Modified hypergeometric equations arising from the Markoff theory

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Abstract

After recalling what is the Markoff theory, the article summarizes some links which exist with the group $GL(2,\mathbb{Z})$ of 2×2 matrices with integer coefficients and determinant ± 1 and with its subgroups $SL(2,\mathbb{Z})$ and the triangle group $\mathbf{T}_3$. Then we visit rapidly the links with conformal punctured toruses. The main part of the article is about the monodromy representation of the Poincaré group of such a torus. We gives the corresponding solution of the associated Riemann-Hilbert problem and the corresponding differential operator whose spectral analysis remains to be done. We conclude quoting the 22^{th} Hilbert's problem and some information about the accessory parameter problem.

1 Introduction

For a real quadratic form $f(x,y)=ax^2+bxy+cy^2\in\mathbb{R}[x,y]$, the problem to know the minimal value of $\mid f(x,y)\mid$ when x and y are non zero integers is classical. When $f(x,y)$ is a definite form i.e. $\Delta(f)=b^2-4ac<0$, the problem was solved by J. L. Lagrange and then C. Hermite [25] :

$$C(f)=\frac{\inf_{(x,y)\in\mathbb{Z}^2-\{(0,0)\}}\mid f(x,y)\mid}{\sqrt{\mid\Delta(f)\mid}}\leq\frac{1}{\sqrt{3}}=C(x^2+xy+y^2).$$

It has been shown ([5] p.33) that for any $\rho\in]0,(1/\sqrt{3})]$, we can find a quadratic form $f(x,y)\in\mathbb{R}[x,y]$ verifying

$$\rho=C(f).$$

When $f(x, y)$ is a indefinite form i.e. $\Delta(f) = b^2 - 4ac > 0$, A. Korkine and G. Zolotareff [30] demonstrated :

$$C(f) \leq \frac{1}{\sqrt{5}} = C(x^2 - xy - y^2) = C(f_0),$$

an isolated value giving also for any other form f not $GL(2, \mathbb{Z})-$equivalent to f_0 :

$$C(f) \leq \frac{1}{\sqrt{8}} = C(x^2 - 2y^2) = C(f_1).$$

Trying to understand this phenomenon A. A. Markoff built its own theory [34]. He described an infinity of values $C(f_i)_{i\in\mathbb{N}}$ situated between $(1/\sqrt{5})$ and $(1/3)$ and having the same properties as $C(f_0)$. These values are isolated and convergent towards $(1/3)$. They can be built thanks to the tree of solutions of the diophantine equation, so called Markoff equation [13] :

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2.$$

For values $C(f)$ less than $(1/3)$ the author has shown than more general diophantine equations give an insight, sometimes with theories similar the the Markoff one [40] but with some complication. Moreover a geometrical interpretation of such results has been found, similar to what was done by H. Cohn [10] for the classical Markoff theory. The general situation can be understood by the Teichmüller theory on the topological punctured torus $\mathcal{T}^\bullet$ (see for example [27] [50]). This topological object is a quite a frequent one in physical problems, for example linked to the KAM theorem [2], and some work has been made after the observation that the Markoff theory could be useful to understand the behavior of some oscillators [41]. It was possible to realize that two types of geometric punctured toruses exist, we called them the hyperbolic and parabolic. The Markoff theory is then linked to the parabolic case, and geometrically to special fuchsian groups, the Fricke groups Γ as defined in [46] [48] :

(1): Γ is isomorphic to a free group with two generators $\mathbf{F}_2 = \mathbb{Z} * \mathbb{Z}$.

(2): The Riemann surface $\mathcal{H}/\Gamma$ (where $\mathcal{H}$ is the Poincaré half-plane) is homeomorphic to a punctured torus.

The closed geodesics on such a Riemann surfaces are linked to indefinite quadratic forms f and to the associated Markoff constant $C(f)$ which can be seen as shorter length of such geodesics [54]. The Markoff theory gives an explanation [49] to the quantification which appears when changing from such a geodesics to another : no continuous deformation is possible on the torus because of the puncture. Such remarks are the basis for the interest of physicians to this subject [24]. In another direction it is known that the Markoff theory has links with the study of exceptional bundles and helices of the projective plane $P_2(\mathbb{C})$ (see [47], [37], [38], [20], [21], [16]), and also with the spectrum of hermitian operators [29] and this is also important for physics [51]. A project that we made a long time ago was to build a common interpretation of such remarks in order to get the set of all Markoff constants of indefinite quadratic forms, the Markoff spectrum, as the spectrum of some operator on a Hilbert space. The reason for it is that the Markoff spectrum seems like the spectrum of some operators. It has a discrete part from $(1/\sqrt{5})$ to $(1/3)$, then a cantorian part from $(1./3)$ to the Freiman number ß which is :

$$ß^{-1} = 4 + \frac{253589820 + 283748\sqrt{462}}{491993569}.$$

From ß to 0 the spectrum is continuous, any real number is a Markoff constant [13]. The present article gives hints towards the possibility to implement such a project, building a possible operator to consider.

Looking at the link with fuchsian groups that we mentioned, the Markoff theory can be described [39] thanks to the two following matrices generating in $SL(2,\mathbb{Z})$ a free group isomorphic to $\mathbf{F}_2$ which is $[SL(2,\mathbb{Z}), SL(2,\mathbb{Z})]$:

$$A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

These matrices give birth to a representation $\rho : Aut(\mathbf{F}_2) \longrightarrow GL(2,\mathbb{Z})$ and we have described in [39] the algebraic importance of this situation. In the present article, the main goal is to give a differential equation whose former representation ρ is the monodromy representation. We hope by such a construction to understand the Lamé equations appearing for the accessory parameters of punctured toruses [28]. Such equations have similarities with hypergeometric equations and also with some Schrödinger equations whose monodromy group has recently been studied [57]. Also we could build soon an hamiltonian interpretation in the spirit of L. D. Fadeev [17] and others giving the way to realize our quoted project.

2 Considering the triangle group :

The Markoff equation gives a complete tree of integer solutions thanks to the solution $(1,1,1)$ and the three transformations

$$X : (m, m_1, m_2) \longmapsto ((3m_1m_2 - m, m_1, m_2),$$
$$Y : (m, m_1, m_2) \longmapsto (m, 3mm_2 - m_1, m_2),$$
$$Z : (m, m_1, m_2) \longmapsto (m, m_1, 3mm_1 - m_2),$$
$$X^2 = Y^2 = Z^2 = Id.$$

The involutions X, Y et Z, give birth to the triangle group $\mathbf{T}_3$ which is the free product of three cyclic groups $\mathbf{C}_2$ with two elements :

$$\mathbf{T}_3 = \mathbf{C}_2 * \mathbf{C}_2 * \mathbf{C}_2.$$

In [39] we showed how this group $\mathbf{T}_3$ is linked to the group of 2×2 matrices $GL(2,\mathbb{Z})$. We used for this an abelianisation morphism π' from the automorphism group $Aut(\mathbf{F}_2)$ to $GL(2,\mathbb{Z})$, and two matrices generating the dihedral group $\mathbf{D}_6$ with 12 elements inside $GL(2,\mathbb{Z})$:

$$\pi'(t) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \ \pi'(o) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

We defined also :

$$\pi'(X) = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}, \ \pi'(Y) = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \ \pi'(Z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The group $\mathbf{T}_3$ acts in $GL(2,\mathbb{Z})$ defining with $ch = ch(X,Y,Z) \in \mathbf{T}_3$:

$$ch(\pi'(X), \pi'(Y), \pi'(Z)) = \pi'(ch(X,Y,Z)) \in \pi'(\mathbf{T}_3).$$

It gives a ternary decomposition in $GL(2,\mathbb{Z})$ using the triangle group :

Proposition 2.1. *Every element $V \in GL(2,\mathbb{Z})$ has a unique decomposition*

$$\pi'(o)^h \pi'(t)^k ch(\pi'(X), \pi'(Y), \pi'(Z)),$$
$$\textit{where } h = 0,1; \quad k = 0,1,...,5; \quad ch \in \mathbf{T}_3.$$

The elements of $\pi'(\mathbf{T}_3)$ are characterized by the conditions $h = 0$ et $k = 0$. The group $\pi'(\mathbf{T}_3)$ is not normal inside $GL(2,\mathbb{Z})$. It is isomorphic by π' to the group $\mathbf{T}_3$. The elements of the group $\mathbf{D}_6$ which is not normal in $GL(2,\mathbb{Z})$ are characterized by the condition

$$ch(\pi'(X), \pi'(Y), \pi'(Z)) = \mathbf{1}_2.$$

The group $\mathbf{D}_6$ introduces two equivalence relations in $GL(2,\mathbb{Z})$

$$V_1 \ \Re_{\mathbf{D}_6}\ V_2 \quad \Leftrightarrow \quad V_1V_2^{-1} \in \mathbf{D}_6 \quad \Leftrightarrow \quad V_2 \in \mathbf{D}_6 V_1,$$

$$V_1 \ {}_{\mathbf{D}_6}\Re\ V_2 \quad \Leftrightarrow \quad V_1^{-1}V_2 \in \mathbf{D}_6 \quad \Leftrightarrow \quad V_2 \in V_1\mathbf{D}_6.$$

The quotients $GL(2,\mathbb{Z})/\Re_{\mathbf{D}_6}$ and $GL(2,\mathbb{Z})/{}_{\mathbf{D}_6}\Re$ are equipotent, but different because $\mathbf{D}_6$ is not a normal subgroup of $GL(2,\mathbb{Z})$. Each $V \in GL(2,\mathbb{Z})$ defines a unique $ch(\pi'(X),\pi'(Y),\pi'(Z)) \in \pi'(\mathbf{T}_3)$ such that

$$V \ \Re_{\mathbf{D}_6}\ ch(\pi'(X),\pi'(Y),\pi'(Z)).$$

Hence we get a description of the complete tree of the Markoff theory :

Proposition 2.2. *The group $\mathbf{T}_3$ is equipotent to the quotient (right or left) of the group $GL(2,\mathbb{Z})$ by its non-normal subgroup $\mathbf{D}_6$. It is an homogeneous $GL(2,\mathbb{Z})$-space. But also it can be considered as a subgroup of $GL(2,\mathbb{Z})$ thanks to the former proposition.*

An easy consequence for the K-theory ([45] (p. 218 and p. 75), [55] (p. 261)) is :

Proposition 2.3. *We have :*

$$H_1(GL(2,\mathbb{Z}),\mathbb{Z}) = GL(2,\mathbb{Z})/[GL(2,\mathbb{Z}),GL(2,\mathbb{Z})] \simeq \mathbf{D}_6/[\mathbf{D}_6,\mathbf{D}_6] \simeq \mathbf{C}_2\times\mathbf{C}_2,$$

$$H_2(GL(2,\mathbb{Z}),\mathbb{Z}) \simeq \mathbf{C}_2.$$

We find a decomposition using the free group $\mathbf{F}_2 \simeq [SL(2,\mathbb{Z}),SL(2,\mathbb{Z})]$:

Proposition 2.4. *Any element $V \in GL(2,\mathbb{Z})$ has a unique decomposition*

$$\pm W(A_0,B_0)O^hW_k(S,T),$$

$$W(A_0,B_0) \in \mathbf{F}_2 \simeq [SL(2,\mathbb{Z}),SL(2,\mathbb{Z})],$$
$$h \in \{0,1\},$$
$$W_k(S,T) \in \{\mathbf{1}_2, S, ST, STS, STST, STSTS\} \text{ with } k = 0,1,...,5.$$

The elements of the normal subgroup $SL(2,\mathbb{Z})$ in $GL(2,\mathbb{Z})$ are characterized by the condition $h = 0$.

For the last proposition we defined in $GL(2,\mathbb{Z})$:

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad O = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

and all the words $W(A_0, B_0)$ are written in a multiplicative way with the two generators of $\mathbf{F}_2$ thanks to [33] (p. 97-98) :

$$A_0 = [(TS)^{-1}, S^{-1}] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \; B_0 = [S^{-1}, (TS)^{-2}] = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

In fact we have a presentation with two generators T and $I = OS$ (see [3]) :

$$GL(2,\mathbb{Z}) =< I, T^{-1} \mid I^2 = ([T^{-1}, I]T^{-1})^4 = ([T^{-1}, I]T^{-1}I)^2 = \mathbf{1}_2 > .$$

The subgroup $\pi'(\mathbf{T}_3)$ is generated by :

$$\pi'(X_0) = T^{-1}IOT^{-1}IOIT^{-1}B_0^{-1}, \;\; \pi'(Y_0) = IOIOA_0^{-1}TS, \;\; \pi'(Z_0) = IS.$$

Moreover [3] the triangle group $\mathbf{T}_3 \simeq \pi'(\mathbf{T}_3)$ is isomorphic to the projective

$$PGL(2,\mathbb{Z}) =< \overline{I}, \overline{T}^{-1} \mid \overline{I}^2 = ([\overline{T}^{-1}, \overline{I}]\overline{T}^{-1})^2 = ([\overline{T}^{-1}, \overline{I}]\overline{T}^{-1}\overline{I})^2 = \mathbf{1} > .$$

We can verify that $\mathbf{F}_2 \simeq [PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})]$ has an index 2 in this last group where we have, with $\mathbf{C}_3$ the cyclic group containing three elements, and $\overline{V_1} = [\overline{I}, \overline{T}^{-1}]$ and $\overline{V_2} = [\overline{I}, \overline{T}]$:

$$[PGL(2,\mathbb{Z}), PGL(2,\mathbb{Z})] =< \overline{V_1}, \overline{V_2} \mid \overline{V_1}^3 = \overline{V_2}^3 = \mathbf{1} > \simeq \mathbf{C}_3 * \mathbf{C}_3.$$

3 Conformal punctured toruses

The conformal punctured toruses are easily built with the Poincaré $\mathcal{H}$ half-plane. We use four geodesics of $\mathcal{H}$ designated by αs, $s\beta$, βp, $p\alpha$, not crossing each other and with α, s, β, p, on the border of $\mathcal{H}$. Any torus is given by transformations

$$t_A : \alpha p \to s\beta, \;\; t_B : \alpha s \to p\beta.$$

These transformations being given by matrices A and B of $SL(2,\mathbb{R})$ acting on $\mathcal{H}$ as conformal transformations we can compute :

$$A = \begin{bmatrix} c\beta & -c\alpha\beta \\ c & (1/c\beta) - c\alpha \end{bmatrix} \text{ where } c \neq 0,$$

$$B = \begin{bmatrix} c'\alpha & -c'\alpha\beta \\ c' & (1/c'\alpha) - c'\beta \end{bmatrix} \text{ where } c' \neq 0,$$

$$A(\alpha) = s, \quad A(p) = \beta, \quad B(\beta) = s, \quad B(p) = \alpha.$$

$$\alpha < 0, \quad \beta > 0, \quad c \neq 0, \quad c' \neq 0.$$

In $SL(2, \mathbb{R})$ the two matrices A and B generate $G = gp(A, B)$ and define a fuchsian group acting on $\mathcal{H}$ where P is the canonical projection from $SL(2, \mathbb{R})$ to $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm \mathbf{1}_2\}$:

$$\Gamma = PG = G/G \cap \{\pm \mathbf{1}_2\} = gp(P(A), P(B)).$$

The Markoff theory with $A = A_0$, $B = B_0$ is given by

$$c = \beta = -c' = -\alpha = 1,$$

For the more general cases we consider the commutator :

$$L = [A, B] = ABA^{-1}B^{-1}.$$

It contents all the necessary information concerning the associated punctured torus because

$$L(s) = ABA^{-1}B^{-1}(s) = ABA^{-1}(\beta) = AB(p) = A(\alpha) = s.$$

Also :

$$tr(L) = tr([A, B]) \leq -2.$$

$$tr(L) + 2 = tr(A)^2 + tr(B)^2 + tr(AB)^2 - tr(A)tr(B)tr(AB) \leq 0.$$

This last condition is due to Fricke. The parabolic case defined by the condition $tr(L) = -2$ gives back the Markoff equation thanks to a factor 3 in the traces which are related by :

$$tr(A)^2 + tr(B)^2 + tr(AB)^2 = tr(A)tr(B)tr(AB).$$

We have now a parametric representation with $(\lambda, \mu) \in \mathbb{R}^2 \backslash \{(0, 0)\}$ due to Fricke [19] :

$$tr(A) = \frac{1 + \lambda^2 + \mu^2}{\mu}, \quad tr(B) = \frac{1 + \lambda^2 + \mu^2}{\lambda}, \quad tr(AB) = \frac{1 + \lambda^2 + \mu^2}{\lambda\mu},$$

All the parabolic punctured toruses are obtained by this way. The Markoff theory is obtained with $\lambda = \mu = 1$. Easily :

Proposition 3.1. *Let* (A, B) *and* (A', B') *generating the two Fricke groups* $\Gamma = Pgp(A, B)$ *and* $\Gamma' = Pgp(A', B')$ *associated to two conformal punctured toruses, we have equivalence of :*

1/ The couples (A, B) *et* (A', B') *are equivalent thanks to an interior automorphism of* $GL(2, \mathbb{R})$ *:*

$$A = DA'D^{-1}, \quad B = DB'D^{-1}, \quad \textit{where } D \in GL(2, \mathbb{R}).$$

2/ The following two triples are equal :

$$\Pi(A, B) = (tr(B^{-1}), tr(A), tr(B^{-1}A^{-1})),$$

$$\Pi(A', B') = (tr(B'^{-1}), tr(A'), tr(B'^{-1}A'^{-1})).$$

3/ The couples (A, B) *and* (A', B') *give the same parameters* λ*,* $\mu \in \mathbb{R}^+$

$$\lambda = (tr(A)/tr(AB)) = (tr(A')/tr(A'B')),$$

$$\mu = (tr(B)/tr(AB)) = (tr(B')/tr(A'B')).$$

Also :

Proposition 3.2. *Any conformal equivalence from a parabolic punctured torus* $\mathcal{T}_\Gamma^\bullet$ *to itself given by an interior automorphism of* $GL(2, \mathbb{R})$ *is equal to identity.*

It is easy to develop a theory of reduction for parabolic toruses and to find a link with quaternions. But the study of the laplacian on such surfaces is not so easy [56], though important for physics [36]. These parabolic punctured toruses are of the form

$$\mathcal{H}/ < A, B, L \mid [A, B]L^{-1} = 1 > .$$

Now if $\mathcal{T}^\bullet$ is the associated topological punctured torus the conformal structure built on it thanks to the choice of A and B is only given [52] by a representation $\rho : \pi_1(\mathcal{T}^\bullet, *) \to SL(2, \mathbb{R})$, where $\pi_1(\mathcal{T}^\bullet, *) \simeq \mathbf{F}_2$ is the Poincaré group of the punctured torus. Introducing the space of deformations

$$\mathcal{R} = \mathcal{R}(\pi_1(\mathcal{T}^\bullet, *), PSL(2, \mathbb{R})),$$

and the morphism $\overline{\rho} = P \circ \rho$, we find by this construction all the possible parabolic conformal punctured toruses

$$\mathcal{H}/\overline{\rho}(\pi_1(\mathcal{T}^\bullet, *)).$$

This approach corresponds to the Teichmüller theory, here specialized to punctured toruses. Replacing $PSL(2,\mathbb{R})$ by $PSL(2,\mathbb{C})$, we have also by the former proposition 3.1. a link with the variety of representations [32] [4] of the group of Poincaré $\pi_1(\mathcal{T}^\bullet,*)$:

$$\rho \in \mathcal{R}(\pi_1(\mathcal{T}^\bullet,*), PSL(2,\mathbb{C})) \to (tr\rho(g_1), tr\rho(g_2), tr\rho(g_3)) \in \mathbb{C}^3.$$

4 Monodromy

A monodromy representation of the group $\pi_1(\mathcal{T}^\bullet,*)$ is a morphism of groups

$$\rho : \pi_1(\mathcal{T}^\bullet,*) \longrightarrow GL(n,\mathbb{C}).$$

Its image is the group of monodromy. These representations are classified with interior automorphisms of $GL(n,\mathbb{C})$. They are considered in Fuchs differential equations ([59] p. 75, [22], [31]) :

$$\frac{d^n f}{dz^n} + a_1(z)\frac{d^{n-1} f}{dz^{n-1}} + ... + a_n(z)f = 0.$$

With $n = 2$ et $\pi_1(\mathcal{T}^\bullet,*) \simeq \mathbf{F}_2$ generated by A et B, the monodromy representations are completely described in [59] (p. 80). The irreducible ones are given thanks to an interior automorphism of $GL(2,\mathbb{C})$ with expressions

$$\rho(A) = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix},\ \rho(B) = \begin{bmatrix} \mu_1 & 0 \\ (\nu_1+\nu_2) - (\lambda_1\mu_1 + \lambda_2\mu_2) & \mu_2 \end{bmatrix},\ \lambda_i\mu_j \neq \nu_k.$$

They are uniquely determined by the couples (λ_1,λ_2), (μ_1,μ_2), (ν_1,ν_2) of eigenvalues of A, B and AB, with the former constraints. Diagonalizing the matrices A_0 et B_0 of the Markoff theory, we can consider :

$$\rho(A_0) = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & 1 \\ 0 & \frac{3+\sqrt{5}}{2} \end{bmatrix},\ \rho(B_0) = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & 0 \\ -4 & \frac{3+\sqrt{5}}{2} \end{bmatrix}.$$

We give now a solution of the corresponding problem of Riemann-Hilbert, which consists to find a differential equation having ρ as a monodromy representation. For this we use [58] (theorem 4.3.2 p.85) in order to compute the associated Riemann scheme. We find this way the following fuchsian equation (modified hypergeometric) given with $\sigma_3+\tau_3=1$ and $\sigma_3+\sigma_3^{-1}=3$ under the following form :

$$z(1-z)\frac{d^2f}{dz^2}+(1-2z)\frac{df}{dz}-(\sigma_3\tau_3)f=\frac{1}{4\pi^2z(1-z)}\log(\frac{3+\sqrt{5}}{2})\log(\frac{3-\sqrt{5}}{2})f.$$

This equation which constitutes the main innovation of the present article can be studied with the methods of [42] [8]. Also we get a differential operator whose spectral analysis is now to realize :

$$\mathfrak{L}=D^2+\frac{(1-2z)}{z(1-z)}D-\frac{(\sigma_3\tau_3)4\pi^2z(1-z)+\log(\frac{3+\sqrt{5}}{2})\log(\frac{3-\sqrt{5}}{2})}{4\pi^2z^2(1-z)^2}.$$

The comparison of the corresponding spectrum with the Markoff spectrum is now to do and will be detailed in a next article. In fact two possibilities exist for $\mathfrak{L}$ owing to the chosen value of σ_3 corresponding to the geometrical phenomenon of Schröder pairs for the same punctured torus, and showing that the two possibilities are linked by an easy transformation. An hamiltonian interpretation could be important in the present case. It is effective for very important physical equation appearing in Physics (Lamé - that is to say periodical Schrödinger in one dimension [18], Sine-Gordon, non linear Schrödinger, Korteweg-de Vries,..., solitons) admitting an hamiltonian representation with states in an Hilbert space. It could give a solution for the project that we mentioned.

5 Conclusion

The comparison with the hypergeometrical approach of Harvey Cohn [9] of the Markoff theory needs to be done. He discovered the link with the following relation between the classical modular function J automorphic for $PSL(2,\mathbb{Z})$ and the Weierstrass function $\wp$:

$$1-J(\tau)=\wp'^2(z)=4\wp^3(z)+1.$$

He explained the link with triples of matrices (A,B,C) associated to the Markoff theory and gave ([11], [12]) the opportunity to look at a formula supposing an hexagonal symmetry

$$dz=const.\times\frac{dJ}{J^{2/3}(J-1)^{1/2}}.$$

It does not seem to the author of the present article that the way between these two formulas has been detailed. The problem is known to be linked to an accessory parameter [7] [28] verifying a Lamé differential equation ([60] p. 110). This question has also a link with the 22^{th} Hilbert's problem [26]. This famous problem is not yet completely solved [53], even if the Lamé equations are much more studied today [1] [57]. We suggest to get insight in this question for punctured toruses through the former developments. Considering the first of the two last equations and differentiating we get the former differential relations :

$$-J'(\tau)d\tau = 12\wp^2(z)\wp'(z)dz, \;\; \wp'(z) = (1-J(\tau))^{1/2}, \;\; \wp^2(z) = (J(\tau)/4)^{2/3}.$$

The difficulties for integration of the differential relation between dz and dJ are known [58] (p. 85 - 90), altogether with the links with the hypergeometric function $F(a,b,c,z)$ solution of the differential equation with two singularities $z=0$ and $z=1$, where $z \in \mathbb{C}$:

$$E(a,b,c): \;\; z(1-z)\frac{d^2F}{dx^2} + (c-(a+b+1)z)\frac{dF}{dx} - abF = 0.$$

When the parameters a, b, c, are real and c, $c-a-b$, $a-b$, non integers, we find on $\mathcal{D} = \mathbb{C}\backslash\{]-\infty,0]\cup[1,\infty[\}$ the Schwarz application :

$$Sch : J \in \mathcal{D} \longrightarrow (F(a,b,c,J) : J^{1-c}F(a+1-c, b+1-c, 2-c, J)) \in \mathbf{P}^1(C).$$

The expression of H. Cohn between dz et dJ leads to consider the case $a=(1/3)$, $b=0$, $c=(5/6)$ giving $\mid 1-c \mid = (1/6)$, $\mid c-a-b \mid = (1/2)$, $\mid a-b \mid = (1/3)$. These values give confirmation that we are in an euclidian hexagonal crystal case. Also we get the known link with the work of R. Dedekind [14] and his function η. Indeed we get $dz = w(\tau)^2 d\tau$ defining $w(\tau)$ with

$$w(\tau) = const.\frac{J'(\tau)^{1/2}}{J(\tau)^{1/3}(1-J(\tau))^{1/4}}.$$

A new hypergeometric equation $E((1/12),(1/12),(2/3))$ appears between w and J. The function η is a square root of w ([6] p. 135 or [35] p. 180) known to verify precisely:

$$\eta(\tau)^{24} = \frac{1}{(48\pi^2)^3}\frac{J'(\tau)^6}{J(\tau)^4(1-J(\tau))^3}.$$

The function η has indeed a tight link with the Markoff equation [43] [44].

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