Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 1, 302–313

New Exact Solutions of Some Two-Dimensional Integrable Nonlinear Equations via $\overline{\partial}$-Dressing Method

V.G. DUBROVSKY †[1], *I.B. FORMUSATIK* ‡ *and Ya.V. LISITSYN* ⋆

† *Universita Degli Studi di Lecce, Via Arnesano, 73100, Lecce, Italy*

‡ *Novosibirsk State Technical University, 20 K. Marx Prosp., 630092, Novosibirsk, Russia*

⋆ *Tomsk State University, 36 Lenin Prosp., 634050, Tomsk, Russia*

Recently obtained via $\overline{\partial}$-dressing method new exact solutions of some $(2+1)$-dimensional integrable nonlinear evolution equations such as Nizhnik–Veselov–Novikov (NVN), generalized Kaup–Kuperschmidt (2DKK) and generalized Savada–Kotera (2DSK) equations are discussed.

1 Introduction

In the last two decades the Inverse Spectral Transform (IST) method has been generalized and successfully applied to various $(2+1)$-dimensional nonlinear evolution equations such as Kadomtsev–Petviashvili, Davey–Stewardson, Nizhnik–Veselov–Novikov, Zakharov–Manakov system, Ishimory, two dimensional integrable sine-Gordon and others (see books [1, 2, 3, 4] and references therein). The nonlocal Riemann–Hilbert [5], $\overline{\partial}$-problem [6] and more general $\overline{\partial}$-dressing method of Zakharov and Manakov [7, 8] are now basic tools for solving $(2+1)$-dimensional integrable nonlinear equations (see also the reviews [10, 11, 12] and books [1, 2, 3, 4]).

In the present short paper new exact solutions calculated via $\overline{\partial}$-dressing method of some two-dimensional integrable nonlinear equations such as Nizhnik–Veselov–Novikov (NVN) [13, 14], generalized Kaup–Kuperschmidt (2DKK) [16, 17] and generalized Savada–Kotera (2DSK) [16, 17] equations are reviewed.

It is well known that $\overline{\partial}$-dressing method is very powerful method for the solution of integrable nonlinear evolution equations. This method has been discovered by Zakharov and Manakov [7, 8] (see also the books [3, 4]) and applies now successfully as to $(1+1)$-dimensional and also to $(2+1)$-dimensional integrable nonlinear evolution equations. The $\overline{\partial}$-dressing method allows to construct Lax pairs (auxiliary linear problems); to solve initial and boundary value problems, to calculate the broad classes of exact solutions of integrable nonlinear equations. By the use of $\overline{\partial}$-dressing method one can construct simultaneously broad classes of exactly solvable potentials (variable coefficients of linear PDE's) and corresponding wave functions of auxiliary linear problems.

Let us remind following to [7, 8] basic ingredients of $\overline{\partial}$-dressing method for $(2+1)$-dimensional case. At first one postulates nonlocal $\overline{\partial}$-problem:

$$\frac{\partial\chi(\lambda,\overline{\lambda})}{\partial\overline{\lambda}} = (\chi * R)(\lambda,\overline{\lambda}) = \iint_C d\lambda' \wedge d\overline{\lambda'}\chi(\lambda'.\overline{\lambda'})R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda}). \tag{1}$$

For the sake of definiteness we restrict the attention to the case of the scalar complex-valued functions χ and R with the canonical normalization ($\chi \to \chi_0 = 1$, as $\lambda \to \infty$). We assume

[1]Permanent address of the author to whom correspondence should be send: Novosibirsk State Technical University, 630092, Novosibirsk, Russia, E-mail: *dubrovsky@online.nsk.su*

also that the problem (1) is uniquely solvable. The equation (1) defines behavior of the wave function χ in the spectral or momentum space.

Then one introduces dependence of kernel R and consequently the function χ on space and time variables ξ, η, t:

$$\begin{aligned}
\frac{\partial R}{\partial \xi} &= I_1(\lambda')R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t) - R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t)I_1(\lambda),\\
\frac{\partial R}{\partial \eta} &= I_2(\lambda')R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t) - R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t)I_2(\lambda),\\
\frac{\partial R}{\partial t} &= I_3(\lambda')R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t) - R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t)I_3(\lambda),
\end{aligned} \tag{2}$$

i.e.

$$R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t) = R_0(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}) \exp(F(\lambda') - F(\lambda)), \tag{3}$$

where

$$F(\lambda) := I_1(\lambda)\xi + I_2(\lambda)\eta + I_3(\lambda)t. \tag{4}$$

Here $I_i(\lambda)$ $(i = 1, 2, 3)$ are some polynomial or rational functions of λ, the choice of these functions depends on concrete integrable equation. The role of the variables ξ, η, t will be played by the usual space and time variables x, y, t or their combinations $\xi = x + \sigma y$, $\eta = x - \sigma y$ with $\sigma^2 = \pm 1$. By introducing the "long" derivatives

$$D_\xi = \partial_\xi + I_1(\lambda), \qquad D_\eta = \partial_\eta + I_2(\lambda), \qquad D_t = \partial_t + I_3(\lambda) \tag{5}$$

dependence of R on ξ, η, t can be expressed in the form

$$[D_\xi, R] = 0, \qquad [D_\eta, R] = 0, \qquad [D_t, R] = 0. \tag{6}$$

By use of derivatives (5) one constructs then linear operators

$$L = \sum u_{lmn}(\xi, \eta, t) D_\xi^l D_\eta^m D_t^n \tag{7}$$

which satisfy to the condition $\left[\frac{\partial}{\partial \overline{\lambda}}, L\right] = 0$ of absence of singularities on λ. For such operators L the function $L\chi$ obeys the same $\overline{\partial}$-equation as the function χ. If there are several operators L_i of this type then by virtue of the unique solvability of (1) one has: $L_i\chi = 0$.

The solution of $\overline{\partial}$-problem (1) with the canonical normalization $\chi_0 = 1$ is equivalent to the solution of the following singular integral equation:

$$\chi(\lambda) = 1 + \iint_C \frac{d\lambda' \wedge d\overline{\lambda'}}{2\pi i(\lambda' - \lambda)} \iint_C d\mu \wedge d\overline{\mu} \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda', \overline{\lambda'}) e^{F(\mu) - F(\lambda')}. \tag{8}$$

From (8) one obtains for the coefficients $\tilde{\chi}_0$, χ_{-1} and χ_{-2} of series expansion of χ near the points $\lambda = 0$ and $\lambda = \infty$ ($\chi = \tilde{\chi}_0 + \chi_1\lambda + \cdots$ and $\chi = \chi_0 + \frac{\chi_{-1}}{\lambda} + \cdots$):

$$\tilde{\chi}_0 = 1 + \iint_C \frac{d\lambda \wedge d\overline{\lambda}}{2\pi i \lambda} \iint_C d\mu \wedge d\overline{\mu} \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) e^{F(\mu) - F(\lambda)}, \tag{9}$$

$$\begin{aligned}
\chi_{-1} &= -\iint_C \frac{d\lambda \wedge d\overline{\lambda}}{2\pi i} \iint_C d\mu \wedge d\overline{\mu} \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) e^{F(\mu) - F(\lambda)},\\
\chi_{-2} &= -\iint_C \frac{d\lambda \wedge d\overline{\lambda}}{2\pi i} \lambda \iint_C d\mu \wedge d\overline{\mu} \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) e^{F(\mu) - F(\lambda)},
\end{aligned} \tag{10}$$

where $F(\lambda)$ is given by the formula (4). Through the coefficients $\tilde{\chi}_0$ and χ_{-1} usually the reconstructions formulas for the potentials are defined. In order to calculate via $\overline{\partial}$-dressing method exact solutions of integrable nonlinear equations and auxiliary linear problems one must to solve for given kernel R of $\overline{\partial}$-problem (3) (usually one chooses the degenerate kernels) singular integral equation (8) for wave function χ. Then by some reconstruction formulas one calculates exact solutions. Going by this way one must to satisfy important reality, potentiality or another conditions for the solutions.

In conclusion of this section let us obtain some useful general formulas for calculations of soliton and rational solutions of integrable nonlinear equations. Soliton solutions can be generated by the following delta-kernel $R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})$ (3) of $\overline{\partial}$-problem (1):

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{p=1}^{N} A_p\delta(\mu-\Lambda_p)\delta(\lambda-\Sigma_p) \tag{11}$$

which has nonzero values at the set of points

$$\Lambda := (\Lambda_1,\ldots,\Lambda_N), \qquad \Sigma := (\Sigma_1,\ldots,\Sigma_N) \tag{12}$$

of complex plane, where A_p are arbitrary complex constants; here and below $\delta(\mu-\Lambda_p)$ and $\delta(\lambda-\Sigma_p)$ are complex δ-functions. Using (11) one obtains from (8) the following linear algebraic system of equations for calculating the quantities $\chi(\Lambda_p)e^{F(\Lambda_p)}$:

$$\sum_{q=1}^{N} A_{pq}\chi(\Lambda_q)e^{F(\Lambda_q)} = e^{F(\Lambda_p)}, \qquad A_{pq} := \delta_{pq} + \frac{iA_q e^{F(\Lambda_p)-F(\Sigma_q)}}{\Lambda_p-\Sigma_q}, \quad (p,q=1,\ldots,N). \tag{13}$$

Coefficients χ_{-1} and χ_{-2} due to (10) are given by expressions:

$$\chi_{-1} = -i\sum_{p=1}^{N} A_p\chi(\Lambda_p)e^{F(\Lambda_p)-F(\Sigma_p)}, \qquad \chi_{-2} = -i\sum_{p=1}^{N} A_p\chi(\Lambda_p)\Sigma_p e^{F(\Lambda_p)-F(\Sigma_p)}. \tag{14}$$

Here as supposed all denominators in the formula (13) have nonzero values.

Rational solutions of integrable nonlinear equations can be generated by another delta-kernel $R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})$ (3) of $\overline{\partial}$-problem (1):

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{p=1}^{N} A_p\,\delta(\mu-\Lambda_p)\,\delta(\lambda-\Lambda_p) \tag{15}$$

which has nonzero values at the set of isolated points

$$\Lambda := (\Lambda_1,\Lambda_2,\ldots,\Lambda_N) \tag{16}$$

of complex plane, where for simplicity we choose A_p as some complex constants. Using (15) in (9) and (10) one obtains for $\widetilde{\chi}_0$ and χ_{-1},χ_{-2} the expressions:

$$\widetilde{\chi}_0 = 1 + \sum_{p=1}^{N}\frac{A_p}{\Lambda_p}\chi(\Lambda_p), \qquad \chi_{-1} = -i\sum_{p=1}^{N} A_p\,\chi(\Lambda_p), \qquad \chi_{-2} = -i\sum_{p=1}^{N} A_p\Lambda_p\chi(\Lambda_p). \tag{17}$$

For the quantities $\chi(\Lambda_p)$ from integral equation (8) a simple algebraic system of equations follows:

$$\sum_{q=1}^{N} A_{pq}\,\chi(\Lambda_q) = 1, \quad A_{pq} = \delta_{pq}(1+i\,A_pF'(\Lambda_p)) + \frac{i\,A_q(1-\delta_{pq})}{\Lambda_p-\Lambda_q}, \quad (p,q=1,\ldots,N). \tag{18}$$

The main problem in constructing soliton and rational solutions is the problem of choice of the sets of points Λ and Σ (12), (16) and constants A_p in (11), (15) in order to satisfy the conditions of reality, potentiality and so on.

2 Exact rational solutions of NVN equations

In this section we present some new rational solutions with constant asymptotic values at infinity of the famous (2+1)-dimensional Nizhnik–Veselov–Novikov (NVN) integrable equations [13, 14]:

$$u_t + \kappa_1 u_{\xi\xi\xi} + \kappa_2 u_{\eta\eta\eta} + 3\kappa_1 \left(u\partial_\xi^{-1} u_\eta\right)_\eta + 3\kappa_2 \left(u\partial_\eta^{-1} u_\xi\right)_\xi = 0 \qquad (19)$$

where $u(\xi, \eta, t)$ is a scalar function; κ_1, κ_2 are arbitrary constants, $\partial_\xi = \partial_x + \sigma\partial_y$, $\partial_\eta = \partial_x - \sigma\partial_y$ and $\sigma^2 = \pm 1$. Equation (19) was first introduced by Nizhnik [13] for $\sigma = 1$ and independently by Veselov and Novikov [14] for $\sigma = i$, $\kappa_1 = \kappa_2 = 1$. Here and below ∂_ξ^{-1}, ∂_η^{-1} denote operators inverse to ∂_ξ, ∂_η: $\partial_\xi^{-1}\partial_\xi = \partial_\eta^{-1}\partial_\eta = 1$.

The integrability of (19) by IST and by another means is based on the representation of this equation as the compatibility condition for two linear auxiliary problems

$$\begin{aligned} &L_1\psi = (\partial^2_{\xi\eta} + U)\psi = 0, \\ &L_2\psi = (\partial_t + \kappa_1\partial_\xi^3 + \kappa_2\partial_\eta^3 + 3\kappa_1(\partial_\xi^{-1}u_\eta)\partial_\xi + 3\kappa_2(\partial_\eta^{-1}u_\xi)\partial_\eta)\psi = 0 \end{aligned} \qquad (20)$$

in the form of Manakov's triad

$$[L_1, L_2] = BL_1, \qquad B := 3(\kappa_1\partial_\xi^{-1}U_{\eta\eta} + \kappa_2\partial_\eta^{-1}U_{\xi\xi}). \qquad (21)$$

Integration of NVN equation (19) has remarkable history. In the work of Nizhnik [13] the equation (19) with $\sigma = 1$ has been integrated by the technique of inverse problems for hyperbolic systems on the plane. In the paper of Veselov and Novikov [14] for the construction of the periodic finite-zone exact solutions of (19) with $\sigma = i$ algebraic geometric methods have been used. There exist several other beautiful works of Grinevich and Manakov, Grinevich and S. Novikov, Grinevich and R. Novikov, Grinevich in which the problem of construction of exact solutions of Veselov–Novikov (VN) equation [14] and transparent potentials for 2D stationary Schrödinger equations via $\overline{\partial}$-problem combined with nonlocal Riemann–Hilbert problem and so on have been discussed (see [18, 19] and references therein).

Here we present some rational solutions of NVN equations (19) obtained recently in the paper [20]. In the paper [20] the $\overline{\partial}$-dressing method is applied to bare operators of linear auxiliary problems (20) with constant asymptotic value of U at infinity

$$U(\xi, \eta, t) \underset{x^2+y^2 \longrightarrow \infty}{\longrightarrow} -\epsilon \neq 0. \qquad (22)$$

In this case the first linear auxiliary problem (20) has the form:

$$(\partial^2_{\xi\eta} + \widetilde{U})\psi = \epsilon\psi. \qquad (23)$$

For $\sigma = 1$ (23) can be interpreted ($\xi \Rightarrow t - x$, $\eta \Rightarrow t + y$) as one-dimensional Klein–Gordon or perturbed telegraph equation; for $\sigma = i$ (23) is nothing but the two-dimensional 2D stationary Schrödinger equation. Construction of exact solutions of (19) with constant asymptotic values at infinity means simultaneously calculation of exact wave function ψ and exactly solvable corresponding potentials for above mentioned classical linear equations; here we present also new exact rational potentials for two-dimensional stationary Schrödinger equation which correspond to two-pole wave functions. Our results partially interplay in the case $\sigma = i$ with that obtained by different methods in the papers of Grinevich and his co-authors (see [18, 19] and references therein). The use of the celebrated $\overline{\partial}$-method of Zakharov and Manakov for the construction of new exact solutions for NVN equations (19) by our opinion is very instructive and useful.

The long derivatives (5) in the case of NVN equations (19) have the form:

$$D_1 = \partial_\xi + i\lambda, \qquad D_2 = \partial_\eta - i\epsilon/\lambda, \qquad D_3 = \partial_t + i\left(\kappa_1\lambda^3 - \kappa_2\epsilon^3/\left(\lambda^3\right)\right). \qquad (24)$$

One can construct in this case two linear auxiliary problems of the type (7):

$$\begin{aligned} &L_1\chi = (D_1D_2 + V_1D_1 + V_2D_2 + U)\chi = 0,\\ &L_2\chi = \left(D_3 + \kappa_1 D_1^3 + \kappa_2 D_2^3 + W_1 D_1^2 + W_2 D_2^2 + W_3 D_1 + W_4 D_2 + W\right)\chi = 0 \end{aligned} \tag{25}$$

satisfying to the condition of absence of singularities in λ. Reconstruction formulae for V_1, V_2, U in the considered case have the form [20]:

$$V_1 = -\chi_{0\eta}/\chi_0, \qquad V_2 = -\tilde{\chi}_{0\xi}/\tilde{\chi}_0, \qquad U = -\epsilon - i\chi_{-1\eta} = -\epsilon + i\chi_{1\xi}. \tag{26}$$

Due to canonical normalization of χ, $\chi_0 = 1$ and $V_1 = 0$. Potentiality condition for the operator L_1 in (25) means $V_2 = 0$ or due to (26) $\widetilde{\chi}_0 = \text{const}$, say $\widetilde{\chi}_0 = 1$, and according to (9) has the form:

$$\iint_C \frac{d\lambda\wedge d\overline{\lambda}}{\lambda} \iint_C d\mu\wedge d\overline{\mu}\,\chi(\mu,\overline{\mu})\,R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})e^{F(\mu)-F(\lambda)} = 0, \tag{27}$$

where due to (4) and (24)

$$F(\lambda) := i\left(\lambda\xi - \frac{\epsilon}{\lambda}\eta\right) - i\left(\kappa_1\lambda^3 - \kappa_2\frac{\epsilon^3}{\lambda^3}\right)t. \tag{28}$$

The conditions of reality U and of potentiality of the operator L_1 give some restrictions on the kernel R_0 of $\overline{\partial}$-problem (1). In Nizhnik case ($\sigma = 1$) of NVN equations (19) with real $\xi = x + y$, $\eta = x - y$ space variables and $\overline{\kappa_1} = \kappa_1$, $\overline{\kappa_2} = \kappa_2$ in the limit of weak fields from (10) and (26) one can easily obtain the following restriction on the kernel R_0 (3) of $\overline{\partial}$-problem:

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \overline{R_0(-\overline{\mu},-\mu;\,-\overline{\lambda},-\lambda)}. \tag{29}$$

To the Veselov–Novikov case ($\sigma = i$, $\kappa_1 = \overline{\kappa_2} = \kappa$) of NVN equations (19) with $z = \xi = x + iy$, $\overline{z} = \eta = x - iy$ the condition of reality of U leads from (10) and (26) in the limit of weak fields to another restriction on the kernel R_0 of $\overline{\partial}$-problem:

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\epsilon}{|\mu|^2|\lambda|^2\overline{\mu}\lambda}\,\overline{R_0\left(-\frac{\epsilon}{\overline{\lambda}} - \frac{\epsilon}{\lambda}; -\frac{\epsilon}{\overline{\mu}}, -\frac{\epsilon}{\mu}\right)}. \tag{30}$$

Various choices for the kernel R of $\overline{\partial}$-problem (1) satisfying to restrictions (27), (29) and (30) lead to various classes of exact solutions of integrable nonlinear NVN equations (19).

In conclusion of this section let us cite several simplest exact rational solutions of NVN equations (19) calculated in the paper [20].

Nizhnik equation, $\boldsymbol{\sigma = 1}$. 1. The kernel R_0 has the form (15) with $N = 2$, $A_2 = \overline{A_1}$, with the set (16) $\Lambda = (\lambda_1, -\lambda_1)$, $\overline{\lambda_1} = \lambda_1$; the potentiality condition (27) is satisfied for $\frac{1}{\overline{A_1}} - \frac{1}{A_1} = \frac{i}{\lambda_1}$. The solution U of Nizhnik version of equations (19) has the form:

$$U = -\epsilon - \frac{2\epsilon}{\left(\xi\lambda_1 + \frac{\epsilon}{\lambda_1}\eta + 3\left(\kappa_1\lambda_1^3 + \kappa_2\frac{\epsilon^3}{\lambda_1^3}\right)t - a_1\lambda_1\right)^2} \tag{31}$$

with the wave function of equation (23) of the following form [20]:

$$\psi = \frac{\exp\left[\pm i\left(\lambda_1\xi - \left(\epsilon/\lambda_1^2\right)\eta + \left(\kappa_1\lambda_1^3 - \kappa_2\epsilon^3/\lambda_1^3\right)t\right)\right]}{\xi + \left(\epsilon/\lambda_1^2\right)\eta + 3\left(\kappa_1\lambda_1^2 + \kappa_2\epsilon^3/\lambda_1^4\right)t - a_1}. \tag{32}$$

2. The kernel R_0 has the form (15) with $N = 2$ and the set (16) $\Lambda = (i\alpha_1, -i\alpha_1)$, $\overline{\alpha_1} = \alpha_1$; the potentiality condition (27) is satisfied for $\frac{1}{A_2} - \frac{1}{A_1} = \frac{i}{\lambda_1}$. The solution U of Nizhnik version of equations (19) has the form:

$$U = -\epsilon - \frac{2\epsilon}{\left(\xi\alpha_1 - \frac{\epsilon}{\alpha_1}\eta - 3\left(\kappa_1\alpha^3 - \kappa_2\frac{\epsilon^3}{\alpha_1^3}\right)t - a_1\alpha_1\right)^2} \tag{33}$$

with the wave function of equation (23) of the following form [20]:

$$\psi = \frac{\exp\left[\pm\left(\alpha_1\xi + \left(\epsilon/\alpha_1^2\right)\eta - \left(\kappa_1\alpha_1^3 - \kappa_2\epsilon^3/\alpha_1^3\right)t\right)\right]}{\xi - \left(\epsilon/\alpha_1^2\right)\eta - 3\left(\kappa_1\alpha_1^2 - \kappa_2\epsilon^3/\alpha_1^4\right)t - a_1}. \tag{34}$$

The solutions (31), (33) and wave functions (32), (34) evidently are singular.

3. The kernel R_0 has the form (15) with $N = 4$, $A_2 = \overline{A_1}, A_4 = \overline{A_3}$, with the set (16) $\Lambda = (\lambda_1, -\overline{\lambda_1}, -\lambda_1, \overline{\lambda_1})$, the potentiality condition (27) is satisfied for $\frac{1}{A_3} - \frac{1}{A_1} = \frac{i}{\lambda_1}$. The solution U of Nizhnik version of equations (19) has the form:

$$U(\xi,\eta,t) = -\epsilon - 2\epsilon\frac{(\lambda_1 X(\lambda_1))^2 + \left(\overline{\lambda_1 X}(\lambda_1)\right)^2 - 1/2\left(\lambda_{1I}^2 - \lambda_{1R}^2\right)^2/\left(\lambda_{1I}^2\lambda_{1R}^2\right)}{\left(|\lambda_1 X(\lambda_1)|^2 + \frac{|\lambda_1|^2}{4}\left(\frac{1}{\lambda_{1I}^2} - \frac{1}{\lambda_{1R}^2}\right)\right)^2} \tag{35}$$

with $X(\lambda_1) = \xi + \frac{\epsilon}{\lambda_1^2}\eta + 3\left(\kappa_1\lambda_1^2 + \kappa_2\frac{\epsilon^3}{\lambda_1^4}\right)t - a_1$ and $\lambda_1 = \lambda_{1R} + i\lambda_{2I}$. The solution (35) evidently nonsingular for $|\lambda_{1I}| < |\lambda_{1R}|$.

Quite analogously one calculates rational solutions of Veselov–Novikov version of equations (19) [20].

Veselov–Novikov equation, $\boldsymbol{\sigma = i}$. 1. The kernel R_0 has the form (15) with $N = 2$, $A_2 = \overline{A_1}\lambda_1/\overline{\lambda_1}$, with the set (16) $\Lambda = (\lambda_1, -\lambda_1)$, $|\lambda_1|^2 = \epsilon$; the potentiality condition (27) is satisfied for $\frac{\overline{\lambda_1}}{\overline{A_1}\lambda_1} - \frac{1}{A_1} = \frac{i}{\lambda_1}$. The solution U of Veselov–Novikov version of equations (19) has the form:

$$U(z,\overline{z},t) = -\epsilon - \frac{2\epsilon}{\left(\lambda_1 z + \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3\right)t - \widetilde{a}_1\lambda_1\right)^2} \tag{36}$$

with the wave function of equation (23) (in this case of 2D stationary Schrödinger equation) of the following form [20]:

$$\psi = \frac{\exp\left[\pm i\left(\lambda_1 z - \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda_1}^3\right)t\right)\right]}{\lambda_1 z + \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3\right)t - \widetilde{a}_1\lambda_1}. \tag{37}$$

2. The kernel R_0 has the form (15) with $N = 2$, $\overline{A_k}\lambda_k/\overline{\lambda_k} = -A_k$, $k = 1, 2$; with the set (16) $\Lambda = (\lambda_1, -\lambda_1)$, $|\lambda_1|^2 = -\epsilon$; the potentiality condition (27) is satisfied for $\frac{1}{A_2} - \frac{1}{A_1} = \frac{i}{\lambda_1}$. The solution U of Veselov–Novikov version of equations (19) has the form:

$$U(z,\overline{z},t) = -\epsilon - \frac{2\epsilon}{\left(\lambda_1 z + \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3\right)t - \widetilde{a}_1\lambda_1\right)^2} \tag{38}$$

with the wave function of equation (23) (in this case of 2D stationary Schrödinger equation) of the following form [20]:

$$\psi = \frac{\exp\left[\pm i\left(\lambda_1 z - \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda_1}^3\right)t\right)\right]}{\lambda_1 z + \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3\right)t - \widetilde{a}_1\lambda_1}. \tag{39}$$

The solutions (36), (38) and wave functions (37), (39) evidently are singular.

3. The kernel R_0 has the form (15) with $N = 4$, $A_{2k} = \epsilon\overline{A_{2k-1}/\lambda_k}^2$, $k = 1, 2$; with the set (16) $\Lambda = (\lambda_1, -\epsilon/\overline{\lambda_1}, -\lambda_1, \epsilon/\overline{\lambda_1})$, $|\lambda_1|^2 = -\epsilon$; the potentiality condition (27) is satisfied for $\frac{1}{A_2} - \frac{1}{A_1} = \frac{i}{\lambda_1}$. The solution U of Veselov–Novikov version of equations (19) has the form:

$$U(z,\overline{z},t) = -\epsilon - 2\epsilon\frac{\lambda_1^2 X(\lambda_1)^2 + \overline{\lambda_1}^2\overline{X}(\lambda_1)^2 + 2\left[\left(\epsilon^2+|\lambda_1|^4\right)^2/\left(\epsilon^2-|\lambda_1|^4\right)^2\right]}{\left(|\lambda_1 X(\lambda_1)|^2 - \left[2\epsilon|\lambda_1|^2\left(\epsilon^2+|\lambda_1|^4\right)/\left(\epsilon^2-|\lambda_1|^4\right)^2\right]\right)^2}. \tag{40}$$

The solution (40) evidently nonsingular for $\epsilon < 0$ and have been obtained earlier (see [18, 19] and references therein) by another method.

Let us mention that one can consider also the kernels R_0 (3) of $\overline{\partial}$-problem (1) with products of delta-functions with derivatives, for example the kernel R_0

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}\Bigg[A_k\,\delta_\mu(\mu-\lambda_k)\delta_\lambda(\lambda-\lambda_k) + \frac{\epsilon^3\overline{A}_k}{|\mu|^2|\lambda|^2\overline{\mu}\overline{\lambda}}\delta_{\epsilon/\lambda}\left(\frac{\epsilon}{\lambda}+\overline{\lambda_k}\right)\delta_{\epsilon/\mu}\left(\frac{\epsilon}{\mu}+\overline{\lambda_k}\right)$$
$$+ B_k\,\delta_\mu(\mu+\lambda_k)\,\delta_\lambda(\lambda+\lambda_k) + \frac{\epsilon^3\overline{B}_k}{|\mu|^2|\lambda|^2\overline{\mu}\overline{\lambda}}\,\delta_{\epsilon/\lambda}\left(\frac{\epsilon}{\lambda}-\overline{\lambda_k}\right)\delta_{\epsilon/\mu}\left(\frac{\epsilon}{\mu}-\overline{\lambda_k}\right)\Bigg] \tag{41}$$

in the form of products of derivatives of the first order of complex delta functions which have non-zero values on the set Λ of complex plane consisting N quartets of complex isolated points $\Lambda := \bigcup_{k=1}^{N}(\lambda_k, -\epsilon/\overline{\lambda_k}, -\lambda_k, \epsilon/\overline{\lambda_k})$ arranged symmetrically near the origin and going to each other by inversion relative to the origin and/or to the circle of radius $\sqrt{|\epsilon|}$; A_k, B_k $(k = 1,\dots,N)$ are some complex constants. Such kernels correspond in the case $\sigma = i$ to so called multiple-pole (to pole of order two in considered case) wave functions of 2D stationary Schrödinger equation (23). Recently following to the paper [21] new exact rational potentials of equation (23) by Dubrovsky and Formusatik have been calculated. For the case $|\lambda_1|^2 = \epsilon > 0$ and one quartet of the points the potentiality condition (27) satisfies for the following choice of parameters $1/B_1 - 1/A_1 = i/\left(2\lambda_1^3\right)$, $\overline{\lambda_1^3/A_1} = -\lambda_1^3/A_1$, $\overline{\lambda_1^3/B_1} = -\lambda_1^3/B_1$ and the corresponding exact potential has the form:

$$U = -2\epsilon\frac{4(\tilde{x}-\tilde{y})^6 - 9\left(\tilde{x}^2-\tilde{y}^2\right)^2}{\left[2(\tilde{x}-\tilde{y})^4 + 3\left(\tilde{x}^2+\tilde{y}^2\right)\right]^2}, \qquad \tilde{x} := \lambda_R(x-\tilde{x}_0), \qquad \tilde{y} := \lambda_I(y-\tilde{y}_0), \tag{42}$$

where

$$\tilde{x}_0 := -\frac{\alpha_1\lambda_I}{|\lambda|^2} + x_0, \qquad \tilde{y}_0 := -\frac{\alpha_1\lambda_R}{|\lambda|^2} + y_0, \qquad \lambda_1 := \lambda_R + i\lambda_I. \tag{43}$$

For another case $|\lambda_1|^2 = -\epsilon > 0$ and one quartet of the points the potentiality condition (27) satisfies for the choice of parameters $1/B_1 - 1/A_1 = i/\left(2\lambda_1^3\right)$, $\overline{\lambda_1^3/A_1} = \lambda_1^3/B_1$ and the corresponding exact potential has the form:

$$U = 2\epsilon\frac{4(\tilde{x}+\tilde{y})^6 + 9\left(\tilde{x}^2-\tilde{y}^2\right)^2}{\left[2(\tilde{x}+\tilde{y})^4 - 3\left(\tilde{x}^2+\tilde{y}^2\right)\right]^2}, \qquad \tilde{x} := \lambda_I(x-\tilde{x}_0), \qquad \tilde{y} := \lambda_R(y-\tilde{y}_0), \tag{44}$$

where

$$\tilde{x}_0 := -\frac{r_1\lambda_R}{|\lambda|^2} + x_0, \qquad \tilde{y}_0 := \frac{r_1\lambda_I}{|\lambda|^2} + y_0, \qquad \lambda_1 := \lambda_R + i\lambda_I. \tag{45}$$

In the formulas (42)–(45) x_0, y_0, α_1, r_1 are some real parameters. It occurs that the main problem in calculating rational solutions corresponding to multiple pole wave functions of 2D stationary Schrödinger equation (23) as in the case of wave functions with simple poles is the fulfillment to the potentiality condition (27), in order to achieve this goal one must to choose in (41) appropriately the constants A_k, B_k and the set Λ.

3 Exact solutions of 2DKK and 2DSK equations

In this section following to the work [22] exact solutions of two-dimensional generalizations of Sawada–Kotera and Kaup–Kuppershmidt equations [16, 17] are considered. The $\overline{\partial}$-dressing method can be applied also to the study of (2 + 1)-dimensional integrable generalizations of Kaup–Kuperschmidt (2DKK)

$$U_t = U_{xxxxx} + \frac{25}{2}U_xU_{xx} + 5UU_{xxx} + 5U_x^2 + 5U_{xxy} - 5\partial_x^{-1}U_{yy} + 5UU_y + 5U_x\partial_x^{-1}U_y \tag{46}$$

and Sawada–Kotera (2DSK)

$$U_t = U_{xxxxx} + 2U_xU_{xx} + 5UU_{xxx} + 5U_x^2 + 5U_{xxy} - 5\partial_x^{-1}U_{yy} + 5UU_y + 5U_x\partial_x^{-1}U_y \tag{47}$$

equations. These equations were discovered in papers [16, 17], now they are known also as a members of the so called CKP hierarchy [16] and can be represented as the compatibility conditions in the Lax form $[L_1, L_2] = 0$; for the 2DKK equation – of the following two linear auxiliary problems [17]:

$$\begin{aligned}
&L_1\Psi = \left(\partial_x^3 + U\partial_x + \frac{1}{2}U_x + \partial_y\right)\Psi = 0,\\
&L_2\Psi = \Bigg[\partial_t - 9\partial_x^5 - 15U^2\partial_x^3 - \frac{45}{2}U_x\partial_x^2\\
&\qquad - \left(\frac{35}{2}U_{xx} + 5U^2 - 5\partial_x^{-1}U_y\right)\partial_x - \left(5UU_x - \frac{5}{2}U_y + 5U_{xxx}\right)\Bigg]\Psi = 0
\end{aligned} \tag{48}$$

and for 2DSK equation – of another two linear auxiliary problems [17]:

$$\begin{aligned}
&L_1\Psi = \left(\partial_x^3 + U\partial_x + \partial_y\right)\Psi = 0,\\
&L_2\Psi = \left[\partial_t - 9\partial_x^5 - 15U^2\partial_x^3 - 15U_x\partial_x^2 - \left(10U_{xx} + 5U^2 - 5\partial_x^{-1}U_y\right)\partial_x\right]\Psi = 0.
\end{aligned} \tag{49}$$

Here and bellow $\partial_x \equiv \partial/\partial_x$, $\partial_y \equiv \partial/\partial_y$, $\partial_t \equiv \partial/\partial_t$ and ∂_x^{-1} is an operator inverse to ∂_x. The first linear auxiliary differential problems in (48) and (49) are of the third order on ∂_x, such problems in general position have several fields as the coefficients at the various degrees of ∂_x, the 2DKK equation (46) and 2DSK equation (47) arise as special reductions of some integrable nonlinear systems for these fields. It is well known that study of special reductions requires more attention and may be more difficult than the consideration of nonlinear equations integrable by auxiliary linear problems in general position. By our opinion application of $\overline{\partial}$-dressing method in nonstandard situations of special reductions may be very instructive and useful (in our case some nonlinear constraint on the wave functions of the linear auxiliary problems must be satisfied).

The long derivatives (5) in the considered case have the form:

$$D_1 = \partial_x + i\lambda, \qquad D_2 = \partial_y + i\lambda^3, \qquad D_3 = \partial_t + 9i\lambda^5. \tag{50}$$

By the use of these derivatives one can construct two linear operators (7):

$$L_1\chi = \left(D_2 + D_1^3 + UD_1 + V\right)\chi = 0, \tag{51}$$

$$L_2\chi = \left(D_3 - 9D_1^5 + w_3D_1^3 + w_2D_1^2 + w_1D_1 + w_0\right)\chi = 0 \tag{52}$$

satisfying to the condition of absence of singularities on λ. After simple calculations the following reconstruction formulas for the potentials U and V:

$$U = -3i\chi_{-1x}, \qquad V = -3i\chi_{-1xx} + 3\chi_{-2x} - 3\chi_{-1}\chi_{-1x} \tag{53}$$

and some formulas for potentials w_0, w_1, w_2 can be obtained [22].

It was shown in the paper [17] that to the (2 + 1)-dimensional integrable generalizations of nonlinear Kaup–Kuperschmidt (46) and Sawada–Kotera (47) equations correspond the reductions:

$$(\text{2DKK}): \quad V = \frac{1}{2}U_x, \qquad (\text{2DSK}): \quad V = 0. \tag{54}$$

In terms of the wave function $\chi = \chi_0 + \chi_{-1}/\lambda + \chi_{-2}/\lambda^2 + \cdots$ the reductions (54) can be expressed as the nonlinear constraint on the coefficients χ_{-1} and χ_{-2} (10):

$$(\text{2DKK}): \quad \chi_{-2x} - \frac{i}{2}\chi_{-1xx} - \chi_{-1}\chi_{-1x} = 0, \tag{55}$$

$$(\text{2DSK}): \quad \chi_{-2x} - i\chi_{-1xx} - \chi_{-1}\chi_{-1x} = 0. \tag{56}$$

As usual the solution of the $\overline{\partial}$-problem (1) with canonical normalization $\chi_0 = 1$ is equivalent to the solution of the singular integral equation (8) with $F(\lambda)$ (4) given due to (4), (5) and (50) by the expression:

$$F(\lambda) := i\left(9\lambda x + \lambda^3 y + 9\lambda^5 t\right). \tag{57}$$

The coefficients χ_{-1} and χ_{-2} of Taylor expansion of $\chi(\lambda)$ near the point $\lambda = \infty$ by the formulas (10) are given.

One can easily obtain the restrictions following from reality $\overline{U} = U$ of U on the kernel R_0 of $\overline{\partial}$-problem (1), one has in the limit of weak fields from (10) and (53):

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \overline{R_0(-\overline{\mu},-\mu;-\overline{\lambda},-\lambda)}, \qquad R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \overline{R_0(\overline{\lambda},\lambda;\overline{\mu},\mu)}. \tag{58}$$

It is evident that the conditions (58) are the same for both 2DKK and 2DSK equations (46) and (47) but the nonlinear constraint (55) and (56) for these equations have different forms. So in order to calculate the exact solutions of 2DKK (46) and 2DSK (47) equations via $\overline{\partial}$-dressing method one must satisfy the conditions of reality (58) and the nonlinear constraint (55) and (56).

Let us consider some new solutions of 2DKK (46) and 2DSK (47) equations obtained recently in the work [22]

Exact solutions of 2DKK equation. 1. In the case of line soliton solutions to the conditions of reality for U (58) the following delta-kernel R_0 of $\overline{\partial}$-problem (1) satisfies:

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N} A_k \delta(\mu - i\alpha_k)\delta(\lambda + i\alpha_k) \tag{59}$$

with nonzero values at the sets (12) of pure imaginary points $\Lambda := (i\alpha_1, \ldots, i\alpha_N)$, $\Sigma := (-i\alpha_1, \ldots, -i\alpha_N)$ of the complex plane; A_p $(p = 1, \ldots, N)$ are arbitrary real constants; α_p are chosen so that $|\alpha_1| < |\alpha_2| < \cdots < |\alpha_N|$ and consequently $\alpha_p + \alpha_q \neq 0$ for all p, q.

As was shown recently [22] the nonlinear constraint (55) for such kernel (59) in the case of 2DKK (46) equation is satisfied and for the N-soliton solution of 2DKK equation one obtains simple determinant formula:

$$U(x,y,t) = 3\frac{\partial^2}{\partial x^2}\ln\det A \tag{60}$$

with matrix A given by (13). In the simplest case $N = 1$ of the kernel R_0 (59) with one term in the sum using (13) and (60) one obtains one-soliton solution of 2DKK equation (46):

$$U(x,y,t) = \frac{3\alpha_1^2}{\cosh^2[\alpha_1 x - \alpha_1^3 y + 9\alpha_1^5 t - a_1]}, \tag{61}$$

where on the constants A_1 and α_1 additional condition $0 < \frac{A_1}{2\alpha_1} := e^{2a_1}$ is imposed. The general formula (60) represents the superposition of N line soliton solutions of the type (61) interacting with each other elastically. The solutions corresponding to the kernel R_0 of the type (59) have been obtained recently [23] for $N = 1, 2$ via the direct Hirota method by adjusting parameters of solutions using symbolic calculations with well known software package Mathematica. The application of $\overline{\partial}$-dressing method leads immediately to general (N-arbitrary) determinant formula (60).

2. By $\overline{\partial}$-dressing method one can also effectively calculate rational solutions of integrable nonlinear equations. To the rational solutions of 2DKK equation leads for example the following delta-kernel R_0 of $\overline{\partial}$-problem (1):

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2} \sum_{k=1}^{N} [A_k \delta(\mu - i\alpha_k)\delta(\lambda - i\alpha_k) + A_k \delta(\mu + i\alpha_k)\delta(\lambda + i\alpha_k)] \tag{62}$$

which has nonzero values at the following set (16) Λ of pure imaginary points of complex plane $\Lambda := (\Lambda_1, \ldots, \Lambda_{2N}) = (i\alpha_1, -i\alpha_1, \ldots, i\alpha_N, -i\alpha_N)$. Constants A_k in (62) are arbitrary real constants. It is evident that such kernel satisfies the conditions of reality for U (58). One can show [22] that for such kernel the constraint (55) is also satisfied. For the rational solutions of 2DKK equation (46) corresponding to the kernel (62) one obtains again the simple determinant formula (60) with the matrix A given by (18). In the simplest $N = 1$ case of two terms in the sum (62) one has from (60) using (18) the following nonsingular rational solution of 2DKK equation:

$$U(x, y, t) = 6 \frac{\frac{1}{4\alpha_1^2} - \left(X(i\alpha_1) - \frac{1}{A_1}\right)^2}{\left[\frac{1}{4\alpha_1^2} + \left(X(i\alpha_1) - \frac{1}{A_1}\right)^2\right]^2}, \qquad X(i\alpha_1) = x - 3\alpha_1^2 y + 45\alpha_1^4 t. \tag{63}$$

The expression (63) represents nonsingular line lump solution of 2DKK equation (46). The general formula (60) with matrix A (18) gives the superposition of N line nonsingular lumps of the type (63) interacting with each other elastically.

3. Quite analogously to previous case one can show that to reality condition (58) and to the constraint (55) also satisfies the following kernel R_0 of $\overline{\partial}$-problem (1):

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2} \sum_{k=1}^{N} [A_k \delta(\mu - \lambda_k)\delta(\lambda - \lambda_k) + A_k \delta(\mu + \lambda_k)\delta(\lambda + \lambda_k)] \tag{64}$$

which has nonzero values at the set Λ (16) with N pairs $(\lambda_k, -\lambda_k)$, $k = 1, \ldots, N$ of real points of complex plane; here A_k are arbitrary real constants. The calculations at the present case are the same as at the previous one [22]. By the general formulas (15)–(18) and (53) one obtains the solution of 2DKK equation in the simple determinant form (60) with some matrix A of the form (18). In the simplest case $N = 1$ of two terms in the sum (64) due to (18) and (60) the solution $U(x, y, t)$

$$U(x, y, t) = -6 \frac{\left(X(\lambda_1) - \frac{1}{A_1}\right)^2 + \frac{1}{4\lambda_1^2}}{\left[\left(X(\lambda_1) - \frac{1}{\lambda_1}\right)^2 - \frac{1}{4\lambda_1^2}\right]^2}, \qquad X(\lambda_1) = x - 3\lambda_1^2 y + 45\lambda_1^4 t \tag{65}$$

of 2DKK equation (46) represents singular line lump. The general formula (60) gives the superposition of N singular line lumps of the type (65) and also is the singular solution of 2DKK equation (46).

Exact solutions of 2DSK equation. $(2+1)$-dimensional integrable generalization of Sawada–Kotera (2DSK) equation (47) differs from 2DKK equation (46) only by the constant coefficient under the term $U_x U_{xx}$. These equations are different reductions (54) ($V = 1/2U_x$ and $V = 0$) of some integrable $(2+1)$-dimensional nonlinear system of equations for the fields U and V. Due to this fact these equations have different linear auxiliary problems (48), (49) and as consequence they have different constraints (55) and (56). The main problem in calculations of exact solution of 2DSK equation (47) (as also for 2DKK equation (46)) is the choice of the kernel R_0 of $\overline{\partial}$-problem (1) in the way that the reality conditions (58) and constraint (56) should be satisfied.

1. In order to calculate line soliton solutions of 2DSK equation (47) let us start from the delta-kernel R_0 of the type (11):

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}[A_k\delta(\mu - i\alpha_k)\delta(\lambda - i\beta_k) + B_k\delta(\mu + i\beta_k)\delta(\lambda + i\alpha_k)] \tag{66}$$

with nonzero values at the sets (12) $\Lambda = (i\alpha_1, -i\beta_1, \ldots, i\alpha_N, -i\beta_N)$ and $\Sigma = (i\alpha_1, -i\beta_1, \ldots, i\alpha_N, -i\beta_N)$; here A_k, B_k, α_k, β_k ($k = 1, \ldots, N$) are arbitrary real constants. Analogously to the calculations in the case of 2DKK equation (46) one can show [22] that constraint (56) with the kernel R_0 (66) is satisfied if the following relation between constants A_k and B_k is fulfilled: $A_k\alpha_k = B_k\beta_k$. The general formulas (11)–(14) and (60) are valid in the present case and the N-soliton solution $U(x,y,t)$ of 2DSK equation is given by the simple determinant formula (60) with some matrix A of the type (13) [22]. In the simplest case $N = 1$ of two terms in the sum (66) using (11)–(14) and (60) one obtains typical line-soliton solution of 2DSK equation (47):

$$U(x,y,t) = \frac{3(\alpha_1 - \beta_1)^2}{2\cosh^2\frac{1}{2}[(\alpha_1 - \beta_1)x - (\alpha_1^3 - \beta_1^3)y + 9(\alpha_1^5 - \beta_1^5)t - 2a_1]}, \tag{67}$$

where on constants A_1, α_1, β_1 the condition $0 < \frac{A_1(\alpha_1+\beta_1)}{2\beta_1(\alpha_1-\beta_1)} := e^{2a_1}$ is imposed. The general formula (60) with matrix (13) represents the superposition of N line soliton solutions of the type (67) which interact with each other elastically.

2. As the second example let us calculate rational solutions of 2DSK equation (47) which correspond to the delta-kernel R_0 of the type (15):

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}[A_k\delta(\mu - i\alpha_k)\delta(\lambda - i\alpha_k) + A_k\delta(\mu + i\alpha_k)\delta(\lambda + i\alpha_k)] \tag{68}$$

with nonzero values at the set (16) $\Lambda = (i\alpha_1, -i\alpha_1, \ldots, i\alpha_N, -i\alpha_N)$; here A_k, B_k, α_k ($k = 1, \ldots, N$) are arbitrary real constants. Analogously to the calculations in the case of 2DKK equation one can show [22] that constraint (56) with the kernel R_0 (68) is satisfied if the following relation between constants A_k and B_k is fulfilled: $\frac{1}{B_k} - \frac{1}{A_k} = \frac{1}{\alpha_k}$; from the last relation follows the parameterizations: $\frac{1}{B_k} = a_k + \frac{1}{2\alpha_k}$, $\frac{1}{A_k} = a_k - \frac{1}{2\alpha_k}$ with a_k ($k = 1, \ldots, N$) – arbitrary real constants. The general formulas (15)–(18) are valid in the present case and the rational solution of 2DSK equation (47) corresponding to the kernel R_0 (68) is given by the simple determinant formula (60) with some matrix A of the type (18). In the simplest case $N = 1$ of two terms in the sum (68) using (18), (60) and (68) one obtains [22] singular rational solution of 2DSK equation (47):

$$U(x,y,t) = \frac{6}{(x - 3\alpha_1^2 y + 45\alpha_1^4 t - a_1)^2}. \tag{69}$$

The general formula (60) represents the superposition of N line lump solutions of the type (69) interacting with each other elastically, these solutions are also singular.

Acknowledgements

V.G.D. acknowledges financial support from the Physical Department of Lecce University in Italy and thanks colleagues at Lecce University Department of Physics for kind hospitality and fruitful discussions. V.G.D and I.B.F. acknowledge financial support from Base Scientific NSTU Grant No. 30. Ya.V.L. acknowledges financial support from the Grant No. E00-1.0-126 of the Ministry of Education of the Russian Federation and from the Grant No. 00-01-00087 of the Russian Federation for Basic Research.

[1] Novikov S.P., Zakharov V.E., Manakov S.V. and Pitaevski L.V., Solition theory: the inverse scattering method, New York, 1984.

[2] Ablowitz M.J. and Clarkson P.A., Solitons, nonlinear evolution equations and inverse scattering, London Mathematical Society Lecture Notes Series, Cambridge, Cambridge University Press, 1991.

[3] Konopelchenko B.G., Introduction to the multidimensional integrable equations: the inverse spectral transform in $2+1$ Dimensions, New York and London, Plenum Press, 1992.

[4] Konopelchenko B.G., Solitons in multidimensions: inverse spectral transform method, Singapore, World Scientific, 1993.

[5] Manakov S.V., The inverse scattering transform for the time-dependent Schrödinger equation and Kadomtsev–Petviashvili equation, *Physica D: Nonlinear Phenomena*, 1981, V.3, 420–427.

[6] Beals R. and Coifman R.R., The D-bar approach to inverse scattering and nonlinear evolutions, *Physica D: Nonlinear Phenomena*, 1986, V.18, 242–249.

[7] Zakharov V.E. and Manakov S.V., The construction of integrable multidimensional nonlinear systems and of their solutions, *Funct. Anal. Appl.*, 1986, V.19, 89–101.

[8] Zakharov V.E., Commutating operators and nonlocal $\overline{\partial}$-problem, in Proc. of the III Intern. Workshop "Nonlinear and Turbulent Processes in Physics" (1987, Kiev), Kiev, Naukova Dumka, 1988, Vol. I , 152–159.

[9] Bogdanov L.V. and Manakov S.V., The nonlocal $\overline{\partial}$-problem and $(2+1)$-dimensional soliton equations, *J. Phys. A: Math. Gen.*, 1988, V.21, L537–L544.

[10] Fokas A.S. and Ablowitz M.J., The inverse scattaring transform for multidimensional $(2+1)$ problems, in Proc. Conf. "Nonlinear Phenomena" (1982, Oaxtepec, Mexico), Editor K.B. Wolf, *Lecture Notes in Physics*, Berlin – Heidelberg – New York, Springer-Verlag, 1983, V.189, 137–183.

[11] Beals R. and Coifman R.R., Linear spectral problems, nonlinear equations and $\overline{\partial}$-method, *Inverse Problems*, 1989, V.5, 87–130.

[12] Zakharov V.E., On the dressing method, in Proc. Conf. "Inverse Methods in Action" (1989, Montpellier), Editor P.C. Sabatier, Berlin, Springer, 1990, 602–623.

[13] Nizhnik L.P., Integration of multidimensional nonlinear equations by inverse problem method, *Sov. Phys. Dokl.*, 1980, V.25, 706–708.

[14] Veselov A.P. and Novikov S.P., Finite-zone two-dimensional Schrödinger potential operators. Explicit formulas and evolution equations, *Sov. Math. Dokl.*, 1984, V.30, 588–591.

[15] Konopelchenko B.G. and Rojers C., On $(2+1)$-dimensional nonlinear system of Loewner type, *Phys. Lett. A*, 1991, V.158, 391–398.

[16] Date E., Jimbo M., Kashiwara M. and Miva T., Transformation groups for soliton equations. VI. KP hierarchies of orthogonal and symplectic type, *J. Phys. Soc. Japan*, 1981, V.50, 3813–3818.

[17] Konopelchenko B.G. and Dubrovsky V.G., Some new integrable nonlinear evolution equations in $2+1$ dimensions, *Phys. Lett. A*, 1984, V.102, 15–17.

[18] Grinevich P.G., Scattering transformation at fixed non-zero energy for the two-dimensional Schrödinger operator with potential decaying at infinity, *Russian Math. Surveys*, 2000, V.55, N 6, 1015–1083.

[19] Grinevich P.G., Doctorate dissertation, Chernogolovka, 1999.

[20] Dubrovsky V.G. and Formusatik I.B., The construction of exact rational solutions with constant asymptotic values at infinity of two-dimensional NVN integrable nonlinear evolution equations via $\overline{\partial}$-dressing method, *J. Phys. A: Math. Gen.*, 2001, V.34, 1837–1851.

[21] Dubrovsky V.G., The construction of exact multiple-pole solutions of $2+1$-dimensional integrable nonlinear evolution equations via $\overline{\partial}$-dressing method, *J. Phys. A: Math. Gen.*, 1999, V.32, 369–390.

[22] Dubrovsky V.G. and Lisitsyn Ya.V., Exact solutions of two-dimensional integrable generalizationsof Kaup–Kuppershmidt and Sawada–Kotera equations via $\overline{\partial}$-dressing method, *Phys. Lett. A*, 2002, to appear.

[23] Xing-Bia Hu, Dao-Liu Wang and Xian-Min Qian, Soliton solutions and symmetries of the $2+1$ dimensional Kaup–Kuppershmidt equation, *Phys. Lett A*, 1999, V.262, 409–415.