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The Euclidean Propagator in Quantum Models with Non-Equivalent Instantons

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We consider in detail the Euclidean propagator in quantum-mechanical models which include the existence of non-equivalent instantons. For such a purpose we resort to the semiclassical approximation in order to take into account the fluctuations over the instantons themselves. The physical effects of the multi-instanton configurations appear in terms of the alternate dilute-gas approximation.

1 Introduction

The tunnelling phenomenon represents one of the most outstanding effects in quantum theory. Starting from the pioneering work of Polyakov on the subject [1], the semiclassical treatment of the tunnelling is presented in terms of the Euclidean version of the path-integral formalism. The basis of this approach relies on the so-called instanton calculus. As usual the instantons themselves correspond to localised finite-action solutions of the Euclidean equation of motion where the time variable is essentially imaginary. In short, one finds the appropiate classical configuration to evaluate the term associated with the quadratic fluctuations. On the other hand the functional integration is solved by means of the gaussian scheme except for the zero-modes which appear in connection with the translational invariances of the system. As expected one introduces collective coordinates so that ultimately the gaussian integration is performed along the directions orthogonal to the zero-modes. A functional determinant includes an infinite product of eigenvalues so that a highly divergent result appears in this context. However one can regularize the fluctuation factors by means of the conventional ratio of determinants.

Next let us describe in brief the instanton calculus for the one-dimensional particle as can be found in [2]. Our particle moves under the action of a confining potential $V(x)$ which yields a pure discrete spectrum of energy eigenvalues. If the particle is located at the initial time $t_i = -T/2$ at the point x_i while one finds it when $t_f = T/2$ at the point x_f, the well-known functional version of the non-relativistic quantum mechanics allows us to write the transition amplitude in terms of a sum over all paths joining the world points with coordinates $(-T/2, x_i)$ and $(T/2, x_f)$. If we incorporate the change $t \to -i\tau$, known in the literature as the Wick rotation, the Euclidean formulation of the path-integral reads

$$\langle x_f|\exp(-HT)|x_i\rangle = N(T)\int [dx]\exp\{-S_e[x(\tau)]\},$$

where H represents as usual the Hamiltonian, the factor $N(T)$ serves to normalize the amplitude while $[dx]$ indicates the integration over all functions which fulfil the corresponding boundary conditions. In addition the Euclidean action S_e corresponds to

$$S_e = \int_{-T/2}^{T/2}\left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]d\tau \tag{2}$$

whenever the mass of the particle is set equal to unity. Now we take care of the octic potential $V(x)$ given by

$$V(x) = \frac{\omega^2}{2}\left(x^2-1\right)^2\left(x^2-4\right)^2 .$$

When considering that $\omega^2 \gg 1$ the energy barriers are high enough to split the physical system into a sum of harmonic oscillators. The particle executes small oscillations around each minima of the potential located at $x = \pm 1$ and $\tilde{x} = \pm 2$. The second derivative of the potential at these points, i.e. $V''(x = \pm 1) = 36\omega^2$ and $V''(\tilde{x} = \pm 2) = 144\omega^2$, gives the frequencies of the harmonic oscillators at issue.

As regards the discrete symmetry $x \to -x$ which the potential $V(x)$ enjoys, we observe how the four minima are non-equivalent since no connection is possible between the two sets represented by $x = \pm 1$ and $\tilde{x} = \pm 2$. We would like to make the description of the tunnelling phenomenon to describe how the symmetry cannot appear spontaneously broken at quantum level. The expectation value of the coordinate x evaluated for the ground-state is zero as corresponds to the even character of the potential $V(x)$.

2 The one-instanton amplitude

In this section we would like to discuss the transition amplitude between the points $x_i = 1$ and $x_f = 2$. For such a purpose we need the explicit form of the topological configuration with $x_i = 1$ at $t_i = -T/2$ while $x_f = 2$ when $t_f = T/2$. To get the instanton $x_{c1}(\tau)$ which connects the points $x_i = 1$ and $x_f = 2$ with infinite euclidean time, we can resort to the well-grounded Bogomol'nyi bound [3]. The situation is solved by integration of a first-order differential equation which derives from the zero-energy condition for the motion of a particle under the action of $-V(x)$. In short

$$x_{c1}(\tau) = 2\cos\left[\frac{\pi}{3} - \frac{1}{3}\arccos\left(\frac{e^{-12\omega(\tau-\tau_c)}-1}{e^{-12\omega(\tau-\tau_c)}+1}\right)\right], \tag{4}$$

where τ_c indicates the point at which the instanton makes the jump. Equivalent solutions are obtained by means of the transformations $\tau \to -\tau$ and $x_{c1}(\tau) \to -x_{c1}(\tau)$. The instanton calculus allows the connection between adjoint minima of the potential. We notice therefore the existence of a second instanton interpolating between $x_i = -1$ and $x_f = 1$. The classical Euclidean action S_1 associated with the topological configuration at issue is computed according to (2) so that $S_1 = 22\omega/15$. Next the standard description of the one-instanton amplitude between $x_i = 1$ and $x_f = 2$ takes over

$$\begin{aligned}&\langle x_f = 2|\exp(-HT)|x_i = 1\rangle \\ &= N(T)\left\{\mathrm{Det}\left[-\frac{d^2}{d\tau^2}+\nu^2\right]\right\}^{-1/2}\left\{\frac{\mathrm{Det}\left[-\left(d^2/d\tau^2\right)+V''[x_{c1}(\tau)]\right]}{\mathrm{Det}\left[-\left(d^2/d\tau^2\right)+\nu^2\right]}\right\}^{-1/2}\exp(-S_1),\end{aligned}$$

where as usual we have multiplied and divided by the determinant of a generic harmonic oscillator of frequency ν. The so-called regularization term is interpreted as a new amplitude given by

$$\langle x_f = 0|\exp(-H_{\mathrm{ho}}T)|x_i = 0\rangle = N(T)\left\{\mathrm{Det}\left[-\frac{d^2}{d\tau^2}+\nu^2\right]\right\}^{-1/2}. \tag{5}$$

Now the explicit evaluation of (5) is made according to the method explained in [4]. To sum up

$$\langle x_f = 0|\exp(-H_{\mathrm{ho}}T)|x_i = 0\rangle = \left(\frac{\nu}{\pi}\right)^{1/2}\left(2\sinh\nu T\right)^{-1/2}.$$

The existence of a zero-mode $x_0(\tau)$ in the spectrum of the stability equation requires the introduction of a collective coordinate. The zero eigenvalue reflects the translational invariance of the system so that there is one direction in the functional space of the second variations which results incapable of changing the action. The explicit form of the zero-mode $x_0(\tau)$ corresponds to the derivative of the topological configuration itself, i.e.

$$x_0(\tau) = \frac{1}{\sqrt{S_1}} \frac{dx_{c1}}{d\tau}.$$

The integral over the zero-mode becomes equivalent to the integration over the center of the instanton τ_c. If the change of variables is incorporated our ratio of determinants corresponds to [2]

$$\left\{ \frac{\mathrm{Det}\left[-\left(d^2/d\tau^2\right) + V''[x_{c1}(\tau)]\right]}{\mathrm{Det}\left[-\left(d^2/d\tau^2\right) + \nu^2\right]} \right\}^{-1/2} = \left\{ \frac{\mathrm{Det}'\left[-\left(d^2/d\tau^2\right) + V''[x_{c1}(\tau)]\right]}{\mathrm{Det}\left[-\left(d^2/d\tau^2\right) + \nu^2\right]} \right\}^{-1/2} \sqrt{\frac{S_1}{2\pi}} d\tau_c,$$

where as usual Det' stands for the so-called reduced determinant once the zero-mode has been removed. Next we take advantage of the Gelfand–Yaglom method of computing ratios of determinants where only the knowledge of the large-τ behaviour of the classical solution $x_{c1}(\tau)$ is necessary [2]. If $\hat{O}$ and $\hat{P}$ represent a couple of second order differential operators whose eigenfunctions vanish at the boundary, the quotient of determinants is given in terms of the zero-energy solutions $f_0(\tau)$ and $g_0(\tau)$ so that

$$\frac{\mathrm{Det}\,\hat{O}}{\mathrm{Det}\,\hat{P}} = \frac{f_0(T/2)}{g_0(T/2)}$$

whenever the eigenfunctions fulfil the initial conditions

$$f_0(-T/2) = g_0(-T/2) = 0, \qquad \frac{df_0}{d\tau}(-T/2) = \frac{dg_0}{d\tau}(-T/2) = 1.$$

As the zero-mode $g_0(\tau)$ of the harmonic oscillator of frequency ν is given by

$$g_0(\tau) = \frac{1}{\nu}\ \sinh[\nu(\tau + T/2)]$$

we need the form of the solution $f_0(\tau)$ associated with the topological configuration written in (4). Starting from $x_0(\tau)$ we can write a second solution $y_0(\tau)$ according to

$$y_0(\tau) = x_0(\tau) \int_0^\tau \frac{ds}{x_0^2(s)}.$$

As regards the asymptotic behaviour of $x_0(\tau)$ and $y_0(\tau)$ we have that

$$x_0(\tau) \sim C\exp(-12\omega\tau) \quad \text{if} \quad \tau \to \infty,$$
$$x_0(\tau) \sim D\exp(6\omega\tau) \quad \text{if} \quad \tau \to -\infty$$

together with

$$y_0(\tau) \sim \exp(12\omega\tau)/24\omega C \quad \text{if} \quad \tau \to \infty,$$
$$y_0(\tau) \sim -\exp(-6\omega\tau)/12\omega D \quad \text{if} \quad \tau \to -\infty,$$

where the constants C and D can be obtained from the derivative of (4). Taking the linear combination of $x_0(\tau)$ and $y_0(\tau)$ given by

$$f_0(\tau) = Ax_0(\tau) + By_0(\tau)$$

the incorporation of the initial conditions allows us to write that

$$f_0(\tau) = x_0(-T/2)y_0(\tau) - y_0(-T/2)x_0(\tau).$$

Now we can extract the asymptotic behaviour of $f_0(\tau)$, i.e.

$$f_0(T/2) \sim \frac{D}{24\omega C}\exp(3\omega T) \quad \text{if} \quad T \to \infty.$$

Next we need to consider the lowest eigenvalue of the stability equation. From a physical point of view we can explain the situation as follows: the derivative of the topological solution does not quite satisfy the boundary conditions for the interval $(-T/2, T/2)$. When enforcing such a behaviour, the eigenstate is compressed and the energy shifted slightly upwards. In doing so the zero-mode $x_0(\tau)$ is substituted for the $f_\lambda(\tau)$, i.e.

$$-\frac{d^2 f_\lambda(\tau)}{d\tau^2} + V''[x_{c1}(\tau)]f_\lambda(\tau) = \lambda f_\lambda(\tau)$$

whenever

$$f_\lambda(-T/2) = f_\lambda(T/2) = 0.$$

Going to the lowest order in perturbation theory we find

$$f_\lambda(\tau) \sim f_0(\tau) + \lambda \left.\frac{df_\lambda}{d\lambda}\right|_{\lambda=0}$$

so that

$$f_\lambda(\tau) = f_0(\tau) + \lambda \int_{-T/2}^{\tau} [x_0(\tau)y_0(s) - y_0(\tau)x_0(s)]f_0(s)ds.$$

The asymptotic behaviour of the zero-modes together with the condition $f_\lambda(T/2) = 0$ allow us to write that

$$\lambda = 12\omega D^2 \exp(-6\omega T).$$

The evaluation of this quotient of determinants requires a choice for the parameter ν so that the frequency of the harmonic oscillator of reference is the average of the frequencies over the non-equivalent minima located at $x = 1$ and $\tilde{x} = 2$. In other words $\nu = 9\omega$. When considering the well-grounded double-well model the two minima of the potential are equivalent so that the aforementioned average is not necessary. However in this case the Gelfand–Yaglom method fixes the frequency ν in order the ratio of determinants to be finite. In addition we have that (see (4))

$$C = \frac{4\sqrt{3}\omega}{\sqrt{S_1}}, \qquad D = \frac{16\omega}{3\sqrt{S_1}}.$$

Now we can write the one-instanton amplitude between the points $x_i = 1$ and $x_f = 2$, i.e.

$$\langle x_f = 2|\exp(-HT)|x_i = 1\rangle = \left(\frac{9\omega}{\pi}\right)^{1/2} (2\sinh 9\omega T)^{-1/2} \sqrt{S_1} K_1 \exp(-S_1)\omega d\tau_c,$$

where K_1 represents a numerical factor given by

$$K_1 = 16\sqrt{\frac{15\sqrt{3}}{11\pi}}.$$

In doing so we get a transition amplitude just depending on the point τ_c at which the instanton makes the jump. This regime seems plausible whenever

$$\sqrt{S_1} K_1 \exp(-S_1)\omega T \ll 1$$

a nonsense condition when T is large enough. However in this situation we can accommodate configurations constructed of instantons and anti-instantons which mimic the behaviour of a trajectory just derived from the euclidean equation of motion.

To finish this section we take care of the second instanton of the octic model. The one-instanton amplitude between $x_i = -1$ and $x_f = 1$ is based on the topological configuration $x_{c2}(\tau)$

$$x_{c2}(\tau) = 2\cos\left[\frac{\pi}{3} + \frac{1}{3}\arccos\left(\frac{e^{12\omega\tau} - 1}{e^{12\omega\tau} + 1}\right)\right]$$

whose classical euclidean action corresponds to $S_2 = 76\omega/5$. This second instanton reminds the case of the double-well potential since connects equivalent minima of the potential. The form of the ratio of determinants at issue should be

$$\left\{\frac{\mathrm{Det}'\left[-\left(d^2/d\tau^2\right) + V''[x_{c2}(\tau)]\right]}{\mathrm{Det}\left[-\left(d^2/d\tau^2\right) + 36\omega^2\right]}\right\}^{-1/2} = \sqrt{S_2} K_2 \omega d\tau_c,$$

where K_2 corresponds to

$$K_2 = 12\sqrt{\frac{15}{38\pi}}.$$

3 The multi-instanton amplitude

In this section we discuss the complete amplitude which incorporates the physical effect of a string of instantons and anti-instantons along the τ axis. The octic potential represents a more complicated case since we need to include the whole scheme of non-equivalent instantons. We wish to evaluate the functional integral by summing over all such configurations with n instantons and anti-instantons centered at points $\tau_1, \ldots, \tau_n$ whenever

$$-\frac{T}{2} < \tau_1 < \cdots < \tau_n < \frac{T}{2}.$$

We can carry things further and assume as usual that the action of the string of instantons and anti-instantons is given by the sum of the n individual actions. This method is well-known in the literature where it appears with the name of dilute-gas approximation [5]. The translational degrees of freedom yield an integral like

$$\int_{-T/2}^{T/2} \omega d\tau_n \int_{-T/2}^{\tau_n} \omega d\tau_{n-1} \cdots \int_{-T/2}^{\tau_2} \omega d\tau_1 = \frac{(\omega T)^n}{n!}.$$

When considering the transition amplitude between $x_i = 1$ and $x_f = 2$ the total number n of topological configurations must be odd. We can split n (odd) into the sum of two contributions n_1 (odd) and n_2 (even) which represent the different possibilities associated with the existence of non-equivalent instantons. Then we have n_1 topological configurations just interpolating between $x = 1$ and $\tilde{x} = 2$ or $x = -1$ and $\tilde{x} = -2$. Identical situation appears in connection with n_2 where now the initial and final points of the trip are $x = \pm 1$. Now we need to include a combinatorial factor F to count the different possibilities that we have of distributing the n instantons. Except for the last step which corresponds to the instanton analyzed in the previous section, we deal with a closed path of topological configurations starting and coming back to the point $x = 1$. As regards the instantons (anti-instantons) belonging to the first type we observe the formation of pairs due to the location of the four minima of the potential along the real axis. Therefore we have $(n_1 - 1)/2 + n_2$ holes to fill bearing in mind that once the $(n_1 - 1)/2$ pairs of instantons and anti-instantons are distributed no freedom at all remains to locate the topological configurations associated with n_2. In short

$$F = \left(\begin{array}{c} (n_1 - 1)/2 + n_2 \\ (n_1 - 1)/2 \end{array} \right).$$

At this point we can discuss the complete transition amplitude we are looking for in terms of the so-called instanton density, i.e.

$$d_i = \sqrt{S_i} K_i \exp(-S_i), \qquad i = 1, 2.$$

To be precise

$$\langle x_f = 2 | \exp(-HT) | x_i = 1 \rangle = \left(\frac{9\omega}{\pi} \right)^{1/2} (2 \sinh 9\omega T)^{-1/2} \sum_{n_1, n_2} [d_1 \omega T]^{n_1} [d_2 \omega T]^{n_2} \frac{F}{n!}. \tag{32}$$

The best way of dealing with the double sum of (32) should be the following

$$S = \sum_{r=0}^{\infty} \frac{[d_1 \omega T]^{2r+1}}{(2r+1)!} \sum_{q=0}^{r} \left(\begin{array}{c} r + q \\ r - q \end{array} \right) (d_2/d_1)^{2q},$$

where we can handle the sum $\tilde{S}$ concerning the variable q taking advantage of [6]

$$\sum_{q=0}^{r} (-1)^q \left(\begin{array}{c} r + q \\ 2q \end{array} \right) = \sec[\arcsin(x/2)] \cos[(2r+1) \arcsin(x/2)]$$

including the transformation $x \to ix$ to obtain that

$$\tilde{S} = \frac{\cosh[(2r+1) \arg\sinh(s/2)]}{\cosh[\arg\sinh(s/2)]},$$

where s stands for the relative instanton density given by $s = d_2/d_1$. In terms of a new variable z defined as

$$z = \arg\sinh(s/2)$$

it is the case that a typical value of r provides us with the final expression for $\tilde{S}$, i.e.

$$\tilde{S} = \frac{\exp[(2r+1)z]}{\sqrt{4 + s^2}}.$$

In other words

$$S = \frac{\sinh[d_1 \omega T \exp(z)]}{\sqrt{4 + s^2}}.$$

In doing so the complete amplitude between the points $x_i = 1$ and $x_f = 2$ reads

$$\langle x_f = 2|\exp(-HT)|x_i = 1\rangle = \left(\frac{9\omega}{\pi}\right)^{1/2} (2\sinh 9\omega T)^{-1/2} \frac{\sinh[d_1\omega T\exp(z)]}{\sqrt{4+s^2}}.$$

To sum up, we have explained the method of dealing with quantum-mechanical models which exhibit a more complicated structure of non-equivalent classical vacua in comparison with the well-grounded cases of the double-well or periodic sine-Gordon potentials where the equivalence of all the minima of $V(x)$ is taken for granted [5]. As regards the octic potential the topological solutions of the system inherit the property of non-equivalence. The global effect of the multi-instanton configurations is discussed in terms of the alternate dilute-gas approximation.

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