Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 2, 730–733

Systems of Linear Differential Equations of Rational Rank with Multiple Root of Characteristic Equation

Svetlana KONDAKOVA

National Aviation University, Kyiv, Ukraine
E-mail: *dc_analit@ukr.net*

A method of the reduction of linear differential equations with multiple root of the characteristic equation to which some multiple elementary divisors correspond to the system, the perturbed characteristic equation of which has the simple roots as well as asymptotic estimation of solutions obtained are presented.

Consider the system of linear differential equations of the following type

$$\varepsilon^{p/q}\frac{dx}{dt} = A(t,\varepsilon)x, \tag{1}$$

where x is an n-dimensional vector, $A(t,\varepsilon) = \sum\limits_{s=0}^{\infty} \varepsilon^s A_s(t)$ is real square $(n\times n)$ dimension matrix, whose elements are infinitely differentiable by t on the segment $[0;L]$, $\varepsilon > 0$ is a small parameter, p and q are natural relatively prime numbers. Besides let the inequality $p < n \leq q$ take place.

Let us denote $\varepsilon^{1/q} = \mu$. Then the system (1) reduces to the form

$$\mu^p\frac{dx}{dt} = \left(A_0(t) + \mu^q A_1(t) + \mu^{2q}A_2(t) + \cdots\right)x, \tag{2}$$

where $\varepsilon = \mu^q$, $\varepsilon^{\frac{p}{q}} = \mu^p$.

The systems for which small parameter has a fractional power were studied by V.K. Grigorenko in [1]. The case of the simple roots of the characteristic equations and the case of the equation having only one multiple n root were studied separately. Let us construct the asymptotic solution of the system (1) by the method of perturbed characteristic equation [2] for the case when the matrix $A_0(t)$ is such that the characteristic equation has one multiple root λ_0, to which $m \geqslant 1$ multiple elementary divisors correspond.

It means that there is non-degenerate matrix $T(t)$ which leads matrix $A_0(t)$ to the matrix with the simplest structure of quasi-diagonal type

$$W(t) = \{H_1(\lambda_0(t)), H_2(\lambda_0(t)), \ldots, H_m(\lambda_0(t))\},$$

where $H_i(\lambda_0(t))$ is Jordan cells, and the length of a cell is equal a multiplicity of elementary divisor, $i = 1, 2, \ldots, m$, m is the number of elementary divisors. Let us put that the elementary divisors with every value of $t \in [0;L]$ have the same multiplicity. Let a set of elementary divisors be $k_1, k_2, \ldots, k_m$ and $k_1 \geq k_2 \geq \cdots \geq k_m$. The substitution $x = T(t)y$ reduces the system (2) to system

$$\mu^p\frac{dy}{dt} = D(t,\mu)y, \tag{3}$$

where

$$D(t,\mu) = D_0(t,\mu) + \sum_{s=1}^{\infty}\mu^{qs}D_s(t),$$
$$D_0(t,\mu) = W(t) - \mu^p T^{-1}(t)T'(t), \qquad D_s(t) = T^{-1}(t)A_s(t)T(t),$$

$T'(t)$ is a derivative of matrix $T(t)$.

Let us consider perturbed equation

$$\det \|D_0(t,\mu) - \lambda E\| = 0. \tag{4}$$

Or opening the determinant (4),

$$(\lambda_0 - \lambda)^n + (\lambda_0 - \lambda)^{n-1} c_{n-1}(t,\mu) + \dots + c_1(t,\mu)(\lambda_0 - \lambda) + c_0(t,\mu) = 0. \tag{5}$$

It is known that coefficients $c_i(t,\mu)$ in expansion (5) will be equal to the sum of all principal minors $n-i$ order of the matrix

$$D_0(t,\mu) - \lambda_0(t)E = W(t) - \mu^p T^{-1}(t)T'(t) - \lambda_0(t)E$$
$$= \begin{pmatrix} \mu^p\bar{t}_{11} & 1+\mu^p\bar{t}_{12} & \dots & \mu^p\bar{t}_{1k_1} & \mu^p\bar{t}_{1k_1+1} & \mu^p\bar{t}_{1k_1+2} & \dots & \mu^p\bar{t}_{1n} \\ \mu^p\bar{t}_{21} & \mu^p\bar{t}_{22} & \dots & \mu^p\bar{t}_{2k_1} & \mu^p\bar{t}_{2k_1+1} & \mu^p\bar{t}_{2k_1+2} & \dots & \mu^p\bar{t}_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mu^p\bar{t}_{k_1,1} & \mu^p\bar{t}_{k_1,2} & \dots & \mu^p\bar{t}_{k_1,k_1} & \mu^p\bar{t}_{k_1,k_1+1} & \mu^p\bar{t}_{k_1,k_1+2} & \dots & \mu^p\bar{t}_{k_1,n} \\ \mu^p\bar{t}_{k_1+1,1} & \mu^p\bar{t}_{k_1+1,2} & \dots & \mu^p\bar{t}_{k_1+1,k_1} & \mu^p\bar{t}_{k_1+1,k_1+1} & 1+\mu^p\bar{t}_{k_1+1,k_1+2} & \dots & \mu^p\bar{t}_{k_1+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mu^p\bar{t}_{n1} & \mu^p\bar{t}_{n2} & \dots & \mu^p\bar{t}_{n,k_1} & \mu^p\bar{t}_{n,k_1+1} & \mu^p\bar{t}_{n,k_1+2} & \dots & \mu^p\bar{t}_{nn} \end{pmatrix},$$

where $\bar{t}_{ij}$ is a matrix element $-T^{-1}(t)T'(t)$, $i,j = \overline{1,n}$.

If m multiple elementary divisors correspond to multiple root, it means that all elements of the given matrix will be of $O(\mu^p)$ order, but it $n-m$ of the first over-diagonal elements will be $1+\mu^p\bar{t}_{ij}$. Proceeding from this, for estimation $\lambda - \lambda_0$ let us draw the first diagram of equation (5).

As ρ_{n-1} corresponds to polynomial power $c_{n-1}(t,\mu) = spD(t,\mu)$, and $\rho_{n-1} = p$. It's easy to see that all main minors, the order of which will be less or equal to k_1 will be of $O(\mu^p)$ order. So, $\rho_{n-k_1} = \rho_{n-k_1+1} = \dots = \rho_{n-1} = p$. The order of the next k_2 polynomials $\rho_{n-k_1-1}, \rho_{n-k_1-2}, \dots, \rho_{n-k_1-k_2}$ will be $O(\mu^{2p})$, because 1 more line of $O(\mu^p)$ order is added, and further $k_2 - 1$ the lines of $O(\mu^0)$ will be added. Estimating further the main minors of matrix we will come to conclusion that the main minors of $n, n-1, \dots, n-k_m+1$ order will be of $O(\mu^{mp})$ order. Thus, minimal power by μ of polynomials $c_i(t,\mu)$ may have the next values:

$$\begin{aligned}
&\rho_n = 0, \qquad \rho_{n-1} = p, \qquad p \le \rho_{n-2} \le 2p, \qquad p \le \rho_{n-3} \le 3p, \quad \dots, \\
&p \le \rho_{n-k_1} \le k_1 p, \qquad 2p \le \rho_{n-k_1-1} \le (k_1+1)p, \quad \dots, \\
&2p \le \rho_{n-k_1-k_2} = \rho_{k_m+k_{m-1}+\dots+k_3} \le (k_1+k_2)p, \quad \dots, \\
&(m-2)p \le \rho_{k_m+k_{m-1}} = \rho_{n-k_1-\dots-k_{m-2}} \le (k_1+\dots+k_{m-2})p = (n-k_m-k_{m-1})p, \\
&(m-1)p \le \rho_{k_m+k_{m-1}-1} \le (n-k_m-k_{m-1}+1)p, \quad \dots, \\
&l(m-1)p \le \rho_{k_m} = \rho_{n-k_1-k_2-\dots-k_{m-1}} \le (n-k_m)p, \\
&mp \le \rho_{k_m-1} \le (n-k_m+1)p, \quad \dots, \\
&mp \le \rho_1 \le (n-1)p, \qquad mp \le \rho_0 \le np.
\end{aligned}$$

Let us draw the obtained results (see Fig. 1).

Here $*$ denotes values meanings of ρ_i if coefficients with the smaller theoretically possible powers μ are equal zero. Figure shows that k_i of solving the equation (5) will be $O\left(\mu^{p/k_i}\right)$ order, $i = 1, 2, \dots, m$, moreover, they all will be different. So, for the case of several multiple elementary divisors the following theorem takes place.

Theorem 1. *If matrices $A_s(t)$ $(s = 0, 1, \dots)$ on the segment $[0; L]$ are infinitely differentiable and proper meanings of matrix $D_0(t,\mu)$ on the given segment are simple when $0 < \mu \le \mu_0$:*

$$\lambda_i(t,\mu) \ne \lambda_j(t,\mu), \qquad i,j = 1,\dots,n, \qquad i \ne j, \qquad \forall\, t \in [0; L],$$

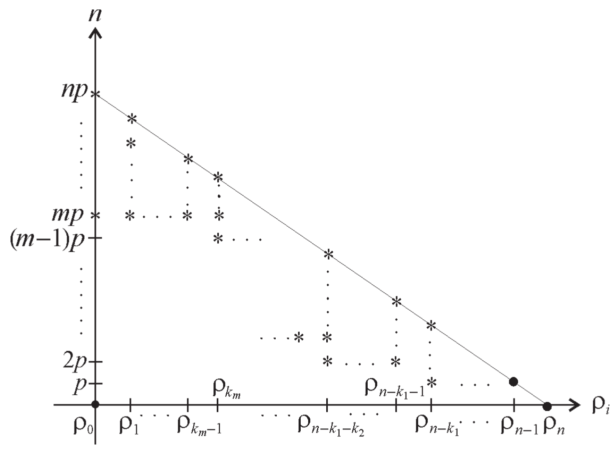

Figure 1.

then the system of differential equations (3) has a formal matrix-solution

$$Y(t,\mu) = U(t,\mu,\mu)\exp\left(\frac{1}{\mu^p}\int_0^t \Lambda(\tau,\mu,\mu)\,d\tau\right),$$

where $U(t,\mu,\mu)$ is a square matrix of n order, $\Lambda(t,\mu,\mu)$ is a diagonal matrix of n order, they are represented by formal series

$$U(t,\mu,\mu) = \sum_{s=0}^{\infty}\mu^s U_s(t,\mu), \qquad \Lambda(t,\mu,\mu) = \sum_{s=0}^{\infty}\mu^s \Lambda_s(t,\mu), \tag{6}$$

where $\mu = \sqrt[q]{\varepsilon}$.

This theorem is proved by the method from [3], as a result we have

$$U_0(t,\mu) = B(t,\mu), \qquad \Lambda_0(t,\mu) = W^*(t,\mu),$$

where $B(t,\mu)$ is the transforming matrix, which leads the matrix $D_0(t,\mu)$ to the diagonal matrix $W^*(t,\mu) = \{\lambda_1(t,\mu), \lambda_2(t,\mu), \ldots, \lambda_n(t,\mu)\}$,

$$\Lambda_s(t,\mu) = G_{1s}(t,\mu), \qquad s = 1,2,\ldots, \tag{7}$$

$G_{1s}(t,\mu)$ is obtained from the diagonal elements of matrix

$$G_s(t,\mu) = B^{-1}(t,\mu)H_s(t,\mu), \tag{8}$$

$$H_s(t,\mu) = \sum_{j=1}^{\left[\frac{s}{q}\right]} D_j(t)U_{s-jq}(t,\mu) - \sum_{i=1}^{s-1} U_i(t,\mu)\Lambda_{s-i}(t,\mu) - U'_{s-p}(t,\mu), \tag{9}$$

$$U_s(t,\mu) = B(t,\mu)Q_s(t,\mu), \tag{10}$$

where $Q_s(t,\mu)$ is the matrix the elements of which are found from the formulas

$$q_{sij}(t,\mu) = \frac{g_{sij}(t,\mu)}{\lambda_j(t,\mu) - \lambda_i(t,\mu)}, \qquad i \neq j, \qquad i,j = \overline{1,n}. \tag{11}$$

The diagonal elements of the matrix $Q_s(t,\mu)$ vanish.

Consider the matrix (9). It is easily seen that for $s < p$ $H_s(t,\mu) \equiv 0$, and from (7)–(11) $U_s(t,\mu) \equiv \Lambda_s(t,\mu) \equiv 0$, $s = 1, 2, \ldots, p-1$ is produced. As in expansion (6) these elements will follow $U_0(t,\mu), \Lambda_0(t,\mu)$, then we will write down the series (6) in the following way

$$U(t,\mu,\mu) = B(t,\mu) + \sum_{s=p}^{\infty} \mu^s U_s(t,\mu) = B(t,\varepsilon) + \sum_{s=p}^{\infty} \varepsilon^{\frac{s}{q}} U_s(t,\varepsilon),$$
$$\Lambda(t,\mu,\mu) = W^*(t,\mu) + \sum_{s=p}^{\infty} \mu^s \Lambda_s(t,\mu) = W^*(t,\varepsilon) + \sum_{s=p}^{\infty} \varepsilon^{\frac{s}{q}} \Lambda_s(t,\mu). \tag{12}$$

The following theorem is true.

Lemma 1. *Let the conditions of Theorem 1 be satisfied be* $\bar{t}_{k_1,1}(t) \neq 0$. *Then the coefficients of the formal series (12) are given by*

$$U_s(t,\mu) = B(t,\varepsilon) + \varepsilon^{-\frac{p}{qk_1}(s-p+1)} U_s^a(t,\varepsilon),$$
$$\Lambda_s(t,\mu) = W^*(t,\varepsilon) + \varepsilon^{-\frac{p}{qk_1}(s-p)} \Lambda_s^a(t,\varepsilon), \qquad s = p, p+1, \ldots, \tag{13}$$

where $U_s^a(t,\mu)$, $\Lambda_s^a(t,\mu)$ *are matrices which do not have asingularity in point* $\mu = 0$.

This Lemma is proved by immediate analysis of the matrixes elements (7)–(11).

Let us substitute (13) for (12). We will have

$$U(t,\mu,\mu) = U_0(t,\varepsilon) + \sum_{s=p}^{\infty} \varepsilon^{\frac{s}{q}} \varepsilon^{-\frac{p}{qk_1}(s-p+1)} U_s^a(t,\varepsilon),$$
$$\Lambda(t,\mu,\mu) = \Lambda_0(t,\varepsilon) + \sum_{s=p}^{\infty} \varepsilon^{\frac{s}{q}} \varepsilon^{-\frac{p(s-p)}{qk_1}} \Lambda_s^a(t,\varepsilon).$$

Lemma 2. *Let the condition of Theorem 1, Lemma 1,*

$$\operatorname{Re}(\lambda_i(t,\mu)) \leq 0$$

be satisfied on the set $\{K : t \in [0; L], 0 < \mu \leq \mu_0\}$, *then on the segment* $[0; L]$ *m-th approximation satisfies the differential system* (1) *up to the order of magnitude* $O\left(\varepsilon^{\frac{1}{q}\left((m+1-p)\left(1-\frac{p}{k_1}\right)+p\right)}\right)$.

Theorem 2. *Let the condition of Theorem 1, Lemma 2 be satisfied and for* $t = 0$

$$y(t,\mu) = y_m(t,\mu),$$

where $y(t,\mu)$ *is the exact solution of the system (3), then for any* $L > 0$ *there is* $c > 0$, *which does not depend upon* μ *and is such that for all* $t \in [0; L]$, $\mu \in (0; \mu_0]$ *the inequality is satisfied*

$$\|y(t,\mu) - y_m(t,\mu)\| \leq \mu^{(m+1-p)\left(1-\frac{p}{k_1}\right)-p+1} c.$$

Lemma 2 and Theorem 2 are proved by the methods from [3].

[1] Grigorenko V.K., About asymptotic expansion solving the linear differential equations systems, PhD Thesis, Kyiv, 1971 (in Russian).

[2] Shkil' N.I., About the periodical solution of the systems of linear differential equations of the second order, *Archivum Mathematicum, Brno*, 1987, V.23, N 1, 53–62 (in Russian).

[3] Feshchenko S.F., Shkil' N.I. and Nikolenko L.D., Asymptotic methods in the theory of linear differential equations, New York, American Elsevier Publishing Company, 1967.