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Melnikov Analysis for Multi-Symplectic PDEs

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In this work the Melnikov method for perturbed Hamiltonian wave equations is considered in order to determine possible chaotic behaviour in the systems. The backbone of the analysis is the multi-symplectic formulation of the unperturbed PDE and its further reduction to travelling waves. In the multi-symplectic approach two separate symplectic operators are introduced for the spatial and temporal variables, which allow one to generalise the usual symplectic structure. The systems under consideration include perturbations of generalised KdV equation, nonlinear wave equation, Boussinesq equation. These equations are equivariant with respect to Abelian subgroups of Euclidean group. It is assumed that the external perturbation preserves this symmetry. Travelling wave reduction for the above-mentioned systems results in a four-dimensional system of ODEs, which is considered for Melnikov type chaos. As a preliminary for the calculation of a Melnikov function, we prove the persistence of a fixed point for the perturbed Poincaré map by using Lyapunov–Schmidt reduction. The framework sketched will be applied to the analysis of possible chaotic behaviour of travelling wave solutions for the above-mentioned PDEs within the multi-symplectic approach.

1 Introduction

Recently it was shown how many nonlinear PDEs can be formulated in a multi-symplectic form [3, 4, 5]. This formulation assigns distinct symplectic structures to the spatial and the temporal coordinates, thereby generalising the usual Hamiltonian formulation. By means of the multi-symplectic approach questions concerning stability of solitary waves, existence of generalised basic state at infinity, equivariant properties of the solutions etc. can be considered in a more general setting, yielding new results on these issues.

In order to analyse chaotic behaviour of travelling wave solutions to Hamiltonian PDEs, Melnikov's method can be used. The problem with its direct application is due to the symmetry in a multi-symplectic formulation of these PDEs, which results in the presence of unit eigenvalue among the spectrum of the Poincaré map. This complication is solved by means of Lyapunov–Schmidt reduction.

Finally, we illustrate the application of the Melnikov method to the study of chaotic behaviour in a perturbed Korteweg-de Vries equation.

2 General setting

We start to consider a perturbed multi-symplectic PDE of the form [5]:

$$\mathbf{M}Z_t + \mathbf{K}Z_x = \nabla S(Z) + \epsilon S_1(Z, x - ct), \qquad Z = \begin{pmatrix} U \\ V \\ W \\ \Phi \end{pmatrix} \in \mathbb{R}^4, \qquad x \in \mathbb{R}, \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are constant skew-symmetric matrices on $\mathbb{R}^4$ and $S : \mathbb{R}^4 \to \mathbb{R}$ is sufficiently smooth (at least twice continuously differentiable). The perturbation S_1 is assumed to be a periodic function of its last argument: $S_1(\cdot, x) = S_1(\cdot, x + T)$ and also C^r, $r \geq 2$.

We suppose that the system (1) is equivariant with respect to a one-parameter Lie group, whose algebra is spanned by the generator ξ. For the unperturbed case ($\epsilon = 0$), multi-symplectic Noether theory provides the existence of the two functionals $P(Z)$ and $Q(Z)$ such that [3]

$$\mathbf{M}\xi(Z) = \nabla P(Z), \qquad \mathbf{K}\xi(Z) = \nabla Q(Z). \tag{2}$$

The state at infinity should satisfy [4, 5]:

$$\nabla S(Z_0) = a\nabla P(Z_0) + b\nabla Q(Z_0) \tag{3}$$

with $P(Z_0) = \mathcal{P}$ and $Q(Z_0) = \mathcal{Q}$ specified real parameters, $a, b \in \mathbb{R}$.

A shape of an unperturbed solitary wave travelling at speed c, $Z(x,t) = Z(x - ct)$, which is biasymptotic to this state should satisfy the equation

$$Z_x = \mathbf{J}_c^{-1}\nabla H_0(Z), \tag{4}$$

where $H_0(Z) = S(Z) - aP(Z) - bQ(Z)$, and $\mathbf{J}_c = \mathbf{K} - c\mathbf{M}$ [4, 5].

To study the existence of travelling waves and their chaotic behaviour, we consider the dynamical system (similar consideration can be found in [7])

$$\frac{d}{dx}Z = f_0(Z) + \epsilon f_1(Z, x), \qquad Z \in \mathbb{R}^4, \qquad 0 < \epsilon \ll 1, \qquad 0 < x < \infty, \tag{5}$$

where $f_0(Z) = \mathbf{J}_c^{-1}\nabla H_0(Z)$, and $f_1(Z, x) = \mathbf{J}_c^{-1}S_1(Z, x)$.

The following hypotheses are imposed on the system:

(H1) **a)** $f_0 : \mathbb{R}^4 \to \mathbb{R}^4$ is C^r $(r \geq 2)$;
b) $f_1 : \mathbb{R}^4 \times S^1 \to \mathbb{R}^4$ is C^r $(r \geq 2)$.

The system (5) can be rewritten as a following suspended system:

$$\frac{dZ}{dx} = f_0(Z) + \epsilon f_1(Z, \theta), \qquad \frac{d\theta}{dx} = \omega, \tag{6}$$

with the frequency $\omega = 2\pi/T$. Its flow $\Phi_t^\epsilon : \mathbb{R}^4 \times S^1 \to \mathbb{R}^4 \times S^1$ is defined for all $t \in \mathbb{R}$.

(H2) **a)** *The unperturbed system*

$$\frac{d}{dx}Z = f_0(Z) = \mathbf{J}_c^{-1}\nabla H_0(Z) \tag{7}$$

is Hamiltonian with energy $H_0 : \mathbb{R}^4 \to \mathbb{R}$.

So, we have the corresponding symplectic form

$$\Omega(Z_1, Z_2) = \langle \mathbf{J}_c Z_1, Z_2 \rangle. \tag{8}$$

b) *The system (7) is equivariant with respect to a one-parameter symmetry group* $\mathcal{G}$ *spanned by the generator* $\xi(Z)$. *This group* $\mathcal{G}$ *is assumed to be either compact or a subgroup of affine translations. We also suppose that the perturbation preserves this symmetry.*

c) *The system (7) has a family of fixed points* ϕp_0, *where* $p_0 = 0$ *and* $\phi \in \mathcal{G}$, *and corresponding (heteroclinic) orbits* $Z_0(x)$ *such that*

$$\frac{d}{dx}Z_0(x) = f_0\left(Z_0(x)\right), \tag{9}$$

and $\lim_{x\to-\infty} Z_0(x) = \phi_1 p_0$ *as well as* $\lim_{x\to+\infty} Z_0(x) = \phi_2 p_0$, *for some* $\phi_1, \phi_2 \in \mathcal{G}$. Correspondingly, the unperturbed suspended system (6) $\epsilon = 0$ has a family of periodic orbits $\phi\gamma_0(x) = (\phi p_0, \omega x)$.

(H3) *$f_1(Z,x) = A_1 Z + f(x) + g(Z,x)$, where A_1 is a linear operator, $f(x) = f(x+T)$, $g(Z,x)$ is time-periodic with period T and also satisfies $g(0,x) = 0$, $Dg(0,x) = 0$.*

(H4) a) *For $\epsilon = 0$ the spectrum $\sigma[\exp(TA)] = \{1, 1, e^{\pm\lambda T}\}$, $\lambda > 0$, where $A = \mathbf{J}_c^{-1} D^2 H_0(p_0)$,*

b) i. (Hamiltonian case) *For $\epsilon > 0$ $\sigma[\exp[T(A+\epsilon A_1)]] = \left\{1, 1, e^{T\lambda_\epsilon^\pm}\right\}$,*

ii. (Dissipative case) *For $\epsilon > 0$ $\sigma[\exp[T(A+\epsilon A_1)]] = \left\{1, \lambda^d, e^{T\lambda_\epsilon^\pm}\right\}$, where $C_1\epsilon \leqslant \operatorname{dist}(\lambda^d, |z| = 1) \leqslant C_2\epsilon$, $C_1 > 0$, $C_2 > 0$.*

Next, one can define the Poincaré map $P^\epsilon : \mathbb{R}^4 \to \mathbb{R}^4$ as

$$P^\epsilon(Z) = \pi_1 \Phi_T^\epsilon(Z, 0), \tag{10}$$

where $\pi_1 : \mathbb{R}^4 \times S^1 \to \mathbb{R}^4$ denotes the projection onto the first factor. Equivalently, one can define

$$P_{x_0}^\epsilon(Z) = \pi_1 \Phi_T^\epsilon(Z, x_0). \tag{11}$$

We rewrite the fixed point equation $P^\epsilon(p_\epsilon) = p_\epsilon$ in the form:

$$\mathcal{P}^\epsilon(p_\epsilon) = 0, \tag{12}$$

where $\mathcal{P}^\epsilon(Z) = P^\epsilon(Z) - Z$, and the operator $L = D\mathcal{P}^0(0)$ is introduced.

3 Main results

Lemma 1. *Let **(H1)**–**(H4)** hold. For ϵ small, there exists a unique group orbit ϕp_ϵ of fixed points of the perturbed Poincaré map near the group orbit ϕp_0 such that*

$$\min_{\phi_1,\, \phi_2 \in \mathcal{G}} \{\phi_1 p_\epsilon - \phi_2 p_0\} = \mathcal{O}(\epsilon).$$

Equivalently, there is a family of periodic orbits $\phi\gamma_\epsilon(x) = (\phi p_\epsilon, \omega x)$ of the perturbed system (6) near $\phi\gamma_0(x)$ for $\phi \in \mathcal{G}$.

Lemma 2. *For $\epsilon > 0$ sufficiently small, we have $\sigma\left[DP^\epsilon(p_\epsilon)\right] = \left\{1, 1, e^{T\lambda_\epsilon^\pm}\right\}$ for the case **(H4b i)** and $\sigma\left[DP^\epsilon(p_\epsilon)\right] = \left\{1, \lambda^d, e^{T\lambda_\epsilon^\pm}\right\}$ for the case **(H4b ii)** respectively.*

Conjecture 3. Corresponding to the eigenvalues $e^{T\lambda_\epsilon^\pm}$, 1 there exist invariant manifolds: $W^{ss}(\gamma_\epsilon(x))$ (the strong stable manifold), $W^u(\gamma_\epsilon(x))$ (the unstable manifold), and $W^c(\gamma_\epsilon(x))$ (the centre manifold) of p_ϵ for the Poincaré map $P^\epsilon(Z)$ such that

i) $W_{\text{loc}}^u(\gamma_\epsilon(x))$ and $W_{\text{loc}}^{ss}(\gamma_\epsilon(x))$ are tangent to the eigenspaces of $e^{T\lambda_\epsilon^\pm}$ respectively at γ_ϵ, while $W_{\text{loc}}^c(\gamma_\epsilon(x))$ is at the same point tangent to the eigenspace corresponding to unity eigenvalue (double in the case of Hamiltonian perturbations). Their global analogues are obtained in the usual way:

$$W^{ss}(\gamma_\epsilon(x)) = \bigcup_{x \leq 0} \Phi_x^\epsilon W_{\text{loc}}^{ss}(\gamma_\epsilon(x)), \qquad W^u(\gamma_\epsilon(x)) = \bigcup_{x \geq 0} \Phi_x^\epsilon W_{\text{loc}}^u(\gamma_\epsilon(x)); \tag{13}$$

ii) they are invariant under $P^\epsilon(\cdot)$;

iii) $W^{ss}(\gamma_\epsilon(x))$ and $W^u(\gamma_\epsilon(x))$ are C^r $\mathcal{O}(\epsilon)$-close to $W^s(\gamma_0(x))$ and $W^u(\gamma_0(x))$ respectively.

Lemma 4. *Let $(Z_\epsilon^{s,u}(x,x_0),\omega x)$ be orbits lying in $W^{ss,u}(\gamma_\epsilon(x))$ and originating in an $\mathcal{O}(\epsilon)$-neighbourhood of $(Z_0(-x_0),0)$. Then the following expressions hold with uniform validity in the indicated time intervals:*

$$\begin{aligned} Z_\epsilon^s(x,x_0) &= Z_0(x-x_0)+\epsilon y_\epsilon^s(x,x_0)+\mathcal{O}\left(\epsilon^2\right), \qquad x\in[x_0,\infty), \\ Z_\epsilon^u(x,x_0) &= Z_0(x-x_0)+\epsilon y_\epsilon^u(x,x_0)+\mathcal{O}\left(\epsilon^2\right), \qquad x\in(-\infty,x_0], \end{aligned} \tag{14}$$

where $y_\epsilon^{s,u}$ satisfy the first variational equation:

$$\frac{dy}{dx}=\mathbf{J}_c^{-1}D^2H_0(Z_0(x-x_0))y+\epsilon f_1(Z_0(x-x_0),\omega x). \tag{15}$$

We introduce Melnikov function as:

$$\begin{aligned} M(x_0) &= \int_{-\infty}^{\infty} DH_0(Z_0(x))\cdot f_1(Z_0(x),x+x_0)dx \\ &= \int_{-\infty}^{\infty} \Omega\left(f_0(Z_0(x)),f_1(Z_0(x),x+x_0\right)dx. \end{aligned} \tag{16}$$

Theorem 1. *Suppose $M(x_0)$ has simple zeros. Then for $\epsilon>0$ sufficiently small $W^{ss}(\gamma_\epsilon)$ and $W^u(\gamma_\epsilon)$ intersect transversely.*

This result implies via the Smale–Birkhoff theorem [6] the appearance of a horseshoe near the saddle-centre point of the perturbed Poincaré map, what results in a chaotic dynamics in the corresponding region of the phase space.

4 Example

We consider the perturbed Korteweg-de Vries equation [1, 2]:

$$u_t+\Delta u_x+\alpha u u_x+u_{xxx}=\epsilon f_x(u,u_x,u_t,x-ct), \tag{17}$$

where the perturbation is assumed to be T-periodic in its last argument. The unperturbed equation can be rewritten in a multi-symplectic form as

$$\mathbf{M}Z_t+\mathbf{K}Z_x=\nabla S, \qquad Z=\begin{pmatrix} u\\ v\\ w\\ \Phi\end{pmatrix}\in\mathbb{R}^4, \qquad x\in\mathbb{R} \tag{18}$$

with

$$\mathbf{M}=\begin{pmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\end{pmatrix}, \qquad \mathbf{K}=\begin{pmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&-1\\ 0&0&1&0\end{pmatrix}, \tag{19}$$

and

$$S=\frac{1}{2}v^2-\frac{1}{2}uw+u\left(\frac{1}{2}\Delta u+\frac{\alpha}{6}u^2\right). \tag{20}$$

The unperturbed solution is defined as:

$$\begin{aligned}
u_0(x) &= 2b + \frac{3}{\alpha}K^2\mathrm{sech}^2\left(\frac{Kx}{2}\right),\\
v_0(x) &= -\frac{3}{\alpha}K^3\sinh\left(\frac{Kx}{2}\right)\mathrm{sech}^3\left(\frac{Kx}{2}\right),\\
w_0(x) &= 2a + 4b(\Delta+\alpha b) + \frac{3}{\alpha}cK^2\mathrm{sech}^2\left(\frac{Kx}{2}\right),\\
\Phi_0(x) &= \frac{3}{\alpha}K\left[1+\tanh\left(\frac{Kx}{2}\right)\right],
\end{aligned} \tag{21}$$

with $K = \sqrt{c-\Delta-2\alpha b}$. Perturbation can be represented in a multi-symplectic setting as:

$$\epsilon f_1(Z,x) = \begin{pmatrix} 0 \\ \epsilon\tilde{f}(Z,x) \\ 0 \\ 0 \end{pmatrix}. \tag{22}$$

Here we expressed the perturbation $\tilde{f}(Z,x) = f(u,v,-cv,x)$ in terms of the components of Z. Therefore Melnikov function (16) for the system (17) yields

$$M(x_0) = \int_{-\infty}^{\infty} v_0(x)\tilde{f}(Z_0(x), x+x_0)dx. \tag{23}$$

For the one-harmonic dissipative driving force this Melnikov function will have simple zeros [2], and therefore one can conclude the chaotic dynamics of u.

5 Conclusions

We have considered a modification of Melnikov's method, which can be used for the analysis of chaotic behaviour of travelling wave solutions to multi-symplectic PDEs. The results are illustrated with the example of the perturbed KdV equation.

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