NONLOCAL CONSTRUCTIONS IN GEOMETRY OF PDE

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This is an overview of recent results obtained by S. Igonin, P. Kersten, and A. Verbovetsky in collaboration with the author and related to using of nonlocal constructions in geometry of partial differential equations. For general references concerning geometry of PDE see [1, 7].

Let $\mathscr{E} \subset J^\infty(\pi) \xrightarrow{\pi_\infty} M$ be an infinitely prolonged differential equation considered as a submanifold in an appropriate manifold of infinite jets. Then $\mathscr{E}$ is endowed with a natural finite-dimensional integrable distribution (the *Cartan distribution* denoted by $\mathscr{C}$) locally spanned by the total derivatives. A fiber bundle $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ is called a *covering* over $\mathscr{E}$ if (a) $\tilde{\mathscr{E}}$ is endowed with an integrable distribution $\tilde{\mathscr{C}}$, $\dim \tilde{\mathscr{C}} = \dim \mathscr{C}$ and (b) $\tau_* \tilde{\mathscr{C}}_y = \mathscr{C}_{\tau(y)}$ for any $y \in \tilde{\mathscr{E}}$. A $\pi_\infty \circ \tau$-vertical vector field X is called a *nonlocal symmetry* of $\mathscr{E}$ if it preserves $\tilde{\mathscr{C}}$.

Nonlocal symmetries can be expressed in finite terms rather rarely. A good (and rather useful) substitute is the notion of a *shadow* (that is often mixed up with nonlocal symmetries themselves!). Naively, in local coordinates a shadow can be introduced as follows. Note that any vector field X lying in the Cartan distribution can be uniquely lifted to a field $\tilde{X}$ lying in $\tilde{\mathscr{C}}$ such that $\tau_* \tilde{X} = X$. Moreover, one has $\widetilde{[X,Y]} = [\tilde{X}, \tilde{Y}]$. Consequently, any linear differential operator Δ on $\mathscr{E}$ expressed in total derivatives (a so-called *$\mathscr{C}$-differential operator*) is lifted to an operator $\tilde{\Delta}$ on $\tilde{\mathscr{E}}$. In particular, assume that $\mathscr{E}$ is given by a differential operator F and $\ell_{\mathscr{E}}$ is the restriction of its linearization to $\mathscr{E}$. Then $\ell_{\mathscr{E}}$ can be lifted to an operator $\tilde{\ell}_{\mathscr{E}}$ on $\tilde{\mathscr{E}}$. Consider a solution φ of the equation $\tilde{\ell}_{\mathscr{E}}(\varphi) = 0$. Then to such a φ there corresponds a derivation $S_\varphi\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$. This derivation (or φ itself) is called a shadow in the covering τ. The *Reconstruction Theorem* (see [8]) states that for any shadow S_φ in τ there exist a covering $\mathscr{E}_\varphi \xrightarrow{\tau_\varphi} \tilde{\mathscr{E}} \xrightarrow{\tau} \mathscr{E}$ and a nonlocal symmetry S in this covering such that $S|_{C^\infty(\mathscr{E})} = S_\varphi$.

In a similar way, one can consider the adjoint operator $\ell^*_{\mathscr{E}}$ and its lifting $\tilde{\ell}^*_{\mathscr{E}}$ to $\tilde{\mathscr{E}}$. In what follows, we assume that the equation under consideration satisfies the conditions of *Vinogradov's Two-Line Theorem* [10]. Then solutions of the equation $\tilde{\ell}^*_{\mathscr{E}}(\psi) = 0$ are called *nonlocal generating functions*.

Nonlocal symmetries (and their shadows) and generating functions naturally arise in various problems related to geometry of PDE. In the talk the following ones will be considered.

Bäcklund transformations [3]**.** Geometrically, a *Bäcklund transformation* between two differential equations $\mathscr{E}$ and $\mathscr{E}'$ is a pair of coverings $\mathscr{E} \xleftarrow{\tau} \tilde{\mathscr{E}} \xrightarrow{\tau'} \mathscr{E}'$. In applications, one of the most important cases is when $\mathscr{E} = \mathscr{E}'$ and τ, τ' belong to a smooth family of coverings τ_λ, $\lambda \in \mathbb{R}$ being a nonremovable[1] parameter. Such a family may be regarded as a *deformation* of the covering τ_{λ_0}.

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[1]I.e., for any $\lambda_1 \neq \lambda_2$ the coverings τ_{λ_1} and τ_{λ_2} are inequivalent.

Theorem 1. *Infinitesimal part of the deformation τ_λ is a τ_0-shadow. If the deformation is infinitesimally nontrivial, then this shadow cannot be extended to a nonlocal symmetry in τ_0.*

Δ-coverings [5]**.** A covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ is said to be *linear* if τ is a vector bundle and for any vector field X on $\mathscr{E}$ the field $\tilde{X}$ preserves the subspace of fiber-wise linear functions in $C^\infty(\tilde{\mathscr{E}})$. If Δ is a $\mathscr{C}$-differential operator over $\mathscr{E}$, then there exists a canonical way to construct a linear covering associated to Δ. This covering is called Δ-*covering*. From *technical* point of view, Δ-coverings are useful to solve the following *factorization problem*: take another $\mathscr{C}$-differential operator ∇ and find $\mathscr{C}$-differential operators V and V' such that the equation

$$\nabla \circ V = V' \circ \Delta \tag{1}$$

holds.

Theorem 2. *Solutions $V \mod (\square\circ\Delta)$ of (1), where $\square$ is a $\mathscr{C}$-differential operator, are in one-to-one correspondence with fiber-wise linear solutions s of the equation $\tilde{\nabla}(s) = 0$, where $\tilde{\nabla}$ is the lifting of ∇ to the Δ-covering.*

There are two canonical coverings associated to a given equation. Namely, taking the operator $\ell_{\mathscr{E}}$, we obtain the $\ell_{\mathscr{E}}$-*covering* and for $\ell_{\mathscr{E}}^*$ one gets the $\ell_{\mathscr{E}}^*$-*covering*. The first one plays the role of the tangent bundle to $\mathscr{E}$ while the second is the counterpart of the cotangent bundle in the category of differential equations. This determines *conceptual* importance of the introduced constructions.

Recursion operators for symmetries [5, 6]**.** Let $\Delta = \nabla = \ell_{\mathscr{E}}$. Then equation (1) takes the form

$$\ell_{\mathscr{E}} \circ V = V' \circ \ell_{\mathscr{E}},$$

i.e., V takes $\ker \ell_{\mathscr{E}}$ (symmetries) to itself. In other words, V is a recursion operator for symmetries. From Theorem 2 we obtain an efficient method to compute such operators:

Corollary. *Recursion operators for symmetries are in one-to-one correspondence with fiber-wise linear solutions of the equation $\tilde{\ell}_{\mathscr{E}}(s) = 0$, i.e., with shadows in the $\ell_{\mathscr{E}}$-covering.*

Recursion operators for conservation laws [5]**.** Let $\Delta = \nabla = \ell_{\mathscr{E}}^*$. Then equation (1) takes the form

$$\ell_{\mathscr{E}}^* \circ V = V' \circ \ell_{\mathscr{E}}^*,$$

i.e., V takes $\ker \ell_{\mathscr{E}}^*$ (generating functions) to itself. In other words, V is a recursion operator for generating functions. From Theorem 2 we obtain an efficient method to compute such operators:

Corollary. *Recursion operators for generating functions are in one-to-one correspondence with fiber-wise linear solutions of the equation $\tilde{\ell}_{\mathscr{E}}^*(s) = 0$, i.e., with nonlocal generating functions in the $\ell_{\mathscr{E}}^*$-covering.*

To obtain a recursion procedure for conservation laws, recall that for 'two-line equation' [10] their conservation laws (the term $E_1^{0,n-1}$ of the $\mathscr{C}$-spectral sequence) are related to generating functions (the term $E_1^{1,n-1}$) by the injective differential $d_1^{0,n-1}$, which in the simplest case of evolution equations coincide with the Euler operator $\mathcal{E}$, and a generating function ψ corresponds to a conservation law if and only if $\ell_\psi = \ell_\psi^*$.

Hamiltonian structures [5]**.** Let $\mathscr{E}$ be an evolution equation, $\Delta = \ell_{\mathscr{E}}^*$, and $\nabla = \ell_{\mathscr{E}}$. Then equation (1) takes the form

$$\ell_{\mathscr{E}} \circ V = V' \circ \ell_{\mathscr{E}}^*,$$

i.e., V takes $\ker \ell_{\mathscr{E}}^*$ (generating functions) to $\ker \ell_{\mathscr{E}}$ (symmetries). We call such operators *pre-Hamiltonian*. As above, Theorem 2 provides an efficient method to compute these operators:

Corollary. *Pre-Hamiltonian operators are in one-to-one correspondence with fiber-wise linear solutions of the equation* $\tilde{\ell}_{\mathscr{E}}(s) = 0$, *i.e., with shadows in the* $\ell_{\mathscr{E}}^*$*-covering.*

A pre-Hamiltonian structure is *Hamiltonian* if (a) $V + V^* = 0$ and (b) $[\![s, s]\!] = 0$, where $[\![\,,]\!]$ is the *variational Schouten bracket* [4]. If $\mathscr{E}$ is of the form

$$u_t = f(x, t, u, \dots, u_k),\ \ u = (u^1, \dots, u^m),\ \ f = (f^1, \dots, f^m),\ \ u_l = \frac{\partial^l u}{\partial x^l}, \qquad (2)$$

and $V = \| \sum_l a_{ij}^l D_x^l \|$, where D_x is the total derivative with respect to x, then conditions (a) and (b) can be reformulated in terms of the function[2] $W_V = \sum_{ijl} a_{ij}^l p_l^i p_0^j$ as follows

$$\sum_i \frac{\delta W_V}{\delta p^i} p_0^i = -2 W_V, \quad \mathcal{E} \sum_i \Big(\frac{\delta W_V}{\delta u^i} \frac{\delta W_V}{\delta p^i} \Big) = 0$$

and are easily checked. Here $\delta/\delta u$, $\delta/\delta p$ are variational derivatives[3].

Symplectic structures [5]**.** Let again $\mathscr{E}$ be an evolution equation, $\Delta = \ell_{\mathscr{E}}$, and $\nabla = \ell_{\mathscr{E}}^*$. Then equation (1) takes the form

$$\ell_{\mathscr{E}}^* \circ V = V' \circ \ell_{\mathscr{E}},$$

i.e., V takes $\ker \ell_{\mathscr{E}}$ (symmetries) to $\ker \ell_{\mathscr{E}}^*$ (generating functions). We call such operators *presymplectic*. Again, Theorem 2 gives an efficient method to compute these operators:

Corollary. *Presymplectic operators are in one-to-one correspondence with fiber-wise linear solutions of the equation* $\tilde{\ell}_{\mathscr{E}}^*(s) = 0$, *i.e., with generating function in the* $\ell_{\mathscr{E}}$*-covering.*

A presymplectic structure is *symplectic* if condition (a) holds and s is 'variationally closed', which for equation (2) means that

$$\mathcal{E} \sum_i \frac{\delta W_V}{\delta u^i} p_0^i = 0.$$

Back to Bäcklund transformations. The above described scheme needs one generalization at least. Since recursion operators arising in practice usually contain terms of the D_x^{-1} type, these nonlocalities should be added in the initial setting[4]. This is being done by substituting equation $\mathscr{E}$ with its covering $\tilde{\mathscr{E}}$ in all constructions. Consequently, the operators obtained cease to be differential, but can be transformed to *$\mathscr{C}$-differential relations* between the objects where they act. Geometrically, these relations are realized as Bäcklund transformations relating:

- $\ell_{\mathscr{E}}$ covering with itself (recursion operators for symmetries);
- $\ell_{\mathscr{E}}^*$ covering with itself (recursion operators for generating functions);
- $\ell_{\mathscr{E}}$ covering with $\ell_{\mathscr{E}}^*$ covering (pre-Hamiltonian and presymplectic structures).

[2]The variables p_l^i, $l = 0, \dots, k$, $i = 1, \dots, m$, should be considered as *odd*.

[3]In the formula above as well as in the condition for symplectic structures below, the Euler operator contains variational derivatives both in u and in p.

[4]Such terms arise also in *nonlocal* Hamiltonian and symplectic structures

The first step in this direction was made in [9], where recursion operators for symmetries were treated as Bäcklund transformations. Clearly, generalized recursion operators for generating functions can be constructed in a similar way. As for the third type of Bäcklund transformations, one can see that they unify the concepts of pre-Hamiltonian and presymplectic structures. Following [2], these transformations may be called *pre-Dirac structures*. Their study is a subject for future research.

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