On algebras generated by linearly dependent generators with certain spectra, their representations and applications

Mariya Vlasenko, Anton Mellit, Yurii Samoilenko

Let $\{X_k\}_{k=1}^n$ be a set of linear operators in separable complex Hilbert space $\mathbb{H}$ with scalar sum $\sum_{k=1}^n X_k = \lambda \mathbb{I}_{\mathbb{H}}$ and a restriction that spectrum of each X_k belongs to a certain finite set $M_k \subset \mathbb{C}$. Such sets of operators play an important role in analysis, algebraic geometry, operator theory and mathematical physics ([1]–[4] and bibliography there). We try an algebraic approach to this situation considering a class of algebras established by generators and relations as

$$\mathcal{P}_{p_1,p_2,\dots,p_n} = \mathbb{C}\langle x_1, x_2, \dots, x_n | x_1 + x_2 + \dots + x_n = 0, p_i(x_i) = 0\rangle,$$

where p_i are certain polynomials with leading coefficient one. With every such algebra we associate a graph $G = G(\mathcal{P}_{p_1,p_2,\dots,p_n})$ which consists of one root vertex and n branches, i-th branch is a sequence of $\deg p_i - 1$ connected vertices. Then:

1. Algebras $\mathcal{P}_{p_1,p_2,\dots,p_n}$ are finite dimensional iff graph G is a Dynkin diagram (A_n, D_n, E_6, E_7 or E_8).
2. The growth of the algebra is polynomial iff corresponding graph G is extended Dynkin diagram ($\widetilde{D_4}$, $\widetilde{E_6}$, $\widetilde{E_7}$ or $\widetilde{E_8}$).
3. Under some conditions on polynomials $p_1, p_2, \dots, p_n$ for extended Dynkin diagrams corresponding algebras are PI-algebras (i.e. having polynomial identities).
4. If graph G is neither Dynkin diagram nor extended Dynkin diagram then corresponding algebra contains a free subalgebra.

For algebras generated by projections sums of which is a multiply of the identity this results see [5].

We also consider representations of our algebras and discuss if there exist more deep relation of these subject with famous H. Weyl problem, Coxeter groups and theory of singularities, physical applications.

References

[1] Fulton W. Eigenvalues, invariant factors, highest weights, and Schubert calculus // Bull. of the AMS. — 2000. — **37**, N 3. — P. 209–249.

[2] Klyachko A.A. Stable bundles, representation theory and Hermitian operators // Selecta Math. — 1998. — **4**. — P. 419–445.

[3] Jones V. Hecke algebra representations of braid groups and link polynomials // Annal. Math. — 1987. — **126**. — P. 335–388.

[4] Ewans D., Kawahigashi Y. Quantum symmetries on operator algebras. Oxford Univ. Press, New York, 1998.

[5] Rabanovich V.I., Samoilenko Yu.S., Strelets A.V. About identities in algebras $\mathbf{Q}_{\mathbf{n},\lambda}$ generated by idempotents. // Ukr. Math. Journn. — 2001. — **53**, N 10. — P. 1380–1390.