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On Differential Invariants of First- and Second-Order of the Splitting Subgroups of the Generalized Poincaré Group $P(1,4)$

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Functional bases of differential invariants of the first-order of the splitting subgroups of the group $P(1,4)$ have been constructed. For majority of these subgroups functional bases of differential invariants of the second-order have also been described.

The differential invariants of the local Lie groups of the point transformations play an important role in the group-analysis of differential equations (see, for example [1–10]). In particular, with the help of these invariants we can construct differential equations with non-trivial symmetry groups. Differential invariants have been studied in many works (see, for example [1–10]).

The present paper is devoted to construction of functional bases of differential invariants of the first- and second-order for the splitting subgroups of generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is the group of rotations and translations of five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics [11, 12, 13]. In order to present some of the results obtained we have to consider the Lie algebra of the group $P(1,4)$.

1 The Lie algebra of the group $P(1,4)$ and its continuous subalgebras.

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ and P'_μ $(\mu,\nu = 0,1,2,3,4)$, satisfying the commutation relations

$$\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,$$
$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

Further we will use following basis elements:

$$G = M'_{40}, \qquad L_1 = M'_{32}, \qquad L_2 = -M'_{31}, \qquad L_3 = M'_{21},$$
$$P_a = M'_{4a} - M'_{a0}, \qquad C_a = M'_{4a} + M'_{a0} \quad (a = 1,2,3),$$
$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \qquad X_k = P'_k \quad (k = 1,2,3), \qquad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

In order to study the subgroup structure of the group $P(1,4)$ we used the method proposed in [14]. Continuous subgroups of the group $P(1,4)$ have been found in [15–19].

One of the important consequences of the description of the continuous subalgebras of the Lie algebra of the group $P(1,4)$ is that the Lie algebra of the group $P(1,4)$ contains as subalgebras the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ [13], i.e. it naturally unites the Lie algebras of the symmetry groups of relativistic and nonrelativistic quantum mechanics.

2 The differential invariants of the first-order for splitting subgroups of the group $P(1,4)$

For all splitting subgroups of the group $P(1,4)$ the functional bases of differential invariants of the first-order have been constructed. Below, we present some of the results obtained.

At first, let us consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \qquad P'_1 = -\frac{\partial}{\partial x_1}, \qquad P'_2 = -\frac{\partial}{\partial x_2}, \qquad P'_3 = -\frac{\partial}{\partial x_3},$$
$$P'_4 = -\frac{\partial}{\partial x_4}, \qquad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right). \tag{1}$$

For this representation of the considered Lie algebra we have obtained the functional bases of differential invariants of the first-order for all its splitting subalgebras.

Below, for some of the splitting subalgebras of the Lie algebra of the group $P(1,4)$ we write its basis elements and corresponding functional basis of differential invariants.

1. $\langle L_3 + eG, X_3,\ e > 0\rangle$,
$$\left(x_0^2 - x_4^2\right)^{1/2}, \quad (x_1^2 + x_2^2)^{1/2}, \quad \ln(x_0 + x_4) + e\arctan\frac{x_1}{x_2}, \quad u, \quad x_1u_2 - x_2u_1,$$
$$(x_0 + x_4)(u_0 + u_4), \quad u_3, \quad u_0^2 - u_4^2, \quad u_1^2 + u_2^2, \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad \mu = 0, 1, 2, 3, 4;$$
2. $\langle L_3 + dG, P_3, X_4,\ d > 0\rangle$,
$$\left(x_1^2 + x_2^2\right)^{1/2}, \quad u, \quad x_1u_2 - x_2u_1, \quad \frac{x_0 + x_4}{u_0 - u_4}, \quad \frac{u_0 - u_4}{x_0 + x_4}x_3 + u_3,$$
$$d\arctan\frac{u_1}{u_2} + \ln(x_0 + x_4), \quad u_1^2 + u_2^2, \quad u_0^2 - u_3^2 - u_4^2;$$
3. $\langle P_1, P_2, P_3, X_4\rangle$,
$$x_0 + x_4, \quad u, \quad \frac{x_1}{x_0 + x_4} + \frac{u_1}{u_0 - u_4}, \quad \frac{x_2}{x_0 + x_4} + \frac{u_2}{u_0 - u_4}, \quad \frac{x_3}{x_0 + x_4} + \frac{u_3}{u_0 - u_4},$$
$$u_0 - u_4, \quad u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$
4. $\langle G, L_1, L_2, L_3, X_4\rangle$,
$$\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \quad u, \quad (x_0 + x_4)(u_0 + u_4), \quad x_1u_1 + x_2u_2 + x_3u_3,$$
$$u_0^2 - u_4^2, \quad u_1^2 + u_2^2 + u_3^2;$$
5. $\langle G, P_1, P_2, X_1, X_2, X_4\rangle$,
$$x_3, \quad u, \quad \frac{x_0 + x_4}{u_0 - u_4}, \quad u_3, \quad u_0^2 - u_1^2 - u_2^2 - u_4^2;$$
6. $\langle L_3, P_1, P_2, P_3, X_1, X_2, X_4\rangle$,
$$x_0 + x_4, \quad u, \quad \frac{x_3}{x_0 + x_4} + \frac{u_3}{u_0 - u_4}, \quad u_0 - u_4, \quad u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

7. $\langle G, L_3, P_1, P_2, X_1, X_2, X_3, X_4\rangle$,

   $u, \quad \dfrac{x_0+x_4}{u_0-u_4}, \quad u_3, \quad u_0^2-u_1^2-u_2^2-u_4^2;$

8. $\langle L_3+bG, P_1, P_2, P_3, X_0, X_1, X_2, X_3, X_4,\ b>0\rangle$,

   $u, \quad u_0^2-u_1^2-u_2^2-u_3^2-u_4^2.$

Now, let us consider an other representation of the Lie algebra of the group $P(1,4)$

$$P_0' = \frac{\partial}{\partial x_0}, \qquad P_1' = -\frac{\partial}{\partial x_1}, \qquad P_2' = -\frac{\partial}{\partial x_2}, \qquad P_3' = -\frac{\partial}{\partial x_3},$$
$$P_4' = -\frac{\partial}{\partial u}, \qquad M_{\mu\nu}' = -\left(x_\mu P_\nu' - x_\nu P_\mu'\right), \qquad x_4 \equiv u. \tag{2}$$

More details about this representation can be found in [20].

Taking into account this representation of the considered Lie algebra we have constructed the functional bases of differential invariants of the first-order for all its splitting subalgebras.

Here, for some of the splitting subalgebras of the Lie algebra of the group $P(1,4)$ we give its basis elements and corresponding functional basis of differential invariants.

1. $\langle L_3+\varepsilon P_3, \varepsilon=\pm 1\rangle$,

   $x_0+u, \quad \left(x_1^2+x_2^2\right)^{1/2}, \quad \left(x_0^2-x_3^2-u^2\right)^{1/2}, \quad \dfrac{x_3}{x_0+u}+\dfrac{u_3}{u_0+1},$

   $\varepsilon \arctan\dfrac{x_1}{x_2}-\dfrac{x_3}{x_0+u}, \quad \dfrac{u_3^2}{(u_0+1)^2}+\dfrac{2}{u_0+1}, \quad \dfrac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad \dfrac{u_1^2+u_2^2}{(u_0+1)^2},$

   $u_\mu \equiv \dfrac{\partial u}{\partial x_\mu}, \quad \mu=0,1,2,3;$

2. $\langle G, L_3\rangle$,

   $x_3, \quad \left(x_1^2+x_2^2\right)^{1/2}, \quad \left(x_0^2-u^2\right)^{1/2}, \quad (x_0+u)^2\dfrac{u_0-1}{u_0+1}, \quad \dfrac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2},$

   $\dfrac{u_0^2-1}{u_3^2}, \quad \dfrac{u_1^2+u_2^2}{u_3^2};$

3. $\langle G, P_1, P_2\rangle$,

   $x_3, \quad \left(x_0^2-x_1^2-x_2^2-u^2\right)^{1/2}, \quad \dfrac{x_0+u}{u_0+1}u_3, \quad x_1+\dfrac{x_0+u}{u_0+1}u_1, \quad x_2+\dfrac{x_0+u}{u_0+1}u_2,$

   $\dfrac{u_0^2-u_1^2-u_2^2-1}{u_3^2};$

4. $\langle G, L_3, P_1, P_2\rangle$,

   $x_3, \quad \left(x_0^2-x_1^2-x_2^2-u^2\right)^{1/2}, \quad \dfrac{x_0+u}{u_0+1}u_3, \quad \dfrac{u_0^2-u_1^2-u_2^2-1}{u_3^2},$

   $\left(x_1+\dfrac{x_0+u}{u_0+1}u_1\right)^2+\left(x_2+\dfrac{x_0+u}{u_0+1}u_2\right)^2;$

5. $\langle G, P_3, L_3, X_1, X_2\rangle$,

   $\left(x_0^2-x_3^2-u^2\right)^{1/2}, \quad x_3+\dfrac{x_0+u}{u_0+1}u_3, \quad \left(u_1^2+u_2^2\right)\left(\dfrac{x_0+u}{u_0+1}\right)^2, \quad \dfrac{u_0^2-u_3^2-1}{u_1^2+u_2^2};$

6. $\langle G, L_1, L_2, L_3, X_0, X_4\rangle$,

   $\left(x_1^2+x_2^2+x_3^2\right)^{1/2}, \quad \dfrac{(x_1u_1+x_2u_2+x_3u_3)^2}{u_0^2-1}, \quad \dfrac{u_1^2+u_2^2+u_3^2}{u_0^2-1};$

7. $\langle G, P_1, P_2, P_3, X_1, X_2, X_4 \rangle$,

$$x_3 + \frac{x_0 + u}{u_0 + 1} u_3, \quad (u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1)\left(\frac{x_0 + u}{u_0 + 1}\right)^2;$$

8. $\langle G, L_3, P_1, P_2, P_3, X_1, X_2, X_4 \rangle$,

$$x_3 + \frac{x_0 + u}{u_0 + 1} u_3, \quad (u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1)\left(\frac{x_0 + u}{u_0 + 1}\right)^2.$$

3 On differential invariants of the second-order for splitting subgroups of the group $P(1,4)$

We have constructed functional bases of differential invariants of the second-order for some splitting subgroups of the group $P(1,4)$. Now, we present some of the results obtained.

Let us consider the representation (1) of the Lie algebra of the group $P(1,4)$. For this representation of the considered Lie algebra we have constructed the functional bases of differential invariants of the second-order for some its splitting subalgebras. Below, for some of the splitting subalgebra of the Lie algebra of the group $P(1,4)$ we write its basis elements and corresponding functional basis of differential invariants.

$$\langle L_3 + eG,\ e > 0 \rangle,$$

$$x_3, \quad \left(x_0^2 - x_4^2\right)^{1/2}, \quad \left(x_1^2 + x_2^2\right)^{1/2}, \quad e \arctan \frac{x_1}{x_2} + \ln(x_0 + x_4), \quad u,$$

$$(x_0 + x_4)(u_0 + u_4), \qquad x_1 u_2 - x_2 u_1, \quad u_3, \quad u_0^2 - u_4^2, \quad u_1^2 + u_2^2,$$

$$e \arctan \frac{u_{13}}{u_{23}} + 2\ln(x_0 + x_4), \quad \ln(u_{00} + u_{44} + \sqrt{2} u_{04}) - 2\sqrt{2} e \arctan \frac{x_1}{x_2},$$

$$\arctan\left(\frac{u_{02} + u_{24}}{u_{01} + u_{14}}\right) + 2\arctan \frac{x_1}{x_2}, \quad 4e \arctan \frac{u_1}{u_2} - \ln\left((u_{01} + u_{14})^2 + (u_{02} + u_{24})^2\right),$$

$$\frac{u_{03} + u_{34}}{(u_0 + u_4)^2}, \quad \arctan\left(\frac{\sqrt{2} u_{12}}{u_{11} - u_{22}}\right) + 2\sqrt{2} \arctan \frac{u_1}{u_2}, \quad u_{33}, \quad u_{00} - u_{44}, \quad u_{11} + u_{22},$$

$$u_{03}^2 - u_{34}^2, \quad u_{13}^2 + u_{23}^2, \quad u_{11}^2 + u_{12}^2 + u_{22}^2, \quad u_{00}^2 - u_{04}^2 + u_{44}^2, \quad u_{01}^2 + u_{02}^2 - u_{14}^2 - u_{24}^2,$$

$$u_{01} u_{24} - u_{02} u_{14}, \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad \mu, \nu = 0, 1, 2, 3, 4.$$

Now, let us consider the representation (2) of the Lie algebra of the group $P(1,4)$. Taking into account this representation of the considered Lie algebra we have obtained the functional bases of differential invariants of the second-order for some its splitting subalgebras.

Here, for some of the splitting subalgebra of the Lie algebra of the group $P(1,4)$ we give its basis elements and corresponding functional basis of differential invariants.

$$\langle L_3 \rangle,$$

$$x_0, \quad x_3, \quad \left(x_1^2 + x_2^2\right)^{1/2}, \quad u, \quad x_1 u_1 + x_2 u_2, \quad u_0, \quad u_3, \quad u_1^2 + u_2^2,$$

$$\left(x_1^2 - x_2^2\right) u_{01} + 2 x_1 x_2 u_{02}, \quad 2\sqrt{2} \arctan \frac{x_1}{x_2} - \arctan\left(\frac{u_{11} - u_{22}}{\sqrt{2} u_{12}}\right), \quad u_{00}, \quad u_{03}, \quad u_{33},$$

$$u_{11} + u_{22}, \quad u_{01}^2 + u_{02}^2, \quad u_{13}^2 + u_{23}^2, \quad u_{11}^2 + u_{12}^2 + u_{22}^2, \quad u_{02} u_{13} - u_{01} u_{23},$$

$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad \mu, \nu = 0, 1, 2, 3.$$

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