

Symmetric Sets of Solutions to Differential Problems

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The presence of a (Lie-point) symmetry for a differential equation leads naturally to the useful notions of symmetric sets of solutions, i.e. of sets which are mapped into themselves by the symmetry, and of orbits of solutions. We introduce the definition of *partial* symmetry, and show that the above notions may be preserved, although the symmetry is not exact. We consider the quite exceptional case of the Liouville equation, which admits an extremely large algebra of symmetries (the conformal symmetry algebra), and we shall see that any modification of this equation destroys this situation, but leaves the possibility of the existence of partial symmetries. Other simple examples are also considered, including a case of generalized (or Lie–Bäcklund) symmetry.

1 Introduction

It is certainly well known that symmetry principles may offer several useful tools and many different implications in the analysis of differential equations (see, e.g., [1–10] and references therein), but probably the most obvious and direct consequence is the fact that any symmetry of a given equation transforms solutions into (generally, different) solutions of the same equation. In other words, given a differential equation, say $\Delta = 0$, with a set of solutions \mathcal{S}_Δ , a symmetry T of this equation is an invertible transformation such that $T(\mathcal{S}_\Delta) = \mathcal{S}_\Delta$; in this sense, we can say that \mathcal{S}_Δ is a *symmetric set of solutions under T* .

For the sake of concreteness and simplicity, we will be concerned here only with the case of partial differential equations, written in the usual form [3]

$$\Delta := \Delta(x, u^{(m)}) = 0, \quad (1)$$

where Δ is a smooth function (or possibly a system of ℓ functions) of the p “independent” variables $x := (x_1, \dots, x_p) \in \mathbb{R}^p$ and of the q “dependent” variables $u := (u_1, \dots, u_q) \in \mathbb{R}^q$, together with the derivatives of the u_α with respect to the x_i ($\alpha = 1, \dots, q$; $i = 1, \dots, p$) up to some order m . Also, we will consider here mainly continuous Lie-point symmetries, in the usual sense and under the usual assumptions (see [3]), although our arguments (in Section 3) could be easily extended e.g. to generalized or Lie–Bäcklund symmetries (as we will briefly show by means of an example in Section 4), or also to discrete symmetries. Denoting by

$$X = \xi_i(x, u) \frac{\partial}{\partial x_i} + \varphi_\alpha(x, u) \frac{\partial}{\partial u_\alpha} \quad (2)$$

the infinitesimal Lie generator of a symmetry of the given problem (1), we can also say that \mathcal{S}_Δ is a symmetric set under X .

A strictly related fact to the presence of a symmetry, is that, given any solution u_0 of $\Delta = 0$, then there is an *orbit* $u^{[\lambda]}$ of solutions obtained under the application to u_0 of the (finite) transformations $T = T^{[\lambda]} = \exp(\lambda X)$ generated by X (here λ is the real Lie parameter, and – as usual – we have generally only a *local* group of transformations $T^{[\lambda]}$, i.e. λ runs only in some interval.) Clearly, orbits provide examples of symmetric sets of solutions under X . Apart from

the trivial case of solutions u_0 invariant under T , any orbit $u^{[\lambda]}$ can be naturally parametrized by the Lie parameter λ , and satisfies the differential equation (in “evolutionary form” [3])

$$Q u^{[\lambda]} = \frac{du^{[\lambda]}}{d\lambda}, \quad (3)$$

where

$$Q = -\xi_i(x, u) \frac{\partial}{\partial x_i} + \varphi_\alpha(x, u) \frac{\partial}{\partial u_\alpha}. \quad (4)$$

Many examples of this situation are well known. In Section 2, we shall consider a rather exceptional case, which is provided by the Liouville equation (both in the “Galilean” or in the “Minkowski” case, see below (5), (6)), which has an enormous relevance in mathematical physics, and which exhibits the quite singular and peculiar (i.e., unique in its class) property of admitting an extremely large algebra of symmetries, the conformal symmetry algebra.

However, the case of Liouville equation is certainly exceptional. In fact, the examples of PDE’s admitting nontrivial symmetries (Lie-point or generalized) are relatively rare. Therefore, one is urged to extend the concept of symmetry. Notions of conditional, nonclassical or similar notions of symmetries are also well known (see e.g. [2, 6, 11, 12, 13, 14, 15]). In Section 3, we shall introduce the notion of *partial symmetry* (see [16]), which is in some sense intermediate between that of exact and of conditional symmetry; we shall show in particular the existence also in this case, although the partial symmetry T is *not* exact, of proper subsets $\mathcal{P} \subset \mathcal{S}_\Delta$ of solutions of the given equation, which are *symmetric sets*, i.e. such that $T(\mathcal{P}) = \mathcal{P}$, meaning that \mathcal{P} is a subset of solutions which are transformed into one another by T . Similarly, the notion of orbit of solutions under the partial symmetry T remains valid, together with its characteristic property expressed by equations (3), (4).

2 The symmetry properties of the Liouville equation

The equation

$$u_{xx} + u_{yy} = \exp(u), \quad u = u(x, y) \quad (5)$$

has a long history. It was introduced by Liouville, studied by Poincaré, Picard, and many others in the past, and reconsidered in recent years. Actually, it enters in many areas of applied mathematics and physics, including fluid vortex theory, problems concerning electric charge distribution round a glowing wire, surface singularities, instantons and solitons theory, whereas the recent interest is concerned mainly with $(2+1)$ -dimensional quantum gravity (see e.g. [17, 18, 19]). The modern applications in classical and quantum field theory deal not only with the “Galilean” version of the Liouville equation (5), but also with its “Minkowski” form

$$u_{xx} - u_{yy} = \exp(u) \quad (6)$$

but, for simplicity, we will consider only the equation (5) (actually, all our conclusions can be suitably extended to the case (6)).

We start considering, instead of (5), the following general equation

$$u_{xx} + u_{yy} = F(u), \quad (7)$$

where $F = F(u)$ is a (smooth) function, and perform the “group theoretical analysis” of this equation, i.e. look for its Lie-point symmetries depending on the choice of $F(u)$ (we can exclude the completely elementary case of “linear” $F = a + bu$). According to standard and well known

procedures [3], one can easily see that, in addition to the obvious translation and rotation symmetries, and apart from the special case

$$F(u) = (u + k)^{1+r}, \quad r, k = \text{const}, \quad r \neq 0$$

admitting the symmetry

$$X = \frac{r}{2} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - (u + k) \frac{\partial}{\partial u}$$

the *unique* case admitting an “interesting” symmetry is just

$$u_{xx} + u_{yy} = \pm \exp(\pm u) \tag{8}$$

which exhibits the following family of symmetries

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \varphi(x, y) \frac{\partial}{\partial u}, \tag{9}$$

where the coefficients $\xi(x, y)$, $\eta(x, y)$ must satisfy

$$\xi_x = \eta_y, \quad \xi_y = -\eta_x \tag{10}$$

which imply that

$$\Delta \xi = 0, \quad \Delta \eta = 0 \tag{11}$$

(in other words, ξ and η must be harmonic conjugated functions), and where φ is given by

$$\varphi = -(\xi_x + \eta_y) = -2\xi_x. \tag{12}$$

We shall then say that (8) admits “full conformal symmetry”. This result may be in some sense reversed and strengthened in the following form (the proof can be obtained by means of direct calculations)

Proposition 1. *A PDE for the function $u = u(x, y)$ of the form $\Delta := \Delta(u, u_{xx}, u_{yy}) = 0$ admits full conformal symmetry if and only if Δ depends on u, u_{xx}, u_{yy} only through the combination $\tilde{u} := (u_{xx} + u_{yy}) \exp(\pm u)$.*

Starting from any solution of the Liouville equation, and using its symmetries, one can write down many different orbits of solutions. Precisely, let $u_0 = u(x_0, y_0)$ be any solution, expressed in terms of the “initial” variables denoted here for convenience by x_0, y_0 ; let us perform a conformal (finite) transformation into the new variables $x = x(\lambda), y = y(\lambda)$, with the infinitesimal generators defined by the harmonic conjugated functions ξ, η , i.e. a transformation satisfying

$$\frac{\partial x}{\partial \lambda} = \xi(x, y), \quad \frac{\partial y}{\partial \lambda} = \eta(x, y) \tag{13}$$

with the “initial conditions”

$$x(0) = x_0, \quad y(0) = y_0. \tag{14}$$

Let us denote by

$$x_0 \rightarrow x \equiv x(\lambda) = p(x_0, y_0, \lambda), \quad y_0 \rightarrow y \equiv y(\lambda) = q(x_0, y_0, \lambda) \tag{15}$$

this transformation, and by

$$x_0 = P(x(\lambda), y(\lambda), \lambda), \quad y_0 = Q(x(\lambda), y(\lambda), \lambda) \quad (16)$$

its inverse, then the orbit of new solutions is given by

$$u^{[\lambda]} := u_0 \left(P(x(\lambda), y(\lambda), \lambda), Q(x(\lambda), y(\lambda), \lambda) \right) + w(x, y; \lambda), \quad (17)$$

where

$$\begin{aligned} w(x, y; \lambda) &= - \int_0^\lambda \left(\xi_x(P(x(\lambda'), y(\lambda'), \lambda'), Q(x(\lambda'), y(\lambda'), \lambda')) + \eta_y(\dots) \right) d\lambda' \\ &= - \ln(\nabla P \cdot \nabla P) = \ln(\nabla Q \cdot \nabla Q). \end{aligned} \quad (18)$$

For instance, an orbit of solutions to the Liouville equation is the following

$$u^{[\lambda]} = - \ln \left(\frac{(1 + 2\lambda x + \lambda^2(x^2 + y^2))^2}{2} \sin^2 \left(\frac{x + \lambda(x^2 + y^2)}{1 + 2\lambda x + \lambda^2(x^2 + y^2)} \right) \right).$$

It has been obtained from (18) choosing in (9) $\xi = x^2 - y^2$, $\eta = 2xy$ and starting from a known solution to the Liouville equation (which can be recognized putting $\lambda = 0$ in the above expression).

3 Partial symmetries

Let us consider a general differential problem, given in the form of a system of ℓ partial differential equations, and shortly denoted, as usual, as in (1). Let

$$X = \xi_i(x, y) \frac{\partial}{\partial x_i} + \varphi_\alpha(x, y) \frac{\partial}{\partial u_\alpha} \quad (19)$$

be a given vector field, where ξ_i and φ_α are respectively p and q smooth functions. We will shortly denote by X^* the “suitable” prolongation of X , i.e. the prolongation which is needed when one has to consider its application to the differential problem in consideration. Alternatively, we may consider X^* as the infinite prolongation of X , it is clear indeed that only a finite number of terms are required and will appear in all the actual computations. The vector field X is (the Lie generator of) an exact symmetry of the differential problem (1) if and only if

$$X^* \Delta \Big|_{\Delta=0} = 0, \quad (20)$$

i.e. if and only if the prolongation X^* (here obviously, $X^* = \text{pr}^{(m)}(X)$, the m -th prolongation of X), applied to the differential operator Δ defined by (1) vanishes once restricted to the set $S^{(0)} := \mathcal{S}_\Delta$ of the solutions to the problem $\Delta = 0$.

We now assume that the vector field X is *not* a symmetry of (1), hence $X^* \Delta \Big|_{S^{(0)}} \neq 0$: let us then put

$$\Delta^{(1)} := X^* \Delta. \quad (21)$$

This defines a differential operator $\Delta^{(1)}$, of order m' not greater than the order m of the initial operator Δ . Assume now that the set of the simultaneous solutions of the two problems $\Delta = 0$ and $\Delta^{(1)} = 0$ is not empty, and let us denote by $S^{(1)}$ the set of these solutions. It can happen

that this set is mapped into itself by the transformations generated by X : *this situation is characterized precisely by the property*

$$X^* \Delta^{(1)} \Big|_{S^{(1)}} = 0.$$

Then, in this case, we can conclude that, although X is not a symmetry for the full problem (1), it generates anyway a transformation which leaves globally invariant a family of solutions of (1): this family is precisely $S^{(1)}$.

But it can also happen that $X^* \Delta^{(1)} \Big|_{S^{(1)}} \neq 0$, we then put

$$\Delta^{(2)} := X^* \Delta^{(1)} \tag{22}$$

and look for the solutions of the system

$$\Delta = \Delta^{(1)} = \Delta^{(2)} = 0$$

and repeat the argument as before: if the set $S^{(2)}$ of the solutions of this system is not empty and satisfies in addition the condition

$$X^* \Delta^{(2)} \Big|_{S^{(2)}} = 0$$

then X is a symmetry for the subset $S^{(2)}$ of solutions of the initial problem (1), exactly as before.

Clearly, the procedure can be iterated, and we can say:

Proposition 2. *Given the general differential problem (1) and a vector field (19), define, with $\Delta^{(0)} := \Delta$,*

$$\Delta^{(r+1)} := X^* \Delta^{(r)}. \tag{23}$$

Denote by $S^{(r)}$ the set of the simultaneous solutions of the system

$$\Delta^{(0)} = \Delta^{(1)} = \dots = \Delta^{(r)} = 0 \tag{24}$$

and assume that there is an integer s such that $S^{(r)}$ is not empty for $r \leq s$, and

$$X^* \Delta^{(r)} \Big|_{S^{(r)}} \neq 0 \quad \text{for } r = 0, 1, \dots, s-1, \tag{25}$$

$$X^* \Delta^{(s)} \Big|_{S^{(s)}} = 0. \tag{26}$$

Then the set $S^{(s)}$ provides a family of solutions to the initial problem (1) which is mapped into itself by the transformations generated by X .

It is clear that, given a differential problem and a vector field X , it can happen that the above procedure gives no result, i.e. that at some k -th step the set $S^{(k)}$ turns out to be empty. Assume instead that a nonempty subset $S^{(s)}$ of solutions has been found according to the above procedure: we shall then say that X is a *partial symmetry* for the problem (1), and the subset of solutions $\mathcal{P} := S^{(s)}$ obtained in this way is globally invariant under X and therefore a symmetric set.

Alternatively, one may also say that this vector field X is an *exact* symmetry for the system

$$\begin{aligned} \Delta &= 0, \\ \Delta^{(1)} &= 0, \\ &\vdots \\ \Delta^{(s)} &= 0. \end{aligned} \tag{27}$$

It must be emphasized that the solutions in this set are, in general, *not* invariant under the action of X : only the set $S^{(s)}$ is globally invariant, while the solutions are transformed into one another under the X action. As in the case of exact symmetries, the set of solutions in $S^{(s)}$ will be constituted by one or more *orbits* under the action of the one-parameter Lie group $T^{[\lambda]} = \exp(\lambda X)$, and each family $u^{[\lambda]}$ satisfies the same differential equation (3), (4). It may happen that the set $S^{(s)}$ contains also solutions u_0 which are *invariant* under $T^{[\lambda]}$, i.e. $T^{[\lambda]}u_0 = u_0$, which can be considered as trivial orbits: if this is the case, then the partial symmetry X is also a *conditional symmetry* (see [2, 12, 13]) for the problem at hand. In this sense, we can say that partial symmetries extend the notion of conditional symmetries.

4 Partial symmetries and symmetric sets: examples

We will briefly propose here some quite simple examples of PDE's admitting partial symmetries and of symmetric sets of solutions under these symmetries. More elaborate examples, including e.g. Boussinesq and Korteweg-de Vries equations, can be found in [16]. The idea can be suitably extended also to ordinary differential equations and to dynamical systems, with an application to Mel'nikov theory for the appearance of chaotic homoclinic (or heteroclinic) motion [20], or to discrete symmetries as well [16].

Example 1. It has been shown in Section 2 that the 2-dimensional Laplace equation with nonlinear additional terms $F(u)$ admits quite exceptionally some symmetry; the same is true in the presence of terms containing higher order derivatives. But partial symmetries may be allowed. Consider e.g. equations of the form

$$u_{xx} + u_{yy} = G(u, u_{x^m}), \quad (28)$$

where u_{x^m} stands for the m -th order derivative $\partial^m u / \partial x^m$, $m > 2$, and with $\partial G / \partial u_{x^m} \neq 0$. Now, the vector field

$$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \quad (29)$$

generating rotations in the plane x, y is clearly not a symmetry for (28), but it is a partial symmetry. Indeed, applying our procedure, one gets at the first step

$$X^* \Delta = \Delta^{(1)} = m \frac{\partial G}{\partial u_{x^m}} \frac{\partial^m u}{\partial x^{m-1} \partial y} = 0. \quad (30)$$

But applying the convenient prolongation X^* to this equation, one obtains $X^*(u_{x^{m-1}y}) \neq 0$ (indeed, (30) does not admit rotation symmetry), and therefore other steps are necessary in order to reach the condition $X^* \Delta^{(s)} = 0$, as requested by Proposition 2. One finds finally that the symmetric set $S^{(s)}$ of solutions must satisfy, together with the initial equation (28), the system of the $m + 1$ equations

$$\frac{\partial^m u}{\partial x^n \partial y^{m-n}} = 0, \quad n = 0, \dots, m$$

(i.e., all the m -th order derivatives must vanish). For instance, if $G = G(u_{xxx})$, the set $S^{(s)}$ has the form

$$S^{(s)} := \left\{ u = A_0 + \frac{c}{4} (x^2 + y^2) + A_1 x + B_1 y \right. \\ \left. + A_2 (x^2 - y^2) + B_2 xy + A_3 (x^3 - 3xy^2) + B_3 (3x^2y - y^3) \right\},$$

where $c = G(0)$, and it is easy to recognize that this set contains a set of rotationally *invariant* solutions, and different families of orbits of solutions which are transformed into themselves under rotations. The presence in this set of rotationally invariant solutions shows that the rotation symmetry is in this example also a conditional symmetry for the equation (28), but the notion of partial symmetry provides clearly a larger set of solutions. Let us emphasize that it should be *not* sufficient to impose only the vanishing of the “symmetry breaking term” in the initial equation (28), or only the first condition obtained above (30), i.e. one or both of the conditions

$$\frac{\partial^m u}{\partial x^m} = 0, \quad \frac{\partial^m u}{\partial x^{m-1} \partial y} = 0$$

indeed, a generic solution of these equations and of the initial one would be transformed by rotations into a $v(x, y)$ which is *not* a solution!

Example 2. As another example, consider vector fields of the form

$$X = \varphi_\alpha(x) \frac{\partial}{\partial u_\alpha}. \quad (31)$$

If this is an exact symmetry of some equation $\Delta = 0$, one has that — given any solution u_0 of this equation — then $u_0 + \lambda\varphi$ is also a solution. But if X is only a partial symmetry, then this is true only for some special u_0 : this gives rise to a “partial linear superposition principle”. For instance, for the equation

$$\Delta := u_x^2 - u_y^2 - u_x - 2u_y - u + x = 0 \quad (32)$$

one can verify that the vector field

$$X = \exp(-x - y) \frac{\partial}{\partial u} \quad (33)$$

is a partial (not exact) symmetry, and in fact

$$u^{[\lambda]}(x, y) = x + \lambda \exp(-x - y)$$

is a symmetric set of solutions to (32). Notice that this set contains just a single orbit, and that there are no invariant solutions under the above (33) in this set: this means that in this example the partial symmetry X is *not* a conditional symmetry.

Example 3. Our final example deals with generalized (or Lie–Bäcklund) symmetries, and illustrates that our method is also applicable to these symmetries. We consider an equation for $u = u(t, x)$ of the form (Burgers, Fisher, Fitzhugh–Nagumo equations are of this form)

$$u_t = u_{xx} + R(u, u_x) \quad (34)$$

and the generalized vector field

$$X = (u_{xx} - 2u) \frac{\partial}{\partial u}. \quad (35)$$

It has been shown by Zhdanov [21] that (35) is a conditional Bäcklund symmetry for equations of the form (34) if and only if the nonlinear term R satisfies a special equation (see [21]). We now choose

$$R = u_x^2 - u^2$$

which does *not* satisfy Zhdanov equation. However, repeating word for word our above procedure, it can be seen that (35) is a partial Bäcklund symmetry for this equation, and in fact

$$u_{\pm}^{[\lambda]} = \exp(t \pm x + \lambda)$$

are two families of solutions to the above equation. As expected, no invariant solution under (35) is contained in this set, and therefore (35) is not a conditional Bäcklund symmetry.

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