

General Relativity as a Symmetry of a Unified Space–Time–Action Geometrical Space

Jaime KELLER

*Departamento de Física y Química Teórica, Facultad de Química,
Universidad Nacional Autónoma de México, AP 70-528, 04510, México D.F., México and
Center for Computational Materials Science (CMS), Technische Universität Wien,
Getreidemark 9/158, A 1060, Vienna, Austria
E-mail: keller@servidor.unam.mx, keller@cms.tuwien.at*

We derive the basic principles of Electromagnetism and general relativity from a common (geometrical) starting formulation we call START, from its geometrical structure as a Space–Time–Action Relativity Theory. Gravitation results from the epistemological approach of defining a test particle which explores the physical world in such a form that its trajectory indicates the influence of the rest of the system. Electromagnetism defines a collection of test particles, we call carriers, in interaction among themselves and with the rest of the system. General Relativity is then derived from a symmetry transformation of the quadratic space geometry corresponding to space–time and action and the philosophical principles of Einstein’s general relativity theory.

1 Introduction

We present a deductive approach to General Relativity (GR) Theory, deriving it from the quadratic space geometry corresponding to the, in our approach fundamental concepts, of space, of time, and of action and from the philosophical principles of Einstein’s general relativity theory. Our basic and more fundamental idea is that the physical world can be described as a distribution of action density in space–time. The properties of matter fields and their interaction are represented by the mathematical properties of this distribution. Action is considered as a fundamental variable, not as a quantity resulting from some calculation.

In [6, 7, 12] we analyzed a classical theory of fields in (complex) space–time geometry and arrived to the conclusion that this geometry corresponds to a unified space–time–action geometry. We started from three basic assumptions: a) The use of space–time as a basic frame of reference; b) The introduction of physical phenomena as an action density function over space–time; and c) The geometrical (vectorial) union of space, time, and action in a quadratic space where a relativistic condition $(dS)^2 = 0$ defines both kinematics and dynamics. The basic principles of this Space–Time–Action Relativity Theory (START) are used to derive General Relativity.

In the construction of General Relativity the procedure is to perform a symmetry transformation which modifies the space–time components, and then the metric, by the allocation of the appropriate amount of action corresponding to the additional action attributed to a test particle as a result of the interaction with the rest of the world. This transformation is made at the level of the quadratic form. This kind of transformations which transfer action into equivalent space–time to modify the metric tensor will be interpreted by the observer as those changes in the metric corresponding to its clocks running slower and the space intervals becoming larger, as is customary in the analysis of general relativity.

1.1 Carriers, Action, Space and Time

Action, as a fundamental variable, is distributed among a set of **carrier (of action) fields**. An action density $a(\mathbf{x}, t)$ is the fundamental concept defining all three space (parametrized by \mathbf{x}), time (parametrized by t) and action density (parametrized by a scalar analytical function $a(\mathbf{x}, t)$) as **primitive** concepts from which all other physical quantities will be derived or at least related directly or indirectly. The density of a carrier field can be defined through a set of scalar constants such that the integral of the product of these constants and the density gives the experimentally attributed value of a property for that carrier. A carrier field identified will have a density $\rho(\mathbf{x}, t)$ and if the property is Q we will obtain the definition $Q = \int q(\mathbf{x}, t)d\mathbf{x} = \int Q\rho(\mathbf{x}, t)d\mathbf{x}$ for all t , which defines that Q is a constant property (in space and time) for that field. The set of properties $\{Q\}$ characterizes a carrier field and in turn establishes the conditions for a density field to correspond to an acceptable carrier.

We already stated that in our theory space and time are fundamental, primordial, concepts. The geometrical unification of these concepts into a space–time coordinate $\mathbf{x} = (\mathbf{x}, ct)$ and an interval ds^2 requires the introduction of a universal constant: the speed of light c . As we will also use action as a fundamental concept we need another universal constant $\kappa = d_0/h$ which we will construct from a fundamental distance d_0 and a fundamental unit of action we will choose to be Planck’s action constant h . In this form we will have a five dimensional, START, manifold $3 + 1 + 1$ with all dimensions in units of length. To agree with standard formulations energy $E = \partial a/\partial t$ and momentum $p_i = \partial a/\partial x^i$ are the fundamental rates of change of the primitive concept of action. It is also appropriate to say that the concept of **matter**, hitherto formally undefined, will acquire proper formal definition in the context of the different structural theories. Then the START theory presented here corresponds to a geometrization beyond Minkowski’s fundamental paper. In fact that author, introducing the space-time interval squared ds^2 , adds: “the axiom signifies that at any world-point the expression $c^2 dt^2 - dx^2 - dy^2 - dz^2$ always has a positive value, or, what comes to the same thing, that any velocity v always proves less than c ”. In our full geometrization scheme action change $dK = P \cdot dX$ is introduced through a series $|dK|^2$ of quadratic terms

$$dS^2 - ds^2 = -|dK|^2 = -\{(E^2/c^2)c^2 dt^2 - p_x^2 dt^2 - p_y^2 dt^2 - p_z^2 dt^2\}, \quad (1)$$

joined in one unified geometrical quadratic form dS^2 . The dK vector, the directional in space-time change of action, is a new theoretical quantity formally defined by (1). It acquires additional relevance because action density will be described as a sum of contributions over carriers, $a = \sum_c a_c$. We are constructing a systematic deductive approach to Physics and it is essential that we derive many of the basic useful structures which have been used.

For a given observer the carrier field c is defined to have an energy density $\frac{1}{N_c}\mathcal{E}_c\rho_c^{(x,t)}$ with \mathcal{E}_c a constant in space and N_c the integer number of carrier units of type c . The density $\rho_c(x, t)$ is required to obey $\int_V \rho_c(x, t)dx = N_c$ in the system volume V .

Action is in our approach a coordinate (expressed in units of distance) and one of the properties of a distribution describing, in relation to an observer, the contents of the physical world in space–time. The concept of Physical Phenomena refers to the existence and change of this distribution. Physics corresponds to the description of the action distribution and its changes in relation to a given observer.

The action density function in space–time $a(X)$, or energy density in space for a given observer, can be considered as a density of action trajectories in space–time. For elementary carriers the trajectories would be elementary trajectories. Both the density function $a(X)$ and the splitting among carrier fields will be considered analytically well behaved functions.

The energy is $\mathcal{E} = \sum_c \mathcal{E}_c$ a sum of constants for a given observer, assumed to be distributed among the different carriers $\{c\}$ and can furthermore be described as a sum of contributions

per carrier. The simplest, almost universal, type of distribution per carrier type is into the constitution energy \mathcal{E}_0 , the position dependent kinetic energy $\mathcal{E}_k(X)$, and the position dependent sum of potential energies $\mathcal{E}_v(X)$

$$\mathcal{E}_c = \mathcal{E}_0^c + \mathcal{E}_k^c(X) + \mathcal{E}_v^c(X) + \mathcal{E}_\Delta^c(X). \tag{2}$$

It is precisely this distribution (2) which defines the carrier for a given observer. \mathcal{E}_0^c defines the basic carrier, $\mathcal{E}_k^c(X)$ the state of motion relative to the observer, and $\mathcal{E}_v^c(X)$ the relation between that carrier and the rest of the system as defined by the observer. The $\mathcal{E}_\Delta^c(X)$ term is required to make \mathcal{E}_c a position independent constant, this is needed to have a meaningful definition of the carriers of type c . The action is considered to be distributed among interacting carriers, and the concept of charges of the carriers has been introduced. Consider the simplest possibility that this action does not depend on the direction, and that at a given distance from the source, in concentric spheres, the total force field per charge should be independent of the distance from the charge then

$$F_Q = F(r)4\pi r^2 = \frac{Q}{\epsilon_0 4\pi r^2} 4\pi r^2 = Q/\epsilon_0 \tag{3}$$

which expresses that a definite capability is attributed to a charge Q . The factor $1/\epsilon_0$ represents any additional condition that the observer has to include to match its definitions.

We use in the analysis a tetrad of, observer dependent, basis vectors $\{e_0, e_1, e_2, e_3\}$, with $e_0^2 = -e_1^2 = -e_2^2 = -e_3^2 = 1$ and the definition property $e_\mu e_\nu = -e_\nu e_\mu$. We also use the notation $e_{0j} = e_0 e_j = e_j$ ($i, j, k = 1, 2, 3$) and $e_5 = e_0 e_1 e_2 e_3 = e_{0123}$. For a given observer a space-time d'Alembertian operator \square has the property $e_0 \square = \frac{1}{c} \partial_t + \nabla = \frac{1}{c} \partial_t + e_i \partial_i$, with ∇ the (ordinary space) Laplacian operator for that observer.

a) In space-time-action the action density $a(x, t)$ is inhomogeneously distributed, corresponding to the different material objects to which this action corresponds, in a possible relative motion in the spatial directions with speeds $0 \leq v \leq c$. Distributions which move with relative velocities $0 \leq v < c$ with respect to a given observer are called matter-like.

b) The matter-like energy distributions are to be considered as sources of (infinite extension, in principle) decaying deformations of action distribution of several types. This second property is not given *a priori* but it is a consequence of the description of the objects, as developed in the previous section, shown below.

c) We introduce now a third fundamental concept: energy-momentum carriers, the definition of identical carriers as a density in a space volume V_s such that at time $t = t'$

$$\int_{V_s} \rho_b dx = n_b, \quad h \int_{V_s} \partial_t a_b dx = \int_{V_s} \rho_b \epsilon_b dx = n_b \epsilon_b = E_b, \tag{4}$$

and $E_{t'} = \left[\sum_b E_b \right]_{t'}$ for a collection $\{b\}$ of (by construction) independent types of carriers. In agreement with our freedom of description we could also allow the n_b not to be integers, provided $E_{t'}$ is not changed. Here

$$\epsilon = m_c c^2 + \mathbf{kin} \left[\rho_0^{(N)}(\mathbf{x}) \right] + \mathbf{V}(\mathbf{x}) + \mathbf{W}_{\text{int},xc} \left[\rho_0^{(N)}(\mathbf{x}) \right] + \epsilon_0 \left[\rho_0^{(N)}(\mathbf{x}) \right], \tag{5}$$

the constitutional energy of the carriers $m_c c^2$, actual local kinetic energy per carrier $\mathbf{kin} \left[\rho_0^{(N)}(\mathbf{x}) \right]$, external potential energy per carrier (in its simplest form) $\mathbf{V}(\mathbf{x})$. Second $\mathbf{W}_{\text{int},xc} \left[\rho_0^{(N)}(\mathbf{x}) \right]$ to define independent carriers and finally a local energy term $\epsilon_0 \left[\rho_0^{(N)}(\mathbf{x}) \right]$ to compensate for any difference in the sum of the previous ones with respect to ϵ . The last two terms define in practice an actual carrier in the system (a pseudo-carrier in condensed matter physics language) as different from an isolated carrier.

2 Space–Time–Action Relativity Theory

Our basic and more fundamental idea [6, 7, 8, 10, 11, 12] is that the physical world can be described as a distribution of action density in space–time. The properties of matter fields and their interaction are represented by the mathematical properties of this distribution. Action is considered as a fundamental variable, not as a quantity resulting from some calculation.

We use the traditional indices 0, 1, 2, 3 for the joint time and space coordinates x_μ , also, the vectors e_μ in the geometry of space–time. They generate G_{ST} a 16 dimensional space–time geometry of multivectors. A special property of the pseudo-scalar (and also volume or inverse volume) in space–time $e_5 = e_0e_1e_2e_3$, and of the linearly independent combination $e_4 \Rightarrow ij e_5$, is that $e_5e_\mu = -e_\mu e_5$ (from $e_\mu e_\nu = -e_\nu e_\mu$, $\mu \neq \nu$). Its use allows the construction of a geometrical framework for the description of physical processes: a unified space–time–action geometry $G_{STA} = G_{ST} \otimes C$, mathematically a vector space with a quadratic form. The auxiliary element j commutes with all e_μ : $e_\mu j = j e_\mu$. In the G_{STA} geometry the coordinates are real numbers.

2.0.1 Formal presentation

The ideas developed in START (Space–Time–Action Relativity Theory) are derived from the systematic use of the following principles and postulates [10, 11, 12].

First Principle: *Principle of Relativity.* Constancy of the **value of the observed velocity of light** in vacuum. Independently of the state of movement of the source or of the observer (Poincaré–Einstein Relativity {Poincaré 1904, Einstein 1905 [1]}). The *Principle of Relativity* in full also requires that the laws of Physics should have the same form for all observers.

First Postulate. There is a geometry, corresponding to space–time, where the First Principle is satisfied (Minkowski space–time with local pseudo-Euclidean structure). Here it is clear that the velocity of light is a fundamental geometrical parameter and the First Principle could be rephrased to assign a unit value to it.

Second Principle: *Principle of Existence.* Constancy of the **action** corresponding to a physical system and in particular to an elementary physical phenomenon. Independently of the state of movement of the phenomenon or of the observer. Each observer considers the physical entity as an amount of action A contained in a given space-time volume V_{ST} , A is a relativistic invariant in the sense of Minkowski’s discussion.

Second Postulate. There is a geometry corresponding to the union of space–time and action where the First and Second Principles are satisfied (pseudo-Euclidean structure).

Main Theorem KT: *Complex Structure Theorem.* The geometry where the Second Postulate is satisfied is a five-dimensional basis geometry, mathematically corresponding to a particular complexification of space–time. The relation between a 5-dimensional geometry and the complexification of the basis set has been briefly presented in the introduction and will be discussed below.

Third Principle: *Principle of Quantization.* The exchanges in action always occur as integer multiples of h . (This has to be a constitutive part of the units and practical use of **KT** theorem). This makes Planck’s principle a universal principle which requires the definition of the entities we have called **action carriers**, because if there are not differentiated action carriers there is not a proper definition of the exchanges of action. This is also a guide and a limitation in the definition of the action carriers and of the equivalent length associated to the time interval in which the system with total action A is defined.

Fourth Principle: *Principle of Choice.* The distribution of action in space–time corresponding to a physical system is unique and any description of this distribution should be equally acceptable. The acceptability of a description, in relation to the Third Principle, is interpreted here as implying an optimization of the action distribution among the available

number of START cells of action and a mechanism to allow the system of carriers to arrive to this optimized state.

Third Postulate. The equivalent acceptable descriptions, for a physical system of carriers, are related by isometries and gauge transformations in the space–time–action geometric space corresponding to the Second Postulate.

Proof of KT. We have the kinematical concept of trajectory $(\mu, \nu = 0, 1, 2, 3)$ with a quadratic form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{6}$$

generated by the dx^μ and we want to include as a fifth coordinate the dynamical concept of action and its distribution at each space–time point $X = x^\mu e_\mu$, use the real scalar quantity

$$dA(X) = p_\mu(X) dx^\mu \quad \text{which defines} \quad p_\mu(X) = \partial A(X) / \partial x^\mu, \tag{7}$$

here $p_\mu(X)$ is a distribution itself, write $p_\mu(X) = \tan \Theta(X, \mu)$ and join formally, defining the (hyper)complex numbers $\mathbf{j}^2 = -\mathbf{1}$ and $i^2 = -1$, into

$$dS^\mu = dx^\mu (\mathbf{1} + \mathbf{j} \kappa_0 i \tan \Theta(X, \mu)), \tag{8}$$

to obtain from the **real** quadratic form (in units of distance square)

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu (\mathbf{1} - \kappa_0^2 \tan \Theta(X, \mu) \tan \Theta(X, \nu)), \tag{9}$$

or, in five-dimensional-like formulation

$$dS^2 = g_{uv} dx^u dx^v = ds^2 - \kappa_0^2 |dK(X)|^2, \quad u, v = 0, 1, 2, 3, 4, \tag{10}$$

where $\kappa_0^2 |dK(X)|^2$ corresponds to the sum of the squares of action contributions. Both quantities i and \mathbf{j} are necessary to explicitly show the complex structure and give simultaneously the desired metric. This has introduced a new 4 – D vector function (remark: e_μ and ie_μ are linearly independent vectors)

$$dK(X) = dK(X)^\mu i e_\mu = \sum_\mu \tan \Theta(X, \mu) dx^\mu i e_\mu,$$

$$dK^\mu = \left(\frac{\partial A}{\partial x^\mu} \right) dx^\mu = \tan \Theta(X, \mu) dx^\mu.$$

It is important to notice that it is not the actual value of the action density which is dynamically important but its space-time variations. Even more important for dynamics is that, when the action density is considered a sum $a = \sum_c a_c$ over carriers, the contributions to $dK = \sum_c (dK)_c$ per carrier could be non-zero even is the sum could itself be null. That is the dynamics could be purely **relative dynamics**. The basic dynamical equation is proposed to be

$$\delta \int dS^2 = 0, \tag{11}$$

in a joint minimization of trajectory and action. The universal constant κ_0 , the ratio of a fundamental distance to the fundamental unit of action, expresses the action as an equivalent distance in such a form that $(dx^4)^2 = |(\kappa_0 dK)|^2$, with $g_{mn} = \text{diag}(1, -1, -1, -1, -1)$ defines the metric of the equivalent five dimensional geometry basis vectors. ■

For the units to be used in the unified geometry consider the definition (m_0 electron rest mass, c speed of light, $h = 2\pi\hbar$ Planck’s constant, e electron charge, r_0 classical electron radius and $\alpha = e^2/\hbar c$)

$$r_{\text{Compton}} = \frac{\hbar}{2m_0 c} = \frac{r_0}{2\alpha}, \quad \kappa_0 = \frac{d_0}{h} = 4\pi r_{\text{Compton}}/h = \frac{1}{m_0 c}. \tag{12}$$

The classical limit of the unification of action to space–time is obtained when $\kappa_0 \rightarrow \infty$ in a form similar to the classical limit of the unification of space and time being obtained when $c \rightarrow \infty$. Note $\kappa_0 \gg c$.

From our definitions we are considering two quantities: energy $\int dV \partial_t a(\mathbf{X})$ and the corresponding momenta $\int dV \partial_{x_i} a(\mathbf{X})$. One of the basic relations in relativistic dynamics is the transformation of the above quantities with respect to observers in relative motion with a relative velocity v_{12} .

For observer 1 $\mathcal{E} = mc^2 = \int dV \partial_t a(\mathbf{X})$, if by definition for this observer that object is at rest and then the energy corresponds to a mass m and no momenta are involved.

For observer 2 the same relations hold. The energy for this observer will be $\mathcal{E}' = m'c^2 = \int dV' \partial_{t'} a'(\mathbf{X}')$, larger than \mathcal{E}

$$\mathcal{E}' = \frac{\partial}{\partial t'} A = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow \text{with } \lim(v \ll c) \rightarrow mc^2 + \frac{1}{2}mv^2 + \dots, \quad (13)$$

and he can call the energy \mathcal{E}' the sum of the rest (mass) energy \mathcal{E} and the kinetic energy \mathcal{E}_k .

2.1 Dynamical principles as symmetries

In space–time–action geometry the main dynamical principle should be that all elementary trajectories be minimal. That is, from our definition of carriers above where $dA = \{\epsilon_c \int \rho_c(X) dx\} dt$, a minimization of a total action A (in the case when we admit that the $\kappa_0 \gg 1$ predominates) or a minimization of a START trajectory. Defining the (square of the) differential $(dS)^2 = (ds)^2 - (da)^2$, where $(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu$ is the space–time differential and $(da)^2$ the action differential, the minimal principle

$$\delta(dS)^2 = 0, \quad (14)$$

could be separated into a kinematical principle similar to the one of (general) relativity and an additional principle of minimum action

$$\delta(ds')^2 = 0, \quad \delta(da')^2 = 0, \quad (15)$$

$$(ds')^2 = (ds)^2 - [(da)^2 - (da')^2], \quad (16)$$

as a modified space–time interval square which, in fact, corresponds to considering a curved effective space–time as will be shown below. The action interval square $(da')^2$ corresponds to some ‘inactive’ part of the action in relation to a given geometrical description.

3 Maxwell and Newton equations from START

Let us formally show that the Maxwell equations in their standard textbook form are analytical properties of the third derivatives of the action density attributed to a test carrier (with ‘electric’ charge).

In the reference frame of a given observer the induced action density, denoted by $a_e(X)$, per unit charge (\Rightarrow push) of a test carrier at space–time point $X = x^\mu e_\mu$ (here the greek indices $\mu = 0, 1, 2, 3$ and $x^0 = ct$ whereas the space vectors $\mathbf{q} = q^i e_i = q_i e^i$, $i = 1, 2, 3$ are written in bold face letters), the related energy density $\mathfrak{E}_e(X)$ and the total (external plus induced) momentum density \mathbf{p}_e , **per unit charge of the test carrier**, would be

$$\mathfrak{E}_e(X) = \frac{\partial a_e(X)}{\partial t}, \quad \mathbf{p}_e = p_{e,i} e^i = \left(\frac{\partial a_e(X)}{\partial x^i} + \Delta_{RP} p_{e,i} \right) e^i, \quad (17)$$

and the, by definition, electric field intensity \mathbf{E} is the force (puch) corresponding to this terms

$$\mathbf{E} = \left(\frac{\partial \mathfrak{E}_e(X)}{\partial x^i} + \frac{\partial p_{e,i}}{\partial t} \right) e^i = \nabla \mathfrak{E}_e(X) + \frac{\partial \mathbf{p}_e}{\partial t},$$

with time dependence

$$\frac{\partial \mathbf{E}}{\partial t} = \left(\frac{\partial^2 \mathfrak{E}_e(X)}{\partial t \partial x^i} + \frac{\partial^2 p_{e,i}}{\partial t \partial t} \right) e^i = 2 \frac{\partial^3 a_e(X)}{\partial t \partial x^i \partial t} e^i + \frac{\partial^2 (\Delta_{RP_{e,i}})}{(\partial t)^2} e^i.$$

By definition of interacting carriers, we have added in (17) the term $\Delta_{RP_{e,i}} e^i$ as the effect of the conservation of **interaction transverse momenta** between the field representing the rest of the carriers with that sort of charges. This is by definition the origin, in START, of a magnetic field intensity $\mathbf{B} = B_k e^k$ that will appear as the curl of the momentum (puch) of an interaction field acting on a carrier of type b. The axial vector

$$\mathbf{B} = \left(\frac{\partial p_{e,i}(X)}{\partial x^j} \right) e^j \times e^i = \nabla \times \mathbf{p}_e, \quad \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial^2 p_{e,i}(X)}{\partial t \partial x^j} e^j \times e^i.$$

Otherwise the space variation of \mathbf{E} , including the **interaction transverse momenta**,

$$\nabla \mathbf{E} = \nabla \cdot \mathbf{E} + \nabla \times \mathbf{E}, \quad (18)$$

will then also include a transversal (rotational) term

$$\nabla \times \mathbf{E} = \frac{\partial^2 p_{e,j}(X)}{\partial x^i \partial t} e^i \times e^j = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2nd \text{ Maxwell Equation})$$

relation which is the direct derivation in START of this well known Maxwell equation. The scalar term $\nabla \cdot \mathbf{E}$ being a divergency of a vector field should be defined to be proportional to a source density

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho = \sum_i \left(\frac{\partial^3 a_e(X)}{\partial x^i \partial x^i \partial t} \right) = \frac{\partial}{\partial t} \nabla^2 a_e(X), \quad (1st \text{ Maxwell Equation})$$

and will be given full physical meaning below. For the space variation of \mathbf{B} we have

$$\nabla \mathbf{B} = \nabla \cdot \mathbf{B} + \nabla \times \mathbf{B}.$$

The first term vanishes identically in our theory because it corresponds to the divergence of the curl of a vector field

$$\nabla \cdot \mathbf{B} = 0, \quad (3rd \text{ Maxwell Equation})$$

while the last term, using $U \times V \times W = V(U \cdot W) - (U \cdot V)W$

$$\nabla \times \mathbf{B} = \nabla (\nabla^2 a_e(X)) - \nabla^2 \mathbf{p}_e = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \quad (4th \text{ Maxwell Equation})$$

where the additional dimensional constant μ_0 is needed to transform from time units (used in the conceptual definition of a current $\mathbf{J} = \nabla (\nabla^2 a_e(X)) / \mu_0$) into distance units and in fact $\epsilon_0 \mu_0 = c^{-2}$.

The derived Maxwell equations are formally equivalent to the original Maxwell equations.

Both the 4th Maxwell Equation, defining \mathbf{J} , related to a Lorentz transformation of the 1st Maxwell Equation, defining ρ , can immediately be integrated using geometric analysis techniques, the standard approach being of fundamental conceptual consequences in START. The

space divergence of a non-solenoidal vector field like \mathbf{E} is immediately interpreted as its ‘source’ given that $\Delta \mathbf{E} = (\nabla \cdot \mathbf{E}) S \Delta \mathbf{x}$, and this equation is integrated using the standard geometric theorem that the volume integral of a divergence $\nabla \cdot \mathbf{E}$ equals the surface integral of the normal (to the surface) component of the corresponding vector field $\mathbf{n} \cdot \mathbf{E}$. Where \mathbf{n} is a unit vector perpendicular to the surface S (in the text-book formula below $S = 4\pi r^2$ corresponding to an integration sphere of radius r containing a spherically symmetric source density $\rho(r)$ generating a force field per unit charge $\mathbf{E} = E(r) \frac{\mathbf{r}}{r}$) of the integration volume $V = 4\pi r^3/3$:

$$\int_V (\nabla \cdot \mathbf{E}) dV = \int_V \frac{4\pi}{\epsilon_0} \rho(r') r'^2 dr' = \frac{1}{\epsilon_0} Q \quad \Longrightarrow \quad \mathbf{E} = E(r) \frac{\mathbf{r}}{r} = \frac{Q}{4\pi\epsilon_0 r^2} \frac{\mathbf{r}}{r}.$$

In the case of the, generated by a current, magnetic force field intensity \mathbf{B} , being a space bivector, it is also a direct geometrical consequence that its source can (must) be a current vector density \mathbf{J} . For a small ($l \ll r$) current source at the origin of coordinates: (in the sphere $\mathbf{r}^t(\theta, \phi) \bullet \mathbf{r}^{ct} = 0$, $(\mathbf{r}^t)^2 = (\mathbf{r}^{ct})^2 = 1$)

$$\int_V 4\pi\mu_0 \mathbf{J} \delta(r') r'^2 dr' = \mu_0 M \mathbf{r}^{ct} = 4\pi r^2 f B(r) \mathbf{r}^{ct} \quad \Longrightarrow \quad \mathbf{B} = B(r) \mathbf{r}^t = \frac{\mu_0 M}{4\pi r^2 f} \mathbf{r}^t.$$

3.1 Newtonian gravity

The analysis above depends only on the assumption of the decomposition of the action and of the energy momentum into contributions per carrier. The solution of the first Maxwell equation, when applied to gravitation considering the mass $M = \mathcal{E}/c^2$, implies (as shown above) the Newtonian gravitational potential equation per unit test mass m :

$$\mathfrak{E} = V = -G \frac{M}{r}, \quad \text{that is} \quad \mathbf{E} = -G \frac{M}{r^2}$$

the usual relations in the textbook formulation of Newtonian gravity. The constant $G = 1/4\pi\epsilon_0^{(g)}$. If we define $c^2 \mu_0^{(g)} \epsilon_0^{(g)} = 1$ then $\mu_0^{(g)} = 4\pi G/c^2$.

4 General relativity and the test particle

The Schwarzschild solution. In our present theory there are two fundamental carrier structures: the massless fields and the massive electron field with basic relation

$$\mathcal{E}^2 = (\mathcal{E}_0 + \Delta \mathcal{E})^2, \quad \mathcal{E}^2 - \mathcal{E}_0^2 = (pc)^2, \quad (19)$$

where $\Delta \mathcal{E}$ is any gauge-free energy contribution and $\mathcal{E}_0 = m_0 c^2$ the energy, at rest relative to some observer, considered to be the mass of the carrier.

The concept of test particle in general relativity in the Schwarzschild solution is compatible with the Newtonian limit for the interaction gravitational energy

$$\Delta \mathcal{E}(r) = -m_0 \frac{GM}{r}, \quad (20)$$

where M is the total mass of ‘the system’ of radius r_s we are exploring with the test particle and, conceptually, with the START use of the action square difference, writing $\mathcal{E} = \mathcal{E}_0 + \Delta \mathcal{E}$ for large (classical limit) values of $r > r_s$

$$\begin{aligned} \mathcal{E}^2 - \mathcal{E}_0^2 &= \mathcal{E}_0^2 + 2\mathcal{E}_0 \Delta \mathcal{E} + (\Delta \mathcal{E})^2 - \mathcal{E}_0^2 = (pc)^2 \\ &= 2\mathcal{E}_0 \Delta \mathcal{E} + (\Delta \mathcal{E})^2 \rightarrow -2m_0 c^2 m_0 \frac{GM}{r} + \left(m_0 \frac{GM}{r} \right)^2, \end{aligned} \quad (21)$$

this corresponds to the energy and radial momentum terms in $(da)^2 - (da')^2$ if $(da')^2 = (m_0 c^2 dt)^2$, and substituting in (16) using $\kappa_0 = 1/m_0 c$ and space spherically symmetric coordinates t, r, θ, ϕ we obtain

$$(dS)^2 = \left(1 - 2\frac{GM}{c^2 r} + \left(\frac{GM}{c^2 r}\right)^2\right) c^2 (dt)^2 - \left(1 + \frac{2GM}{c^2 r} - \left(\frac{GM}{c^2 r}\right)^2\right) (dr)^2 - r^2 \left[(d\theta)^2 + \sin^2 \theta (d\phi)^2\right], \tag{22}$$

which is the Schwarzschild metric in the limit of $r \gg GM/c^2$.

It is customary to write [15] the interval square using in our case $f(r) = 1 + b^2(r)$ and $h(r) = 1 - b^2(r)$

$$f(r) = \left(1 - 2\frac{GM}{c^2 r} + \left(\frac{GM}{c^2 r}\right)^2\right) \quad \text{and} \quad h(r) = \left(1 + \frac{2GM}{c^2 r} - \left(\frac{GM}{c^2 r}\right)^2\right), \tag{23}$$

for $c^2 r \gg GM$ we obtain the Schwarzschild relation $f \cong h^{-1}$.

The result (22) shows that our approach provides a conceptual understanding of the role of sources carriers and test particles in general relativity. It also shows the possibility of extending the analysis to circumstances more difficult to consider within the traditional approaches.

Once we have obtained the Schwarzschild metric we can now find the **curved hypersurface in START corresponding to the curved space–time** where the test particles are assumed to move. Formally we need to define a set of vectors $e_\mu, \mu = 0, 1, 2, 3, g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, and their reciprocal, in terms of a vierbein using the Minkowski space reference vectors. From (22) it is clear (see [15]) how to construct an orthonormal system of vectors.

One of the possible symmetries in START is the transformation of position vectors \mathbf{y} in START to a new set $\{\mathbf{y} = x^u e_u; u = 0, 1, 2, 3, 4\}$

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) = x'^u e'_u. \tag{24}$$

which describes the curvature of the space–time part necessary for representing physical interactions, at the expense of defining ‘test’ carriers.

4.0.1 General relativity in START

From our previous analysis, the structure equivalent to Einstein’s general relativity is the following:

- In the flat space–time–action geometry a distribution of action is given and analyzed as corresponding to the total matter and interaction fields (radiation) content.
- Basically one obtains the structure corresponding to general relativity by the process of transforming this 1 + 3 + 1 geometrical description into an equivalent 1 + 3 description given by a curved space–time.
- Even if the projection of the surface in five dimensions as a four-dimensional space corresponds to the curved space–time of general relativity, the physical meaning of this curved space–time is given by defining the trajectories of ‘test’ particles as the geodesics in this 4-D space.

The analysis we have presented here corresponds to changing the status of general relativity from a physical model to a part of a deductive theory.

4.0.2 A charged carrier as a test particle in general relativity

A charged particle at rest which is acted on by gravitational and electromagnetic interactions will have for the (attributed) total energy (at distances large enough such that the collection of masses with which the test carrier interacts are collectively represented by the volume integral of a mass density $\mathcal{M}(\mathbf{r})$) in the presence of the mass $M = \int_D \mathcal{M}(\mathbf{r}) dV$, the following description:

$$\varepsilon = m_0 c^2 - m_0 \frac{GM}{r} + e \frac{Q}{r}.$$

Substituting this in (19)–(23) will change the functions $f(r)$ and $h(r)$ into

$$f(r) = 1 - 2 \frac{GM}{c^2 r} + \left(\frac{GM}{c^2 r} \right)^2 - \frac{e}{m_0} Q \frac{GM}{c^4 r^2} + 2 \frac{eQ}{m_0 c^2 r} + \left(\frac{eQ}{m_0 c^2 r} \right)^2,$$

$$h(r) = 1 + 2 \frac{GM}{c^2 r} - \left(\frac{GM}{c^2 r} \right)^2 + \frac{e}{m_0} Q \frac{GM}{c^4 r^2} - 2 \frac{eQ}{m_0 c^2 r} - \left(\frac{eQ}{m_0 c^2 r} \right)^2.$$

The analysis of these functions would lead to the following conclusions:

1. Besides the attractive gravitational term there is a (quadratic) repulsive term which will dominate at intermediate distances. Time coordinates do not become imaginary or discontinuous.
2. The electric part of the interaction depends explicitly in the e/m_0 ratio of the test particle, and it can then not be a universal behavior of a test particle.

Otherwise, when the relations corresponding to general relativity are derived from START, those entering into the experimental proofs of the validity of general relativity (considered this far) are not changed and retain their validation status.

4.1 The mathematical structure of general relativity from START

Once we have seen that an electron used as a test particle in the START geometry allows us to obtain the Schwarzschild metric we can now proceed to a systematic derivation of the structure of general relativity from START.

The main considerations are the following.

- a) General relativity is a geometric theory describing the trajectories of test particles as the natural trajectories, geodesics, in curved space–time geometry.
- b) The curved space–time is obtained by incorporating, within STA, equivalent distances from the action part into the ST part. That is, general relativity is a theory where the geometry describes everything that is to be described, through the curved space–time, and the test particle is only an auxiliary in this description.
- c) The quadratic form obtained was afterwards analyzed using intrinsic geometrical techniques to have a purely geometrical theory. The basic equations, everywhere in space, are the transfer of the intervals corresponding to the relevant action (squared) to the flat quadratic form of space–time.
- d) We can directly consider that the quadratic form defines the metric tensor of the new geometry, and then use the definition of the curvature from the metric tensor in the generated curved space–time, to obtain a relation between the curvature and the energy–momentum–stress tensor.

The metric in GR. Once we have created the equivalent curved space–time the metric in GR is given through the use of the line element (here $g_{\mu\nu} = g_{\mu\nu}^{\text{GR}}$ from the choice of action allocation to geometry and $g_{\mu\nu}^0$ corresponds to flat space–time)

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu}^0 (1 + \Delta g_{\mu\nu}) dx^\mu dx^\nu, \tag{25}$$

which in turn defines local vector frames (up to a gauge transformation)

$$e_\mu^{\text{GR}} = \mathbf{h}(e_\mu), \quad \text{such that} \quad g_{\mu\nu} = e_\mu^{\text{GR}} \cdot e_\nu^{\text{GR}},$$

with $\mathbf{h}(x)$ a vector-valued function of vectors usually represented through a vierbein h_μ^ν .

In practice the metric appears as an independent field in START which is defined according to the Principle of Choice of Acceptable Descriptions, then once it is chosen the condition of flat STA is that the total curvature vanishes. Otherwise (from the integral of the selected contributions to action)

$$\kappa_0 \frac{\delta A}{\delta g^{\mu\nu}(x)} \equiv \frac{\kappa_0}{2} \mathcal{T}_{\mu\nu}(x), \tag{26}$$

(the factor $\frac{1}{2}$ is needed for convention reasons); also, from the Ricci scalar curvature \mathcal{R} which results from the chosen line elements

$$\frac{\delta \mathcal{R}}{\delta g^{\mu\nu}(x)} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}, \quad \text{with} \quad \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} + \frac{\kappa_0}{2} \mathcal{T}_{\mu\nu} = 0 \tag{27}$$

to obtain the equivalent to the GR basic equation.

4.2 Rumer (Kaluza–Klein) theory deduced from START

The Rumer form of the Kaluza–Klein–Einstein–Bergmann theory is deduced from START when besides deriving the metric tensor from the square of the line element dS , as the symmetric part of dS^2 , the antisymmetric, then imaginary, elements are kept and considered in turn as as real elements of an extended metric tensor in a 5-D geometry. That is consider again the complex line element and compute again the complex square, keeping now the scalar and the bivector parts

$$(dS)^2 = dS^2 + e_{\mu\nu} dx^\mu dx^\nu ij \kappa_0 (p(X, \mu) - p(X, \nu))$$

where the antisymmetric product of two vectors, the bivectors $e_{\mu\nu}$ are also the generators of spin angular momentum.

From the principles of General Relativity of considering the changes in energy-momentum for the test particle, consider that in the case of an electromagnetic interaction the test particle of charge e receives and additional energy momentum $p(X, \mu) = eA_\mu(X)$

$$ij \kappa_0 p(X, \mu) = ij \frac{e}{m_0 c} A_\mu(X)$$

using the action equivalent distance $\kappa_0 = 1/m_0 c$.

Besides the many papers which have been written about the Kaluza–Klein proposition and their extension to the idea of hyper-space with one additional dimension (at least) for each additional interaction included, the direct inclusion of action as a fifth dimension was proposed as early as the 1949–1956 by the Russian physicist Y.B. Rumer [13, 14] under the name of “Action as a spatial coordinate. I–X”. In the work of Rumer the main foreseen application is to the case of optics in what he called 5-optics. We should remember that in this case the action $dA = 0$ and then the fifth coordinate turns out to be identically null.

5 Hypothesis and principles in START

The set of hypothesis and principles which are explicitly included are:

Physics is the science which describes the basic phenomena of Nature within the procedures of the Scientific Method. We consider that the mathematization of the anthropocentric primary concepts of space, time and the existence of the physical objects (action carriers), is a suitable point of departure for creating intellectual structures which describe Nature.

We introduce a set of principles: Relativity, Existence, Quantization and Choice as the operational procedure, and a set of 3 mathematical postulates to give this principles a formal, useful, structure.

In START, because of its equivalent complex structure and its quadratic forms we have, besides the geometrical space-time Poincaré group \mathcal{P} of transformations leaving the finite difference $(dx^0)^2 - (d\mathbf{x})^2$ invariant, an additional set of transformations related to the complex structure. The additional operations are: a translation in the e^4 direction, three rotations in the $e^i e^4$, $i = 1, 2, 3$ planes and one ‘boost’ in the $e^0 e^4$ plane.

It is clear that most of the here presented relations are known relations as far as we are **deriving** the structures and theories from START.

- [1] Einstein A., Zur elektrodynamik bewegter Körper, *Annalen der Physik, Vierte folge*, 1905, V.17, 891–921.
- [2] Einstein A., Zur allgemeine Relativitätstheorie, *Sitz. Preuss. Akad. Wiss.*, 1915, 778–786, 799–801.
- [3] Einstein A., Die Feldgleichungen der Gravitation, *Sitz. Preuss. Akad. Wiss.*, 1915, 844–847.
- [4] Einstein A., Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie, *Sitz. Preuss. Akad. Wiss.*, 1915, 831–839.
- [5] Einstein A., Die Grundlage der allgemeinen Relativitätstheorie, *Annalen der Physik, Vierte folge*, 1916, V.17, 769–822.
- [6] Keller J., The geometric content of the electron theory II, *Advances in Applied Clifford Algebras*, 1999, V.9, N 2, 309–395.
- [7] Keller J., Complex space-time formulation of a theory of elementary particles, in Proceedings Conference “Symmetry in Physics” (1997, Dubna), to appear.
- [8] Keller J., Space-time-action geometrical unity as origin of a structural theory of matter, in Proceedings Int. Conference “Geometrization of Physics IV” (4–8 October, 1999, Kazan State University, Kazan), Kazan Univ. Press, 2000, 22–39.
- [9] Keller J., Space-time-action-relativity theory, in Proceedings of the Wigner Conference (1999, Istanbul), to appear.
- [10] Keller J., General relativity from START, in the Proceedings of the Conference “Clifford Analysis and Related Topics” (2000, Beijing), Editors Gürlebeck K., Ji X. and Sprössig W., *Advances in Applied Clifford Algebras*, 2001, V.11(S2), 183–204.
- [11] Keller J., The foundations of density functional theory and wave quantum mechanics, *Rev. Soc. Quim. Mex.*, 2000, V.44, N 1, 22–28.
- [12] Keller J., The theory of the electron. A theory of matter from START, *Foundations of Physics Series*, Vol.115, Dordrecht, Kluwer Academic Publishers, 2001.
- [13] Rumer Y.B., Action as space coordinate I–X, *Journal of Experimental and Theoretical Physics*, 1949, V.19, 86–94, 207–214, 868–875; 1953, V.25, 271; 1959, V.31, 1894–1902 (in Russian).
- [14] Rumer Y.B., Studies on 5-optics, *Uspekhi Mat. Nauk*, 1953, V.8, N 6, 1–151 (in Russian).
- [15] Snygg J., Clifford algebra – a computational tool for physicists, New York, Oxford University Press, 1997.