

Aspects of Symmetry in Sine-Gordon Theory

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As a prototype of powerful non-Abelian symmetry in an Integrable System, I will show the appearance of a Witt algebra of vector fields in the SG theory. This symmetry does not share anything with the well-known Virasoro algebra of the conformal $c = 1$ unperturbed limit. Although it is quasi-local in the SG field theory, nevertheless it gives rise to a local action on N -soliton solution variables. I will explicitly write the action on special variables, which possess a beautiful geometrical meaning and enter the Form Factor expressions of quantum theory. At the end, I will also give some preliminary hints about the quantisation.

1 Introduction

Nowadays the very peculiar rôle of symmetries is clearly recognised in all the areas of Mathematical Physics also thanks to the recent developments of Quantum Physics. In fact, it was in the context of Classical Physics that Liouville defined as integrable a system having a number of local integrals of motion in involution (LIMI's) equal to the degrees of freedom and proposed a theorem (Liouville–Arnold theorem [1]) to *solve the motion up to quadratures* – in the case of finite number of degrees of freedom. Nevertheless, there is no equivalent theorem when the degrees of freedom become infinite as well as the number of Abelian symmetries: the classical field theories represent an important example which attracted more and more interest. The situation is even more complicated when the system is a quantum field theory: in this case we may be interested, for instance, in the energy spectrum [2, 3] or in the spectrum of fields or in the correlation functions of those fields [4], as the usual meaning of motion is definitely lost. In fact, in systems with infinite degrees of freedom non-Abelian symmetries revealed to be more useful: let us think of Classical Inverse Scattering Method [5] and Bethe Ansatz [2, 3] as two illustrative examples among the others. Moreover, the Virasoro algebra in two Dimensional Conformal Field Theories (CFT's) represents perhaps the most successful example of how a non-Abelian symmetry can *solve* a quantum field theory and in this case a theory realising a physical system at the very important *critical point* [6].

Unfortunately, this Virasoro algebra does not exist any longer if the system is pushed out of the critical point, still preserving Liouville integrability [6]. For instance, the Sine-Gordon (SG) theory is one of the simplest massive Integrable Field Theories (IFT's), although it is the first theory in a series of structure richer theories, the Affine Toda Field Theories (ATFT's) [7] and possesses all the features peculiar to the more general IFT's [8]. Actually, non-Abelian infinite-dimensional symmetries were found in all Toda theories and they are called dressing symmetries at classical level [9] and become (level 0) affine quantum algebras after quantisation [10]. Nevertheless, because of their affine and highly non-local characters those symmetries are not of large use.

In this talk I present the appearance of infinitesimal symmetry transformations (vector fields) acting on the boson field of the classical Sine-Gordon theory. These vector fields turn out to close a Witt (centerless Virasoro) algebra. Since the only *ingredient of the recipe* is the Lax pair formulation of SG equation, it is clear how to generalise the construction to more general field theories like, for instance, ATFT's. Nevertheless, I rather would like to focus my attention on the origin and form of the infinitesimal transformations in the particular case of SG theory.

Specifically, I will show how to introduce the SG theory starting from the simpler Korteweg-de Vries (KdV) theory and how to frame this symmetry inside the KdV theory. Actually, I will not give a complete proof of all the statements I will formulate, leaving this part to a more systematic publication [11]. On the contrary, the restriction of these vector fields on the variables of the N -soliton solutions was described and analysed in [12]: in this talk I sketch only how to derive this action on a more intuitive ground. In the soliton phase space the infinitesimal transformations are realised in a much simpler form and in particular they become local contrary to the field theory case (in which these are quasi-local). At the end, I will deliver few comments about how much easier quantisation of the soliton phase space might appear.

2 The action of the Witt symmetry on fields

Let me recall the construction of the Witt symmetry in the context of (m)KdV theory [13, 14]. It was shown in [14], following the so-called matrix approach, that it appears as a generalisation of the ordinary dressing transformations of integrable models. As integrable system the mKdV equation enjoys a zero-curvature representation

$$[\partial_t - A_t, \partial_x - A_x] = 0, \quad (1)$$

where the Lax connections A_x, A_t belong to a finite dimensional representation of some loop algebra and contain the fields and their derivatives. In this particular case the first Lax operator \mathcal{L} is given by

$$A_x = \begin{pmatrix} \phi' & \lambda \\ \lambda & -\phi' \end{pmatrix}, \quad (2)$$

where I have denoted with ϕ' the mKdV field (prime means derivative with respect to the *space variable* x), with λ the $A_1^{(1)}$ loop algebra parameter (spectral parameter) and A_t can be found using the dressing procedure I am going to describe [15]. The KdV variable $u(x)$ is connected to the mKdV field ϕ' by the Miura transformation:

$$u = -(\phi')^2 - \phi''. \quad (3)$$

Key objects in the following construction are solutions $T(x, \lambda)$ of the so-called associated linear problem

$$(\partial_x - A_x(x, \lambda))T(x, \lambda) = 0, \quad (4)$$

which may be called monodromy matrices. A formal (suitably normalised) solution of (4) can be formally expressed by

$$T_{\text{reg}}(x, \lambda) = e^{H\phi(x)} \mathcal{P} \exp \left(\lambda \int_0^x dy \left(e^{-2\phi(y)} E + e^{2\phi(y)} F \right) \right). \quad (5)$$

Of course, this solution is just an infinite series in positive powers of $\lambda \in \mathbb{C}$ with an infinite radius of convergence. I shall often refer to (5) as *regular expansion*. It is also clear from (5) that any solution $T(x, \lambda)$ possesses an essential singularity at $\lambda = \infty$ where it is governed by the corresponding *asymptotic expansion*. In consequence, an asymptotic expansion has been derived in detail in [15]

$$T_{\text{asy}}(x, \lambda) = KG(x, \lambda) e^{-\int_0^x dy D(y)}, \quad (6)$$

where K and G and D are written explicitly in [15]. In particular the matrix

$$D(x, \lambda) = \sum_{i=-1}^{\infty} \lambda^{-i} d_i(x) H^i, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (7)$$

contains the local conserved densities $d_{2n+2}(x)$.

Obviously, a gauge transformation for A_x

$$\delta A_x(x, \lambda) = [\theta(x, \lambda), \mathcal{L}] \quad (8)$$

preserves the zero-curvature form (1) if an analogous one applies to A_t : the vector field δ defines a symmetry of *the equation of motion* (1) in the usual sense, mapping a solution into another solution. Moreover, to build up a consistent gauge connection $\theta(x, \lambda)$ for the previous infinitesimal transformation, I must pay attention to the fact that the r.h.s. needs to be independent of λ since the l.h.s. is, as consequence of (2). Hence, a suitable choice for the gauge connection goes through the construction of the following object

$$Z^X(x, \lambda) = T(x, \lambda) X T(x, \lambda)^{-1}, \quad (9)$$

where X is such that

$$[\partial_x, X] = 0. \quad (10)$$

Indeed, it is obvious from the previous definition that it satisfies the *resolvent condition*

$$[\mathcal{L}, Z^X(x, \lambda)] = 0, \quad (11)$$

for the first Lax operator $\mathcal{L} = \partial_x - A_x(x, \lambda)$. Now, this property implies

$$[\mathcal{L}, (Z^X(x, \lambda))_-] = -[\mathcal{L}, (Z^X(x, \lambda))_+], \quad (12)$$

where the subscript $-$ ($+$) means that I restrict the series only to negative (non-negative) powers of λ , and hence yields the construction of a consistent gauge connection defined as

$$\theta^X(x, \lambda) = (Z^X(x, \lambda))_- \quad \text{or} \quad \theta^X(x, \lambda) = (Z^X(x, \lambda))_+. \quad (13)$$

Further, I have to impose one more consistency condition implied by the explicit form of A_x (2), namely δA_x must be diagonal

$$\delta^X A_x = H \delta^X \phi'. \quad (14)$$

This implies restrictions about the indices of the transformations [16]. After posing $T = T_{\text{reg}}$ I obtain the so-called dressing symmetries [15] and the indices are even for $X = H$ and odd for $X = E, F$. Instead, after posing $T = T_{\text{asy}}$ I get for $X = H$ the commuting (m)KdV flows (or the (m)KdV hierarchy), which define the different *time* t_{2k+1} , $k = 0, 1, 2, \dots$ *evolutions* and in particular (1) with $t = t_3$ [16].

At this point I want to make an important observation. Let me consider the KdV variable x as a *space direction* x_- of some more general system (and $\partial_- = \partial_x$ as a space derivative). Let me introduce the *time* variable x_+ through the corresponding evolution flow

$$\partial_+ = (\delta_{-1}^E + \delta_{-1}^F), \quad (15)$$

defined by a zero curvature condition of the form (8)

$$\partial_+ A_{x_-}(x_-, x_+; \lambda) = [\theta_+(x_-, x_+; \lambda), \mathcal{L}]. \quad (16)$$

This specific $\theta_+(x_-, x_+; \lambda)$ is derived using (13) with the regular expansion. Then, it can be proved [16] that the equation of motion for ϕ becomes:

$$\partial_+ \partial_- \phi = 2 \sinh(2\phi), \quad \text{or if } \phi \rightarrow i\phi, \quad \partial_+ \partial_- \phi = 2 \sin(2\phi), \quad (17)$$

i.e. the Sine-Gordon equation. As we will see later, this observation will appear very fruitful for my purpose since it provides an introduction of Sine-Gordon dynamics as a vector field in the powerful algebraic framework of the KdV hierarchy and its symmetries. For instance, I obtain as simple by-product the fact that mKdV hierarchy is a symmetry for SG equation. Of course, the Hamiltonians – given by the part of the dressing charges corresponding to the (16) and the higher flows $\partial_{2k+1} = \delta_{-2k-1}^E + \delta_{-2k-1}^F$ [15] – coincide with the well-known ones [5].

Now, let me explain how the Witt symmetry appears in the KdV system [14]. The main idea is that one may dress not only the generators of the underlying $A_1^{(1)}$ algebra but also an arbitrary differential operator in the spectral parameter. I take for example $\lambda^{m+1} \partial_\lambda$ which are the well known vector fields of the diffeomorphisms of the unit circumference and close a Witt algebra. Then I proceed in the same way as above defining the resolvent associated to the circumference diffeomorphisms

$$Z^V(x, \lambda) = T(x, \lambda) \partial_\lambda T(x, \lambda)^{-1}. \quad (18)$$

When I consider the asymptotic case, i.e. I take $T = T_{\text{asy}}$ in (18), I obtain the non-negative Witt flows. In general they are written in terms of recursive quasi-local expressions $\alpha_{2m}^V(x)$, $m \geq 0$ as

$$\delta_{2m}^V \phi(x) = \alpha_{2m}^V(x), \quad (19)$$

where

$$\alpha_0^V(x) = -x\phi'(x), \quad \alpha_{2m+2}^V(x) = \left[-\phi' \partial_x^{-1} \phi' \partial_x + \frac{1}{4} \partial_x^2 \right] \alpha_{2m}^V(x). \quad (20)$$

Let me highlight the appearance of the pseudodifferential operator ∂_x^{-1} , acting on a function $f(x)$ as

$$\partial_x^{-1} f(x) = \int^x dy f(y), \quad (21)$$

which is responsible (together with the form of the initial condition (20)) for the non complete locality. From these vector fields I can deduce the action on $u(x)$ using (3)

$$\delta_{2m}^V u(x) = 2 \partial_x \beta_{2m+1}^V(x) \quad (22)$$

again in terms of recursive quasi-local expressions $\beta_{2m-1}^V(x)$

$$\beta_{-1}^V = -x, \quad \beta_{2m+1}^V(x) = \left[\frac{1}{2} \left(u + \partial_x^{-1} u \partial_x + \frac{1}{2} \partial_x^2 \right) \right] \beta_{2m-1}^V(x). \quad (23)$$

For instance, the first two of (19) can be written as

$$\delta_0^V \phi = -x\phi', \quad \delta_2^V \phi = \frac{1}{2} \phi' (\partial_x^{-1} \phi'^2) - \frac{1}{2} \phi'' - \frac{1}{2} x \left[\frac{1}{2} \phi''' - (\phi')^3 \right]. \quad (24)$$

The negative Witt transformations can also be built up by taking $T = T_{\text{reg}}$ in (18) in such a way to complete the algebra [14]. Unfortunately, those vector fields do not act as gauge transformations on the SG equation of motion (16), actually they are not true symmetry transformations [16].

Nevertheless, thanks to the way (16) I have introduced SG theory through, I am in the position to extend the (half) Witt symmetry algebra jumping from (m)KdV to the SG theory. For obvious reasons I will rename in the following the KdV variable u with

$$u^-(x_-, x_+) = -(\partial_- \phi(x_-, x_+))^2 - \partial_-^2 \phi(x_-, x_+). \quad (25)$$

Hence, after looking at the symmetric rôle that the derivatives ∂_- and ∂_+ play in the Sine-Gordon equation, I can obtain *the negative (m)KdV hierarchy*, acting on the fields $\phi(x_-, x_+)$ and

$$u^+(x_-, x_+) = -(\partial_+ \phi(x_-, x_+))^2 - \partial_+^2 \phi(x_-, x_+), \quad (26)$$

in the same way as above but with the change of rôles $x_- \rightarrow x_+$ (and consequently $\partial_- \rightarrow \partial_+$). Similarly, I obtain the other half of a Witt algebra by using the same construction already showed, but with x_- substituted by x_+ .

Of course, it is not obvious at all that the two different halves will recombine into a unique Witt algebra. Actually, even the first Witt vector field in the original construction (24) needs a *symmetrising improvement* to leave exactly invariant the zero curvature form of SG equation (16):

$$\delta_0^V \phi = -x_- \partial_- \phi - x_+ \partial_+ \phi. \quad (27)$$

Nevertheless, I have checked this statement brute force in the case

$$[\delta_2^V, \delta_{-2}^V] \phi = 4\delta_0^V \phi, \quad (28)$$

and it works in a peculiar manner, simply using the transformation definitions (27) and the second of (24). I would like to leave for future publication the detailed explanation of how a complete proof of this proposition may be elaborated along smart lines [11].

In conclusion, I have found an entire Witt algebra of transformations acting as gauge symmetries on SG equation (16). Moreover, I sketch now how the restriction of the action on soliton solution phase space yields the result argued in [12] following a slightly different procedure.

3 The Witt symmetry acting on the soliton solution variables

I start with a brief description of the well known soliton solutions of SG equation and (m)KdV hierarchy in the infinite *times* formalism. To see how a N -soliton solution can be parametrised, I need to go through the expression of the so-called *tau-function*. This can be written as a determinant

$$\tau(X_1, \dots, X_N | B_1, \dots, B_N) = \det(1 + V), \quad (29)$$

where V is a $N \times N$ matrix

$$V_{ij} = 2 \frac{B_i X_i}{B_i + B_j}, \quad i, j = 1, \dots, N, \quad (30)$$

and $X_i(\{t_{2k+1}\} | x_i, B_i)$ are exponential functions of all the *times* $\{t_{2k+1}\}$, $k \in \mathbb{Z}$ (e.g. in the previous notation $t_{-1} = x_+$, $t_1 = x_-$, $t_3 = t$)

$$X_i(\{t_{2k+1}\} | x_i, B_i) = x_i \exp \left(2 \sum_{k=-\infty}^{+\infty} B_i^{2k+1} t_{2k+1} \right). \quad (31)$$

The constant parameters B_i and x_i describe the soliton velocities and positions respectively. Now the SG or the mKdV field solution is expressed in a beautiful unitary way as

$$e^\phi = \frac{\tau_-}{\tau_+}, \quad (32)$$

where simply

$$\tau_\pm = \tau(\{\pm X_i\}|\{B_i\}), \quad (33)$$

in the sense that after putting all the negative (positive) times to zero, I end up with the N -soliton solution of the mKdV hierarchy (the negative mKdV hierarchy), whereas after the position to zero of all the times but $t_{-1} = x_+$, $t_1 = x_-$, I end up with the N -soliton solution of the SG equation.

The main goal of this Section is to find the action of the Witt symmetry on the N -soliton solution and this is more conveniently achieved introducing other variables $\{A_i, B_i\}$, expressed implicitly by the old variables $\{X_i, B_i\}$ through the implicit formulae

$$X_j \prod_{k \neq j} \frac{B_j - B_k}{B_j + B_k} = \prod_{k=1}^N \frac{B_j - A_k}{B_j + A_k}, \quad j = 1, \dots, N. \quad (34)$$

In fact, the $\{A_i, B_i\}$ are the soliton limit of certain variables describing the more general quasi-periodic finite-zone solutions of (m)KdV [17], being the B_i the limit of the branch points of the hyperelliptic curve describing a particular solution and the A_i the limit of the zeroes of the so-called Baker–Akhiezer function defined on the curve. Actually, even for the description of the quantum physics of Form Factors these variables are apparently more natural and suitable [18]. Although, in terms of these variables the tau functions have still a cumbersome form

$$\begin{aligned} \tau_+ &= 2^N \prod_{j=1}^N B_j \left\{ \frac{\prod_{i < j} (A_i + A_j) \prod_{i < j} (B_i + B_j)}{\prod_{i,j} (B_i + A_j)} \right\}, \\ \tau_- &= 2^N \prod_{j=1}^N A_j \left\{ \frac{\prod_{i < j} (A_i + A_j) \prod_{i < j} (B_i + B_j)}{\prod_{i,j} (B_i + A_j)} \right\}, \end{aligned} \quad (35)$$

the SG (mKdV) field (32) enjoys a simple expression

$$e^\phi = \prod_{j=1}^N \frac{A_j}{B_j}. \quad (36)$$

In consequence, the two components of the stress-energy tensor (25) and (26) take a wieldy form as well

$$u^- = -2 \left(\sum_{j=1}^N A_j^2 - \sum_{j=1}^N B_j^2 \right), \quad u^+ = -2 \left(\sum_{j=1}^N A_j^{-2} - \sum_{j=1}^N B_j^{-2} \right). \quad (37)$$

Now I am in the position to restrict the Witt symmetry of SG equation developed in the previous Section to the case of soliton solutions. Although these transformations have been derived in [12], here I will follow a more intuitive path, which underlines the geometrical meaning of this symmetry. In other words our starting point consists in the transformations of the rapidities under the Witt symmetry: I do expect that they change the conformal structure of the Riemann

surface describing the finite-zone solutions. Actually, in the (m)KdV theory the soliton limit of the Witt action on the Riemann surface reads simply [13]

$$\delta_{2n}B_i = B_i^{2n+1}, \quad n \geq 0, \quad (38)$$

where I have forgotten the superscript V for indicating the action on soliton variables. Further, the action of negative transformations should not be different

$$\delta_{-2n}B_i = -B_i^{-2n+1}, \quad n > 0, \quad (39)$$

save an additional $-$ sign in the r.h.s. [12] which takes into account the Witt algebra commutation relations. I have to show now the transformations of the A_i variables as consequences of (38) and (39) once applied to the implicit map (34) by using the expression (31) of X_i in terms of B_i . The problem is simplified by the fact that I know from the field theory that the symmetry algebra is a Witt algebra, and hence I need to compute only the transformations δ_0 , $\delta_{\pm 2}$ and $\delta_{\pm 4}$, for the higher vector fields are then furnished by commuting. In this way it is evident why the Witt transformations become *local* when restricted on the soliton solutions, though the transformations of ϕ and u^\pm in the SG theory are quasi-local. Actually, I think more natural and more compact to express the Witt action on A_i by using the *equations of motion* of A_i derived from (31) and (34), like for instance

$$\begin{aligned} \delta_{-1}A_i &= \partial_+A_i = \prod_{j=1}^N \frac{(A_i^2 - B_j^2)}{B_j^2} \prod_{j \neq i} \frac{A_j^2}{(A_i^2 - A_j^2)}, \\ \delta_1A_i &= \partial_-A_i = \prod_{j=1}^N (A_i^2 - B_j^2) \prod_{j \neq i} \frac{1}{(A_i^2 - A_j^2)}, \\ \delta_3A_i &= 3 \left(\sum_{j=1}^N B_j^2 - \sum_{k \neq i} A_k^2 \right) \partial_-A_i, \\ \frac{1}{5}\delta_5A_i &= \left(\sum_{j=1}^N B_j^4 - \sum_{k \neq i} A_k^4 \right) \partial_-A_i - \sum_{j \neq i} (A_i^2 - A_j^2) \partial_-A_i \partial_-A_j. \end{aligned} \quad (40)$$

In conclusion, the direct calculation is quite tiresome and I present here only few results:

$$\begin{aligned} \delta_{-2}A_i &= \frac{1}{3}x_+\delta_{-3}A_i - A_i^{-1} - \partial_+A_i \sum_{j=1}^N A_j^{-1} - x_-\partial_+A_i, \\ \delta_{-4}A_i &= \frac{1}{5}x_+\delta_{-5}A_i - A_i^{-3} - \left\{ \sum_{j \neq i} \frac{1}{A_i} \left(\frac{1}{A_i^2} - \frac{1}{A_j^2} \right) + \sum_{j=1}^N \frac{1}{A_j} \sum_{k=1}^N \frac{1}{B_k^2} \right\} \partial_+A_i - x_-\delta_{-3}A_i \end{aligned} \quad (41)$$

and for non-negative vector fields

$$\begin{aligned} \delta_0A_i &= (x_-\partial_- - x_+\partial_+ + 1)A_i, \quad \delta_2A_i = \frac{1}{3}x_-\delta_3A_i + A_i^3 - \left(\sum_{j=1}^N A_j \right) \partial_-A_i - x_+\partial_-A_i, \\ \delta_4A_i &= \frac{1}{5}x_-\delta_5A_i + A_i^5 - \left\{ \sum_{j \neq i} A_i (A_i^2 - A_j^2) + \sum_{j=1}^N A_j \sum_{k=1}^N B_k^2 \right\} \partial_-A_i - x_+\delta_3A_i. \end{aligned} \quad (42)$$

At this point, I need to carry out two important checks. First, I have to calculate the commutators of the δ_{2m} (with $m \in \mathbb{Z}$) with the light-cone SG flow ∂_{\pm} , acting on A_i . These are always zero and represent an equivalent way to express the symmetry action. Second, I have to verify the algebra of the δ_{2m} (with $m \in \mathbb{Z}$) on A_i and this is a very non trivial check for I have derived all the transformations (41) and (42) from the Witt algebra on B_i , written in (38) and (39), and from the implicit map (34). Nevertheless the action on A_i is again a representation of the Witt algebra:

$$[\delta_{2n}, \delta_{2m}]A_i = (2n - 2m)\delta_{2n+2m}A_i, \quad n, m \in \mathbb{Z}. \quad (43)$$

4 Comments about quantisation

Of course, I might be interested in the quantum Sine-Gordon theory. In the case of solitons there is a standard procedure: the canonical quantisation of the N -soliton solutions. Indeed, let me introduce the variables canonically conjugated to the A_i :

$$P_j = \prod_{k=1}^N \frac{B_k - A_j}{B_k + A_j}, \quad j = 1, \dots, N. \quad (44)$$

In these variables one can perform the canonical quantisation of the N -soliton system introducing the deformed commutation relations between the operators \hat{A}_i and \hat{P}_i :

$$\begin{aligned} \hat{P}_j \hat{A}_j &= q \hat{A}_j \hat{P}_j, \\ \hat{P}_k \hat{A}_j &= \hat{A}_j \hat{P}_k, \quad \text{for } k \neq j, \end{aligned} \quad (45)$$

where $q = \exp(i\xi)$, $\xi = \frac{\pi\gamma}{\pi-\gamma}$ and γ is the coupling constant of the SG theory. Understanding how the Witt symmetry is deformed after quantisation is a very seductive problem.

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