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Asymptotics of the Coupled Solutions of the Modified Kadomtsev–Petviashvili Equation

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We determine a subset of $\mathbb{R}^2$ and a measure on this set which allow to construct coupled non-localized solutions $u^+(x,y,t)$ and $u^-(x,y,t)$ of the modified KP-I equation, which are connected by the relation $u^-(x,y,t) = -u^+(-x,y,-t)$, and split into asymptotic solitons as $t \to \infty$ in the neighbourhood of the leading edge of the solutions. The solitons corresponding to each of the solutions have different amplitudes and lines of constant phase, and are not connected by the above relation.

1 Introduction

In 1974 V.E. Zakharov and A.B. Shabat [1] proposed a very effective scheme of the inverse scattering method for the integration of nonlinear evolution equations with one and two spatial variables, so called *dressing method*. It turns out very convenient to obtain wide classes of solutions avoiding the difficult stage of the solving of the inverse scattering problem for the corresponding differential operator.

We consider the modified Kadomtsev–Petviashvili (mKP) equation [2]

$$u_t + \frac{1}{4}u_{xxx} - \frac{3}{2}\alpha^2\left(u^2u_x - \frac{1}{2\alpha}\hat{D}^{\pm}[u_{yy}] + u_x\hat{D}^{\pm}[u_y]\right) = 0, \tag{1}$$

with $u = u(x,y,t)$, $\alpha = i$ for the mKP-I equation, $\alpha = 1$ for the mKP-II equation, and operators $\hat{D}^{\pm}$ choosen as $\hat{D}^-[u(x,y,t)] = \int_{-\infty}^{x} u(s,y,t)ds$ or $\hat{D}^+[u(x,y,t)] = -\int_{x}^{\infty} u(s,y,t)ds$ simultaneously in both of the summands. These equations plays an important role in the understanding of the various properties of the Kadomtsev–Petviashvili equation and generalized Miura transformation. In both cases ($\alpha^2 = \pm 1$) (1) has exact solutions in the form of plane solitons

$$u(x,y,t) = \pm\frac{2q^2}{p + \sqrt{p^2+q^2}\cosh\left[2q(x - 2py - (q^2-3p^2)t - \beta)\right]},$$

($\beta = \text{const}$, $(p,q) \in \mathbb{R}^+$ – upper half-plane of $\mathbb{R}$) the lines of constant phase of which $x = 2py + \left(q^2 - 3p^2\right)t + \beta$ are straight lines in (x,y)-plane.

In 1986, V.E. Zakharov constructed some solutions of the KP-II equation that were interpreted as curved solitons [3]. In 1994, using the dressing method, there were constructed KP-II non-localized solutions vanishing as $x \to \infty$, which split in the neighbourhood of the leading edge into infinite series of curved solitons for $t \to \infty$ [4]. These solitons are represented in the form of plane solitons, but their lines of constant phase are curves in the (x,y)-plane, and the depend on the parameter $Y = y/t$. Analogous solutions of the KP-I, mKP-I and Johnson equation (also called cylindrical KP) were constructed in [5, 6, 7], and their long-time asymptotic behaviour was investigated.

In this note we use the scheme of the dressing method for the mKP equation introduced in [5], and the fact that each constructed solution $u^+(x,y,t)$ of the mKP-I with operator $\hat{D}^+$ generates another solution $u^-(x,y,t) = -u^+(-x,y,-t)$ of the mKP-I with operator $\hat{D}^-$. So each solution $u^+(x,y,t)$ vanishing as $x \to +\infty$ [5, 7] generates $u^-(x,y,t)$, which vanishes as $x \to -\infty$. Such pair of solutions we call *coupled solutions*. However, direct application of the change of variables to the asymptotic formulae is not correct, and the investigation of the asymptotics of the new constructed solution requires special consideration.

We determine some subset in $\mathbb{R}^2$ and a measure on this set which allow to construct real coupled solutions of the mKP-1 equation, which split into asymptotic solitons as $t \to \infty$ in the neighbourhood of the leading edge of the solutions. The asymptotic solitons corresponding to each of both solutions have a different form and are not connected by the above transformation.

2 Construction of the mKP-I equation solution

According to the scheme of the dressing method [5] the mKP solution can be represented as follows:

$$u(x,y,t) = \frac{1}{\alpha}\frac{d}{dx}\left(1 \mp \hat{D}^{\pm}[K(x,s,y,t)]\right), \tag{2}$$

where $K(x,z,y,t)$ is a solution of Marchenko integral equation

$$K(x,z,y,t) + F(x,z,y,t) \mp \hat{D}^{\pm}[K(x,s,y,t)F(s,z,y,t)] = 0, \tag{3}$$

and $\hat{D}^{\pm}$ are operators with respect to the argument s. The kernel $F(x,z,y,t)$ of (3) satisfies the system of linear differential equations

$$\begin{aligned} &F_t + F_{xxx} + F_{zzz} = 0, \\ &\alpha F_y + F_{xx} - F_{zz} = 0. \end{aligned} \tag{4}$$

We start from the solution $u^+(x,y,t)$ corresponding to the operator $\hat{D}^+$ in (1)–(3). A wide class of solutions of (4) for $\alpha = i$ in this case can be found by the Fourier method in the form:

$$F(x,z,y,t) = \iint_{\Omega} \exp\left[ip(x-z) - q(x+z) + 4pqy + 2q\left(q^2 - 3p^2\right)t\right] d\mu(p,q), \tag{5}$$

where $\Omega \subset \mathbb{R}^2$ and $d\mu(p,q)$ is some measure on Ω.

To construct a mKP-I solution by the scheme (2)–(5) we must define the set Ω in (5) and the measure $d\mu(p,q)$ over this set. For this goal we introduce the functions $C^{\pm}(s)$ and $g(s)$ which play an important role in the construction of the solution and in the investigation of its asymptotic behaviour. For the sake of simplicity we restrict ourselves by a special choice of these functions in this note.

Let $b = \text{const} > 0$. The functions $C^+(s)\colon \mathbb{R} \to \mathbb{R}^+$ and $C^-(s)\colon \mathbb{R} \to \mathbb{R}^+$ are defined by

$$C^+(s) = s^2 + b^2, \qquad C^-(s) = (|s| + b)^2.$$

We denote

$$f^{\pm}(p,q,s) = 2ps \pm \left(q^2 - 3p^2\right).$$

The curve $q = h^+(p) = \sqrt{2p^2 + b^2}$ is envelope of the family of hyperbolas

$$E^+(p,q;s) = \left\{(p,q) \in \mathbb{R}^2 \mid f^+(p,q,s) = C^+(s)\right\}_{s\in\mathbb{R}}$$

with a contact at $(p_0^+(s), q_0^+(s))$ defined by

$$p_0^+(s) = \frac{C_s^+(s)}{2} = s,$$
$$q_0^+(s) = \sqrt{C^+(s) + (3/4)(C_s^+(s))^2 - sC^+(s)} = \sqrt{2s^2 + b^2}. \tag{6}$$

The curve $q = h^-(p) = \sqrt{4p^2 - 2|p|\,b}$ $(|p| \geq b/2)$ is envelope of the family of hyperbolas

$$E^-(p, q; s) = \left\{(p, q) \in \mathbb{R}^2 \mid f^-(p, q, s) = C^-(s)\right\}_{s\in\mathbb{R}},$$

with a contact at $(p_0^-(s), q_0^-(s))$ defined by

$$p_0^-(s) = \frac{C_s^+(s)}{2} = \operatorname{sign} s(|s| + b),$$
$$q_0^-(s) = \sqrt{\frac{3}{4}(C_s^-(s))^2 + sC^-(s) - C^-(s)} = \sqrt{2(|s| + b)(2|s| + b)}. \tag{7}$$

We consider a subset $\Omega \subset \mathbb{R}^2$ of the form

$$\Omega = \left\{(p, q) \in \mathbb{R}^2 \mid -\infty < p < \infty,\ q \in Q\right\}, \tag{8}$$

where

$$Q = \left\{q \mid q \geq \delta > 0\right\} \cap \left\{q \mid q \geq h^-(p)\right\} \cap \left\{q \mid q \leq h^+(p)\right\}$$

and $\delta < b$. It is easy to show that this choice of Ω implies

$$C^+(s) = \max_{(p,q)\in\Omega} f^+(p, q, s), \qquad C^-(s) = \max_{(p,q)\in\Omega} f^-(p, q, s). \tag{9}$$

Moreover, the maximum value of $f^\pm(p, q, s)$ is attained at the unique point $(p_0^\pm(s), q_0^\pm(s))$ respectively.

About the function $g(s)$ and the measure $d\mu$ we assume that

$$g(s)\colon \mathbb{R} \to \mathbb{R}^+ \quad \text{is} \quad \mathcal{C}^\infty, \qquad \tilde{g}\colon \Omega \to \mathbb{R}^+ \quad \text{is} \quad \mathcal{C}^\infty,$$
$$\tilde{g}(p_0(\kappa), q_0(\kappa)) = g(\kappa), \qquad d\mu(p, q) = \sqrt{\frac{p - iq}{p + iq}}\,\tilde{g}(p, q)dpdq. \tag{10}$$

Lemma 1. *Assume that Ω has the form (8) and $d\mu$ satisfies (9). Then the scheme (2)–(5) determines smooth real coupled solutions of the mKP-I equation vanishing as $x \to \pm\infty$ respectively, and bounded for all fixed x, y, t.*

Proof. The proof is based on the fact that the function $F(x, z, y, t)$ (5) generates a self-adjoint positive compact operator $\hat{F}[z]\colon L^2([x, \infty)) \to L^2([x, \infty))$. Then, by Fredholm theory [8] (4) has a unique solution $K(x, z, y, t)$, which is $\mathcal{C}^\infty$ with respect to all variables. Moreover, the self-adjointness of F and the special choice of the measure $d\mu$ (10) leads to the reality of $u(x, y, t)$. Thus the solution of the mKP-I equation constructed by (2) has all properties of the Lemma. We denote this solution $u^+(x, y, t)$.

This solution $u^+(x, y, t)$ of the mKP-I equation generates a new solution $u^-(x, y, t)$ by the change $u^-(x, y, t) = -u^+(-x, y, -t)$. This solution has the same properties as $u^+(x, y, t)$, boundedness, smoothness and reality, but it vanishes as $x \to -\infty$. ■

In Section 3 we describe the long time asymptotic behaviour of these solutions.

3 Theorem about soliton asymptotics of $u^{\pm}(x, y, t)$

The asymptotic behaviour of $u^{\pm}(x, y, t)$ for $t \to \infty$ is investigated in the following domains of the leading edge of the solutions ($M > 2$):

$$G^{\pm}(M) = \left\{ (x, y, t) \in \mathbb{R}^3 \,\middle|\, t > t_0(M),\, Y = \frac{y}{t} \in I^{\pm},\, x \gtrless C^{\pm}(Y)t \mp \frac{M+1}{2q_0^{\pm}(Y)} \ln t \right\}, \tag{11}$$

where $I^+ = \left[-\frac{1+\sqrt{3}}{2}b + \varepsilon,\, \frac{1+\sqrt{3}}{2}b - \varepsilon\right]$, $I^- = \left[-\frac{\sqrt{3}-1}{2}b + \varepsilon,\, -\varepsilon\right] \bigcup \left[\varepsilon,\, \frac{\sqrt{3}-1}{2}b - \varepsilon\right]$, $\varepsilon > 0$, and $t_0(M)$ is large enough.

It is described by the following theorem.

Theorem 1. *The solution $u^{\pm}(x, y, t)$ of the mKP-I equation constructed in Lemma 1 have the following asymptotics in the domains $G_M^{\pm}(t)$ as $t \to \infty$:*

$$u^{\pm}(x, y, t) = \mp \sum_{n=1}^{[M-1]} u_n^{\pm}(x, y, t) + O\left(\frac{1}{t^{1/2-\varepsilon_1}}\right), \qquad (0 < \varepsilon_1 < 1/2), \tag{12}$$

$$u_n^{\pm}(x, y, t) = \frac{2q_0^{\pm}(Y)^2}{p_0^{\pm}(Y) + \sqrt{(p_0^{\pm}(Y))^2 + (q_0^{\pm}(Y))^2} \cosh\left[2q_0^{\pm}(Y)\kappa_n^{\pm}(x, y, t)\right]}, \tag{13}$$

where

$$\kappa_n(x, y, t) = x \mp C^{\pm}(Y)t \pm \frac{1}{2q_0^{\pm}(Y)} \left(\ln t^{n+1/2} - \ln g(Y)\phi_n^{\pm}(Y)\right),$$

$$\phi_n^{\pm}(Y) = \frac{(C_{YY}^{\pm}(Y))^{n-1/2}((q_0^{\pm}(Y))^2 + (3p_0^{\pm}(Y) - Y)^2)^{n-1}\Omega^{(n)}I^{(n)}}{2^{4n+1}(q_0^{\pm}(Y))^{5n-3/2}[(n-1)!]^2\Omega^{(n-1)}I^{(n-1)}},$$

$p_0^{\pm}(Y)$, $q_0^{\pm}(Y)$ are defined in (6), (7), $C_{YY}^{\pm} = \frac{d^2C^{\pm}(Y)}{dY^2}$, and $\Gamma^{(n)}$, $Q^{(n)} > 0$ are determinants of $n \times n$ matrices with entries ($0 \le i, k \le n-1$)

$$\Gamma_{i+1,k+1}^{(n)} = \Gamma\left(\frac{i+k+1}{2}\right)\left(1 + (-1)^{i+k}\right), \qquad Q_{i+1,k+1}^{(n)} = \Gamma(i+k+1).$$

Here the asymptotic representation (12) is uniform with respect to x and y in $G_M^{\pm}(t)$ for any fixed $M > 2$.

Proof. The statement of the theorem concerning the solution $u^+(x, y, t)$ was proved in [5, 7]. We present here a scheme of the proof for the case of $u^-(x, y, t)$. First of all we apply the transformation $(x, t) \mapsto (-x, -t)$ to (2)–(5) with Ω (8) and $d\mu$ (10), and obtain the corresponding formulas for the solution $u^-(x, y, t)$. (3) is transformed into the equation containing integration from $-\infty$ to x with the kernel

$$F^-(x, z, y, t) = \iint_{\Omega} \exp\left[-ip(x-z) + q(x+z) + 4pqy + 2q\left(3p^2 - q^2\right)t\right] d\mu(p, q). \tag{14}$$

On the next stage we study the asymptotics of (14) as $t \to \infty$. After change of variables $x = -C^-(Y)t - \xi$, $z = -C^-(Y)t - \zeta$ (14) acquires the form:

$$\begin{aligned}\tilde{F}^-(\xi, \zeta, y, t) &= F^-(C^-(Y)t + \xi, C^-(Y)t + \zeta, y, t) \\ &= \iint_{\Omega} \exp\left[-ip(\xi - \zeta) + q(\xi + \zeta) - 2q(C^-(Y) - f^-(p, q, Y))t\right] d\mu(p, q).\end{aligned} \tag{15}$$

Using property (9), we apply Laplace method to (15) and prove that

$$\tilde{F}^{-}(\xi,\zeta,y,t) = F_N(\xi,\zeta,y,t) + \tilde{G}(\xi,\zeta,y,t),$$

in the domains

$$\zeta > \xi < -\frac{1}{2q_0^-(Y)} \ln t^M, \qquad Y = \frac{y}{t}, \qquad t \to \infty, \tag{16}$$

where $F_N(\xi,\zeta,y,t)$ is a degenerate kernel ($N = [2M-3]$, $M > 2$ is an arbitrary integer), and $\|\tilde{G}(\xi,\zeta,y,t)\|_{L_2([x,\infty))} = O\left(1/t^{\varepsilon_1}\right)$, $0 < \varepsilon_1 < 1/2$.

On the third stage we prove that the degenerate kernel $F_N(\xi,\zeta,y,t)$ brings the main contribution into the asymptotics of Marchenko's equation in (16). After the solving of the equation and analysis of the corresponding determinant formulae we obtain the statement of the theorem. ■

Thus we have constructed non-localized coupled solutions $u^{\pm}(x,y,t)$ of the mKP-I equation, which split into infinite series of the curved asymptotic solitons in the domains of the leading edge of the solutions as $t \to \infty$. These asymptotic solitons are generated by the neighbourhoods of the curves $q = h^{\pm}(p)$ respectively. Both of them have the varying amplitude, width, and are diverged as t increases, but they are not connected by the transformation $u^+(x,y,t) \mapsto u^-(x,y,t)$.

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