gamma^2} - \Delta \right] \Phi \phi = 0.$$

In accordance with Definition 1 it means that the Galilei symmetry of d'Alembert equation is the generalized symmetry. The weight function belongs to the set [7]:

$$\Phi_{G_1}(x) = \frac{\phi'(x' \rightarrow x)}{\phi(x)} \in \left\{ \frac{\phi(x')}{\phi(x)}; \frac{1}{\phi(x)}; \frac{P'_j \phi(x')}{\phi(x)}; \frac{[\square', H'_1] \phi(x')}{\phi(x)}; \dots \right\},$$

where  $H'_1 = i t' \partial_{x'}$  is the generator of the Galilei transformations. For plane waves the weight function  $\Phi(x)$  is [6, 7, 8]:

$$\Phi_{G_1}(x) = \frac{\phi(x' \rightarrow x)}{\phi(x)} = \exp \left\{ -\frac{i}{\gamma} \left[ (1 - \gamma) k \cdot x - \beta \omega \left( n_{xt} - \frac{x}{c} \right) \right] \right\},$$

where  $k = (\mathbf{k}, k_4)$ ,  $\mathbf{k} = \omega \mathbf{n}/c$  is the wave vector,  $\mathbf{n}$  is the wave front guiding vector,  $\omega$  is the wave frequency,  $k_4 = i\omega/c$ ,  $k'_1 = (k_1 + i\beta k_4)/\gamma$ ,  $k'_2 = k_2/\gamma$ ,  $k'_3 = k_3/\gamma$ ,  $k'_4 = k_4$ ,  $\mathbf{k}'^2 = \mathbf{k}^2$  - inv. (For comparison, in the relativistic case we have  $k'_1 = (k_1 + i\beta k_4)/(1 - \beta^2)^{1/2}$ ,  $k'_2 = k_2$ ,  $k'_3 = k_3$ ,  $k'_4 = (k_4 - i\beta k_1)/(1 - \beta^2)^{1/2}$ ,  $\mathbf{k}'^2 + k_4'^2 = \mathbf{k}^2 + k_4^2$  - inv as is well-known).

The results obtained above we illustrate by means of the Table 1:

Group	$P_{10}$	$W_{11}$	$I$	$C_4$	$G_1$
WF $\Phi(x)$	1	$\rho^l$	$x^2$	$\sigma(x)$	$\exp\{-i[(1 - \gamma)\mathbf{k} \cdot \mathbf{x} - \beta\omega(n_x t - x/c)]/\gamma\}$

For the different transformations  $x' = x'(x)$ , the weight functions  $\Phi(x)$  may be found in a similar way.

Let us note that in the symmetry theory of d'Alembert equation, the conditions (4) for transforming this equation into itself combine the requirements formulated by various authors, as can be seen in the Table 2:

Author	Coordinates transform.	Group	Conditions of invariance	Fields transform.
Voigt [4]	$x'_j = A_{jk}x_k$	$L_6X\Delta_1$	$A'_{ji}A'_{ki} = \rho'^2\delta_{jk}$	$\phi' = \phi$
Umov [12]	$x'_j = x'_j(x)$	$W_{11}$	$\frac{\partial x_j}{\partial x'_i} \frac{\partial x_k}{\partial x'_i} = \rho'^2\delta_{jk}$ , $\square'x_j = 0$	$\phi' = \phi$
Di Jorio [13]	$x'_j = L_{jk}x_k + a_j$	$P_{10}$	$L'_{ji}L'_{ki} = \delta_{jk}$ , $\frac{\partial^2 \phi'}{\partial \phi_\alpha \partial \phi_\beta} = 0$	$\phi' = m_\alpha \phi_\alpha + m_0$ $\alpha = 1, \dots, n$
Kotel'nikov [6, 7, 8]	$x'_j = x'_j(x)$	$C_4$	$\frac{\partial x_j}{\partial x'_i} \frac{\partial x_k}{\partial x'_i} = \rho'^2(x')\delta_{jk}$	$\phi'_\alpha = \psi D_{\alpha\beta}\phi_\beta$
			$\frac{\partial^2 \phi'_\alpha}{\partial \xi_\beta \partial \xi_\gamma} = 0$ , $\square' \phi'_\alpha = 0 \rightarrow$	$\xi_\alpha = \psi \phi_\alpha$
			$\hat{A}\phi'_\alpha(\psi\phi_1, \dots, \psi\phi_6) = 0$	$\alpha, \beta = 1, \dots, 6$
			$\square\phi_\beta = 0$	
	$x'_j = x'_j(x)$	$G_1$	$\frac{\partial^2 \phi'_\alpha}{\partial \xi_\beta \partial \xi_\gamma} = 0$ , $\square' \phi'_\alpha = 0 \rightarrow$	$\phi'_\alpha = \psi M_{\alpha\beta}\phi_\beta$
			$\hat{B}\phi'_\alpha(\psi\phi_1, \dots, \psi\phi_6) = 0$	$\xi_\alpha = \psi \phi_\alpha$
			$\square\phi_\beta = 0$	$\alpha, \beta = 1, \dots, 6$

Here  $m_\alpha$ ,  $m_0$  are some numbers,  $D_{\alpha\beta}$  and  $M_{\alpha\beta}$  are the  $6 \times 6$  numerical matrices.

According to this Table for the field  $\phi' = \phi$  with conformal dimension  $l = 0$  and the linear homogeneous coordinate transformations from the group  $L_6 \times \Delta_1 \in W_{11}$  with  $\rho = (1 - \beta^2)^{1/2}$ , the formulae were proposed by Voigt [4, 9]. In the plain waves case they correspond to the transformations of the 4-vector  $k = (\mathbf{k}, k_4)$  and proper frequency  $\omega_0$  according to the law  $k'_1 = (k_1 + i\beta k_4)/\rho(1 - \beta^2)^{1/2}$ ,  $k'_2 = k_2/\rho$ ,  $k'_3 = k_3/\rho$ ,  $k'_4 = (k_4 - i\beta k_1)/\rho(1 - \beta^2)^{1/2}$ ,  $\omega'_0 = \omega_0/\rho$ ,  $k'x' = kx$  - inv. In the case of the  $W_{11}$ -coordinate transformations belonging to the set of arbitrary transformations  $x' = x'(x)$  the requirements for the one component field with  $l = 0$  were found by Umov [12]. The requirement that the second derivative  $\partial^2 \phi' / \partial \phi_\alpha \partial \phi_\beta = 0$  with  $\Phi = 1$  be turned into zero was introduced by Di Jorio [13]. The weight function  $\Phi \neq 1$  and the set (4) were proposed by the author of the present work [6, 7, 8].

By now well-studied have been only the d'Alembert equation symmetries corresponding to the linear systems of the type (5), (6), (8). These are the well-known relativistic and conformal symmetry of the equation. The investigations corresponding to the linear conditions (4) are much more scanty and presented only in the papers [6, 7, 8]. The publications corresponding to the non-linear conditions (1) are absent completely. The difficulties arising here are connected with analysis of compatibility of the set (1) containing the non-linear partial differential equation.

### 3 Conclusion

It is shown that under generalized understanding of the symmetry according to Definition 1, d'Alembert equation for one component field is invariant with respect to any arbitrary reversible coordinate transformations  $x' = x'(x)$ . In particular, they contain transformations of the conformal and Galilei groups realizing the type of standard and generalized symmetry for  $\Phi(x) = \phi'(x' \rightarrow x)/\phi(x)$ . The concept of partial differential equations symmetry is conventional.

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