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Group Classification with Respect to Hidden Symmetry

Irina YEHORCHENKO

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *iyegorch@imath.kiev.ua*

We present approaches to systematic description of PDE that display hidden symmetry, with reduced equations having additional symmetry operators compared to the initial equations.

1 Introduction

The concept of conditional symmetry was extensively developed recently following the papers [1] and allowed finding many new solutions of nonlinear differential equations. Here we will consider another facet of the conditional symmetry – additional symmetry of the reduced equations.

"Hidden symmetry" in various contexts is usually a symmetry not obtainable by some standard and straightforward procedure applicable to the models in this context. This term shares the usage of other related terms like "conditional symmetry", "approximate symmetry", and "symmetry" or "invariance" when the same words may denote rather different concepts. Here we will consider hidden symmetry of partial differential equations similarly to Type II hidden symmetry of ordinary differential equations according to [2]. With respect to ODE such symmetry arises as symmetry of equations with reduced order that is not a symmetry of the original equations. In the same way, for a PDE it is symmetry of the reduced equation (with reduced number of independent variables) not present in the original equation.

Definition 1. A differential equation is said to have *hidden symmetry* under an operator X if after the process of reduction of the number of independent variables the resulting reduced equation is invariant under the operator X_1 (being the projection of the operator X in new variables) while the original equation is not invariant under the operator X.

Such symmetry is "classical" in the sense that full hidden symmetry of either ODE or PDE may be found by consecutive reduction of the original equation and investigation of Lie symmetries of the reduced equations, as provided by L. Ovsyannikov's *Submodels* programme [3]. "Symmetry" or "Lie symmetry" is determined in accordance to the procedures that may be found e.g. in [4, 5].

The definition of hidden symmetry can be reformulated in terms of the conditional symmetry according to the book [6]. Here we also will take into account the result proved in the paper [7] that conditional invariance of a differential equation under an involutive family of first-order differential operators Q_a is equivalent to possibility of reduction of this equation by means of the ansatz corresponding to this family of operators. Thus, the additional symmetry under the operator X_1 of the reduced equation (hidden symmetry under the operator X for the original equation) turns out to be a conditional symmetry of the original equation under conditions $Q_a[u] = 0$ ($Q_a[u]$ designates characteristics of the vector field Q_a) with all appropriate differential consequences. Note that X_1 is a Lie symmetry of the reduced equation, and we do not add the condition $X[u] = 0$ to the set of conditions, so such operator will not present a purely Q-conditional symmetry in the sense of [6].

Q-conditional symmetry can also be hidden – that is being a new Q-conditional symmetry of the reduced equation. For examples of such symmetries see [8].

In this paper we give an outlook of description of partial differential equations possessing certain hidden symmetries.

Group classification for classes of differential equations is aimed at identification of equations having wider symmetries than the equations of the class in general. For an overview of the group classification problems and the extensive list of related references see [9]. Usually two types of such problems are considered – finding the equations within a general class that are invariant under specific symmetry group, and description of all symmetries (up to appropriately chosen equivalence) of equations that belong to the specific class. On the basis of the known algorithms for group classification of differential equations in the Lie's sense we develop approaches for a systematic description of classes of nonlinear PDEs that display hidden symmetry.

Definition 2. A class of equations can be regarded as *general* if any local transformations of dependent and independent variables transform an equation from this class into another equation within the same class.

An example is the class of all kth-order PDEs $F = F(x, u, \underset{1}{u}, \ldots, \underset{k}{u}) = 0$ with x, u being respectively n- and m-dimensional independent and dependent variables, $\underset{r}{u}$ being the set of all rth-order partial derivatives of the function $u = \left(u^1, u^2, \ldots, u^m\right)$. Group classification even with respect to the Lie symmetry for the general classes is usually an overwhelming task, and, to our knowledge, such problem was completely solved only for single ordinary differential equations by S. Lie [10]. A restricted, but practically important problem for the general classes of equations would be description of all equations within the class invariant under some specified symmetry group that can be done by describing all differential invariants for such group. Similarly, description of equations having specified Lie symmetry and specified hidden symmetry may be done by means of conditional differential invariants, as shown in Section 2.

For a more specific class it may be possible to make full group classification of a system consisting of the original equation together with the reduction conditions of the type $Q_a[u] = 0$ (with appropriate prolongations of the conditions), as shown in the Section 3.

2 Description of all equations in a general class possessing particular hidden symmetry

The first example is the general first-order partial differential equation with one dependent and three independent variables of the form

$$F = F(x, y, z, u, u_x, u_y, u_z) = 0. \tag{1}$$

We describe all such equations having Lie symmetry with respect to the operator ∂_x and hidden symmetry with respect to the operator ∂_y after reduction by means of the operator ∂_x. The condition of such Lie and hidden symmetry, according to Definition 1, is invariance of the equation (1) under the operator ∂_y on condition that $u_x = 0$:

$$\partial_x F\big|_{F=0} = 0, \qquad \partial_y F\big|_{F=0,\ u_x=0} = 0. \tag{2}$$

The general solution of the condition (2) will be a function of all invariants of the operators ∂_x and ∂_y, that is z, u, u_x, u_y, u_z, and of the conditional invariant $r = u_x R(y, z, u, u_x, u_y, u_z)$ (being an absolute invariant of ∂_x), where R is an arbitrary function that is reasonably determined on the manifold $u_x = 0$:

$$F(r, z, u, u_x, u_y, u_z) = 0, \tag{3}$$

F has to be a function of the invariants of the hidden symmetry operator on the manifold determined by the reduction condition, and have arbitrary form elsewhere. Note that the function r in (3) is not arbitrary: we cannot take e.g.

$$r = R_1(y, z, u, u_x, u_y, u_z) = u_x \frac{R_1}{u_x}$$

as such $R = \frac{R_1}{u_x}$ will be in the general case undetermined on the manifold $u_x = 0$.

Definition 3 ([12]). A function $F(x, u, \underset{1}{u}, \dots, \underset{k}{u})$ is a *conditional differential invariant* of the operator Q, if under the conditions $G(x, u, \underset{1}{u}, \dots, \underset{r}{u}) = 0$ the relations $Q[F] = 0$, $Q[G] = 0$ are satisfied. We take prolongations of the operators of the order $\max(k, r)$.

A set of invariants of the order $r \leq k$ of the operator Q with the conditions $G = 0$ is called a *generating set* of the kth-order conditional differential invariants of Q if all other invariants can be represented as functions of invariants from this set.

Invariants in such generating set may be both absolute invariants of Q and G-conditional of the form $G^{(l)} \times R_{(l)}$, where $G^{(l)}$ are derivatives of G of the order $l \leq k - r$ and $R_{(l)}$ are arbitrary functions determined on the manifold $G^{(k)} = 0$ for all values of k.

Number of functionally independent Q-absolute invariants in the generating set of conditional differential invariants can be calculated similarly to the number of invariants in a functional basis of absolute differential invariants, as $m - 1$, where m is the number of variables in the set $x, u, \underset{1}{u}, \dots, \underset{k}{u}$. Number of linearly independent purely G-conditional invariants is equal to the number of independent conditions of the type $G^{(l)} = 0$.

In some cases we would be able to construct a functional basis of conditional invariants, i.e. the maximal set of functionally independent conditional invariants. That is possible in the case when we put a requirement that our conditional invariants should be also absolute invariants of some Lie algebra L, and additional conditions $G = 0$ in the Definition 3 and their relevant differential consequences are not invariant under L.

Equation (3) is reduced by means of the operator ∂_x to the equation $F(0, z, u, 0, u_y, u_z) = 0$ that is invariant with respect to ∂_y. If $R_y \neq 0$ in the expression for r in (3) then this equation is not invariant with respect to the operator ∂_y, so such symmetry is purely hidden.

This example is easily generalised for larger order of equations or number of reduction operators.

The second example is hidden projective symmetry presented by the operator ($m \neq 0$)

$$\begin{aligned} A = {} & (x_0 - x_2)^2(\partial_0 - \partial_2) + (x_0 - x_2)x_1\partial_1 \\ & + \frac{imx_1^2}{2}(u\partial_u - u^*\partial_{u^*}) - (x_0 - x_2)(u\partial_u + u^*\partial_{u^*}) \end{aligned} \tag{4}$$

for the d'Alembert equation

$$\Box u = \lambda u|u|^4, \tag{5}$$

here $u = u(x_0, x_1, x_2)$ is a complex-valued function, $|u| = (uu^*)^{1/2}$.

The maximal invariance algebra of the equation (5) is the Poincaré algebra extended by the dilation and charge operators (see e.g. [6] and references therein), with basis operators

$$\begin{aligned} & p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \\ & D = x_\mu p_\mu - \frac{i}{4}(u\partial_u + u^*\partial_{u^*}), \quad J = i(u\partial_u - u^*\partial_{u^*}), \end{aligned} \tag{6}$$

where μ, ν take the values 0, 1, 2; the summation is implied over the repeated indices (small Greek letters) in the following way: $x_\nu x_\nu \equiv x_\nu x^\nu \equiv x^\nu x_\nu = x_0^2 - x_1^2 - x_2^2$, $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1)$.

This equation can be reduced by means of the ansatz ($m \neq 0$)

$$u = \exp\left(\frac{-im(x_0+x_2)}{2}\right)\Phi(x_0-x_2, x_1) \tag{7}$$

to the Schrödinger equation

$$2im\Phi_t + \Phi_{xx} = \Phi|\Phi|^4, \qquad t = x_0 - x_2, \qquad x = x_1. \tag{8}$$

The ansatz (7) corresponds to the following additional condition with its conjugate:

$$u_0 + u_2 + imu = 0. \tag{9}$$

Such reduction allowed construction of numerous new solutions for the nonlinear wave equation by means of the solutions of a nonlinear Schrödinger equation [11]. Here reduction gives additional symmetry properties for equation (5), related to the symmetry properties of the equation (8). This equation is invariant, beside the Galilei algebra and the dilation operator

$$\begin{aligned} &\partial_t, \quad \partial_x, \quad J_1 = i(\Phi\partial_\Phi - \Phi^*\partial_{\Phi^*}), \quad G = t\partial_x + xJ_1, \\ &\hat{D}_1 = 2t\partial_t + x\partial_x - I \quad (\text{here } I = i(\Phi\partial_\Phi + \Phi^*\partial_{\Phi^*})) \end{aligned} \tag{10}$$

(see e.g. [6]), under the projective operator equal to the operator A in the new variables t, x, Φ

$$A_1 = t^2\partial_t + tx\partial_x + \frac{im}{2}x^2J_1 - tI. \tag{11}$$

This is one of the immediate examples of hidden symmetry: when presence of some conformal- or projective-type symmetry in a nonlinear equation depends on both the degree of the polynomial nonlinearity and the number of independent variables, such symmetry will exist as hidden in equations with the degree of the polynomial nonlinearity corresponding to the smaller number of independent variables. Note that the operator $\hat{D}_1$ in (10) is not a projection of D in (6), but leads to another hidden symmetry operator $\hat{D} = x_\mu p_\mu - i(u\partial_u + u^*\partial_{u^*})$.

Here we will present full description of Poincaré-invariant equations possessing such hidden symmetry with respect to the operator (4). This is done by means of conditional differential invariants introduced in [12]. Note that in our example we do not require Poincaré invariance of the additional condition itself, so we are able to construct a functional basis of conditional differential invariants.

We adduce functional bases of differential invariants that may be utilised for construction of classes of Poncaré-invariant equations with hidden symmetry (4).

First we present according to [13] a functional basis of absolute differential invariants of the second order for the Poincaré algebra $\langle p_\mu, J_{\mu\nu}\rangle$ (6) and the complex-valued scalar function $u = u(x_0, x_1, x_2)$. It consists of 17 invariants

$$\begin{aligned} &u,\ u^*,\ u_\mu u_\mu,\ u^*_\mu u^*_\mu,\ \Box u,\ \Box u^*,\ u_{\mu\nu}u_{\mu\nu},\ u_{\mu\nu}u^*_{\mu\nu},\ u^*_{\mu\nu}u^*_{\mu\nu},\ u_\mu u_{\mu\nu}u_\nu,\ u^*_\mu u_{\mu\nu}u^*_\nu, \\ &u_\mu u_{\mu\nu}u_{\nu\lambda}u_\lambda,\ \ u^*_\mu u_{\mu\nu}u_{\nu\lambda}u^*_\lambda,\ \ u_{\mu\nu}u_{\nu\lambda}u_{\lambda\mu},\ \ u_{\mu\nu}u_{\nu\lambda}u^*_{\lambda\mu},\ \ u_{\mu\nu}u^*_{\nu\lambda}u^*_{\lambda\mu},\ \ u^*_{\mu\nu}u^*_{\nu\lambda}u^*_{\lambda\mu}. \end{aligned} \tag{12}$$

A functional basis of differential invariants for the Galilei algebra (10), mass $m \neq 0$, of the second order for the complex-valued scalar function $\Phi = \Phi(t,x)$ consists of 10 invariants that can be checked by direct calculation.

For simplification of the expressions for differential invariants we introduced the following notations: $\phi = \ln\Phi$, and to ensure that ϕ is single-valued we choose it so as $\operatorname{Im}\Phi = \arctan\frac{\operatorname{Re}\phi}{\operatorname{Im}\phi}$.

The elements of the functional basis may be chosen as follows:

$$\phi + \phi^*, \quad \phi_x + \phi^*_x, \quad M_1 = 2im\phi_t + \phi_x^2, \quad M_1^*, \quad \phi_{xx}, \quad \phi^*_{xx},$$

$$\theta = im\phi_{tx} + \phi_x\phi_{xx}, \quad \theta^*, \quad M_2 = -m^2\phi_{tt} + 2im\phi_x\phi_{tx} + \phi_x^2\phi_{xx}, \quad M_2^*. \tag{13}$$

A functional basis of differential invariants for the Galilei algebra extended by the dilation operator (10) and the projective operator (11) may be chosen as follows:

$$N_1 e^{-4(\phi+\phi^*)}, \quad \frac{N_1}{N_1^*}, \quad \frac{N_2}{N_1^2}, \quad \frac{N_2^*}{(N_1^*)^2}, \quad \frac{\phi_{xx}+\phi_{xx}^*}{N_1}, \quad \frac{\theta}{N_1^{\frac{3}{2}}}, \quad \frac{\theta^*}{N_1^{\frac{3}{2}}}, \quad \frac{(\phi_x+\phi_x^*)^2}{N_1},$$

where

$$N_1 = M_1 + \phi_{xx} = 2im\phi_t + \phi_{xx} + \phi_x^2, \qquad N_2 = \phi_{xx}N_1 + \frac{\phi_{xx}^2}{2} + M_2. \tag{14}$$

An algorithm for construction of conditional differential invariants may be derived directly from the Definition 3. We can construct differential invariants of the Poincaré algebra $\langle p_\mu, J_{\mu\nu}\rangle$ (6) being conditional differential invariants of the projective operator (4) solving the system

$$\underset{2}{A}_1 F(\mathrm{Inv}_P) = 0, \qquad u_0 + u_2 + imu = 0, \qquad \frac{\partial}{\partial x_\mu}(u_0 + u_2 + imu) = 0, \quad \mu = 0, 1, 2,$$

where Inv_P are all differential invariants (12) of the Poincaré algebra $\langle p_\mu, J_{\mu\nu}\rangle$ (6). Using the fact that the ansatz (7) is the general solution of the additional condition (9), we can directly substitute this ansatz into differential invariants (12). The expression $\Box u$ transforms into uN_1, where N_1 is an expression entering into expression for differential invariants (14). Substituting the ansatz (7) into all elements of the fundamental basis (12) of second-order differential invariants of the Poincaré algebra, we can obtain a reduced basis of differential invariants that may be used for construction of all equations reducible by means of this ansatz to equations possessing the projective symmetry. We can give the following representation of the Poincaré invariants using expressions M_k (13) and N_k (14):

$$u_\mu u_\mu = u^2 M_1, \ u_\mu u_\mu^* = \frac{uu^*}{2}\big(M_1 + M_1^* - (\phi_x+\phi_x^*)^2\big), \ u_{\mu\nu}u_{\mu\nu} = u^2\big(2M_2 + M_1^2 + \phi_{xx}^2\big),$$

$$u_{\mu\nu}u_{\mu\nu}^* = uu^*\Big(M_2 + M_2^* + \frac{1}{4}\big(M_1 + M_1^* - (\phi_x+\phi_x^*)^2\big)^2 - 2(\theta+\theta^*)(\phi_x+\phi_x^*)$$
$$+ \big(\phi_{xx}+\phi_{xx}^*\big)(\phi_x+\phi_x^*)^2 + \phi_{xx}\phi_{xx}^*\Big), \quad u_\mu u_\nu u_{\mu\nu} = u^3\big(M_2 + M_1^2\big),$$

$$u_\mu^* u_\nu^* u_{\mu\nu} = uu^{*2}\Big(M_2 - 2\theta(\phi_x+\phi_x^*) + \phi_{xx}(\phi_x+\phi_x^*)^2 + \frac{1}{4}\big(M_1 + M_1^* - (\phi_x+\phi_x^*)^2\big)^2\Big),$$

with similar, however more cumbersome expressions for other invariants. Note that it is convenient to use for expression of the projective conditional invariants not only invariants from the fundamental basis (12) but also other invariants (that are anyway functions of (12)).

We construct Poincaré-invariant conditional differential invariants of the projective operator (4) under the condition (9) using differential invariants (13)

$$I_1 = N_1 e^{-4(\phi+\phi^*)} = \frac{\Box u}{u(uu^*)^2}, \quad I_2 = \frac{N_1}{N_1^*} = \frac{u^*\Box u}{u\Box u^*},$$

$$I_3 = \frac{N_2}{N_1^2} = \left(uu_\mu u_\nu u_{\mu\nu} + \frac{3}{2}u^2(\Box u)^2 + \frac{1}{2}(u_\mu u_\mu)^2 - 2u\Box u(u_\mu u_\mu)\right)\big(u^2(\Box u)^2\big)^{-1}, \quad I_3^*,$$

$$I_4 = \frac{(\phi_x+\phi_x^*)^2}{N_1} = \frac{u_\mu u_\mu u^{*2} + u_\mu^* u_\mu^* u^2 - 2u_\mu u_\mu^* uu^*}{uu^{*2}\Box u},$$

$$I_5 = \frac{(\phi_{xx}+\phi_{xx}^*)}{N_1} = \frac{u^*\Box u + u\Box u^* - 2u_\mu u_\mu^*}{u^*\Box u}, \quad I_6 = \frac{\theta}{2N_1} = \big(- u^3 u_\mu^* u_\nu^* u_{\mu\nu}$$

$$+ u^{*2}\big(uu_\mu u_\nu u_{\mu\nu} - (u_\mu u_\mu)^2\big) + (u\Box u - u_\mu u_\mu)\big(u_\mu u_\mu u^{*2} + u^*_\mu u^*_\mu u^2 - 2u_\mu u^*_\mu uu^*\big)$$
$$+ (uu_\mu u^*_\mu)^2\big)\big(u^2u^*\Box u(u_\mu u_\mu u^{*2} + u^*_\mu u^*_\mu u^2 - 2u_\mu u^*_\mu uu^*)^{1/2}\big)^{-1}, \quad I^*_6. \qquad (15)$$

The expressions (15) form a functional basis of invariants that are absolute differential invariants for the operators of the extended Poincaré algebra (6) and conditional invariants of the operator A (4) with the additional condition (9). Functional independence of the expressions (15) can be checked directly, most easily by application of the additional conditions, that is by substitution of the ansatz (7) into (15), with the "reduced basis" obtained as a result. The reduced invariants in the basis will depend on new functions without conditions and will be absolute invariants of the operators $\langle G, J, \hat{D}_1, A_1\rangle$.

Calculation of the number of invariants in such basis may be also considered as follows:

$$\begin{aligned}\text{Number of invariants} &= \text{Number of (variables + derivatives)}\\ &\quad - \big(\text{Rank of the set of operators}\\ &\quad + \text{Number of independent conditions and their derivatives}\big).\end{aligned}$$

We have 20 dependent variables and derivatives, 8 conditions ((9) with the conjugate and derivatives), and the rank of the set of operators $\langle J_{\mu\nu}, \hat{D}, J, A\rangle$ under the conditions (9) with the conjugate and derivatives is equal to 6 with all conditions taken into account.

However, the number of conditional differential invariants in the considered example is 8 and not $6 = 20 - 8 - 6$ as it may seem to be. However, if we apply the additional condition (9) and its first derivatives with respective conjugates (8 conditions) to the fundamental basis of invariants of the Poincaré algebra (12) (17 invariants), we will obtain 11 independent reduced invariants, as may be checked by direct calculation. So two of the conditions appear to be dependent on the manifold of invariants, and with other 3 conditions of invariance with respect to operators $\langle \hat{D}, J, A\rangle$ we obtain 8 invariants in the functional basis of conditional differential operators. Thus, we arrived to the following statement.

Proposition 1. *All equations of the form*

$$F(I_1, I_2, I_3, I^*_3, I_4, I_5, I_6, I^*_6) = 0 \qquad (16)$$

are conditionally invariant with respect to the operator A (4) *with the additional condition* (9). *All equations that are invariant with respect to the algebra* (6) *and conditionally invariant with respect to the operator A* (4) *with the additional condition* (9) *have the form* (16).

Note that in this example we required the equations to be invariant in the Lie sense under the algebra (6), and the condition (9) is not invariant with respect to this algebra. So we do not have conditional invariants with arbitrary functions of the type $(u_0+u_2+imu)\times$(arbitrary function). In general, if we require some hidden or conditional invariance producing conditional invariants with arbitrary functions together with some Lie invariance, we may require Lie invariance of an equation depending on such conditional invariants. Moreover, the operators $\langle J_{\mu\nu}, \hat{D}, J, A\rangle$ do not form an algebra, and we would not be able to use absolute differential invariants in the Lie sense to describe invariant equations similarly to the example of the equation (1) and its reduction under the operator ∂_y.

3 Finding hidden symmetries

"Simple" hidden symmetries (Lie symmetries of reduced equations) for a particular class of equations can be found by means of consecutive Lie reductions and consecutive finding Lie symmetries of the reduced equations. Group classification of such class with respect to hidden symmetries of reduced equations will involve description of all possible reductions and group classification of the respective classes of reduced equations. Here we will consider a rather simple example of the nonlinear wave equation in two spatial dimensions

$$\Box u = f(x_0, x_1, x_2, u) \tag{17}$$

that will allow us to illustrate the proposed algorithm. We present group classification of this equation with respect to hidden symmetries for reduction by means of the operator ∂_{x_2}.

Such reduction leads to the two-dimensional wave equation $u_{00} - u_{11} = f(x_0, x_1, u)$. The next step is the usual group classification of the reduced equation. It will be more convenient to use the substitution $\tau = x_0 + x_1$, $\omega = x_0 - x_1$ and to go to the equation

$$u_{\tau\omega} = F(\tau, \omega, u). \tag{18}$$

The equivalence group of equation (18) (see [14]) consists of the continuous transformations (for the definitions and calculation procedures of equivalence transformation groups see [4, 15])

$$\tau' = T(\tau), \quad \omega' = W(\omega), \quad u' = e^{\lambda} u + U(\tau, \omega), \quad F' = \frac{1}{\dot{T}(\tau)\dot{W}(\omega)}(e^{\lambda} F + U_{\tau\omega}), \tag{19}$$

where T, W, U are arbitrary (T, W do not take constant values simultaneously) functions of their arguments and dots over the letters mean derivatives, and discrete transformations $\tau' = \epsilon\epsilon_1\omega + (1-\epsilon)\epsilon_2\tau$, $\omega' = \epsilon\epsilon_3\tau + (1-\epsilon)\epsilon_4\omega$, $u' = \epsilon_5 u$, $F' = \epsilon_5(\epsilon\epsilon_1\epsilon_3 + (1-\epsilon)\epsilon_2\epsilon_4)\, F(\tau', \omega', u')$ ($\epsilon = 0, 1$, $\epsilon_k = \pm 1$, $k = 1, 2, \ldots, 5$), together with products of continuous and discrete transformations.

Group classification of the equation (18) is given in [4] ($F_{uu} = 0$) and in [14] ($F_{uu} \neq 0$). Such group classification was performed up to transformations from the equivalence group of the equation. However, if we multiply the resulting inequivalent invariant equations by means of transformations from the equivalence group (19) that do not belong to the equivalence group of equation (17) and then add u_{22} ("reverse reduction"), we will obtain description of equations with hidden symmetry.

For example, when $F = F(\rho_1(\tau) - \rho_2(\omega), u)$, equation (18) is invariant with respect to the operator $Q = \rho_1'\partial_\tau + \rho_2'\partial_\omega$ (primes here designate derivatives). So, the equations from the class (17) $\Box u = f(\rho_1(\tau) - \rho_2(\omega), x_2, u)$ will have hidden invariance with respect to the operator Q with appropriate change of variables in Q. It quite obvious that such transformed Q is not a Lie symmetry operator for the above equation except if $\rho_1 = a\tau + b$, $\rho_2 = a\omega + c$, a, b, c are constants.

Similarly, for $f = \exp(\rho_1(\tau) + \rho_2(\omega))\, \phi(u \exp(-\rho_1(\tau) - \rho_2(\omega)), x_2)$, there is hidden invariance of (17) with respect to the operators $Q_1 = \rho_1'\partial_\tau + u\partial_u$, $Q_2 = \rho_2'\partial_\omega + u\partial_u$. Here ρ_1, ρ_2 are arbitrary but fixed functions.

We present an algorithm for group classification with respect to hidden symmetry:

Step 1. Obtain reduced equations of for the class of initial PDE.

Step 2. Perform group classification of the reduced equations.

Step 3. Multiply inequivalent invariant reduced equations by means of transformations from the equivalence group of this class.

Step 4. Reverse of the equation: find equations from the initial class of PDE corresponding to the multiplied reduced equations.

Step 5. Find all inequivalent equations with respect to transformations from the equivalence group of the initial class of PDE.

4 Discussion

We see that a nontrivial hidden symmetry for partial differential equations stems from the reduced equations having wider equivalence group than the original equations. So group classification of the classes of equations with respect to hidden symmetry involves study of equivalence groups of such classes, in the similar way as it is done for classification with respect to the Lie symmetry. Description of hidden symmetry of physically interesting classes of equations would allow to identify wider classes of reducible equations than it is possible by means of group classification in the Lie sense. The special interest in such classification lies in that reduced equations often have infinite-dimensional equivalence groups that allows to obtain classes of equations with hidden symmetry that have large degree of arbitrariness.

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[1] Bluman G.W. and Cole J.D., The general similarity solution of the heat equation, *J. Math. Mech.*, 1969, V.18, 1025–1042; Olver P.J. and Rosenau P., The construction of special solutions to partial differential equations, *Phys. Lett. A*, 1986, V.114, 107–112; Fushchych W.I. and Tsyfra I.M., On a reduction and solutions of the nonlinear wave equations with broken symmetry, *J. Phys. A*, 1987, V.20, L45–L48; Fushchych W.I. and Zhdanov R.Z., Symmetry and exact solutions of nonlinear spinor equations, *Phys. Reports*, 1989, V.172, 123–174; Clarkson P. and Kruskal M.D., New similarity solutions of the Boussinesq equation, *J. Math. Phys.*, 1989, V.30, 2201–2213; Levi D. and Winternitz P., Non-classical symmetry reduction: example of the Boussinesq equation, *J. Phys. A*, 1989, V.22, 2915–2924.

[2] Abraham-Shrauner B., Hidden symmetries and nonlocal group generators for ordinary differential equations, *IMA J. Appl. Math.*, 1996, V.56, 235–252; Hidden symmetries, first integrals and reduction of order of nonlinear ordinary differential equations, *J. Nonlin. Math. Phys.*, 2002, V.9, Suppl. 2, 1–9.

[3] Ovsyannikov L.V., Program SUBMODELS. Gas dynamics, *Appl. Math. Mech.*, 1994, V.58, N 4, 30–55.

[4] Ovsyannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.

[5] Olver P., Application of Lie groups to differential equations, New York, Springer Verlag, 1987; Bluman G.W. and Kumei S., Symmetries and differential equations, New York, Springer Verlag, 1989.

[6] Fushchych W.I., Shtelen W.M. and Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kyiv, Naukova Dumka, 1989 (in Russian); Kluwer Publishers, 1993 (in English).

[7] Zhdanov R.Z., Tsyfra I.M. and Popovych R.O., A precise definition of reduction of partial differential equations, *J. Math. Anal. Appl.*, 1999, V.238, N 1, 101–123.

[8] Yehorchenko I. and Vorobyova A., Sets of conditional symmetry operators and exact solutions for wave and generalised heat equations, Contribution in these Proceedings.

[9] Popovych R.O. and Ivanova N.M., New results on group classification of nonlinear diffusion-convection equations, math-ph/0306035.

[10] Lie S., Über die Integration durch bestimmte Integrale von einer Klasse linear partieller Differentialgleichung, *Arch. for Math.*, 1881, V.6, N 3, 328–368 (Translation by N.H. Ibragimov: Lie S., On integration of a class of linear partial differential equations by means of definite integrals, in CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 2, Boca Raton, CRC Press, 1995, 473–508).

[11] Basarab-Horwath P., Barannyk L. and Fushchych W.I., New solutions of the wave equation by reduction to the heat equation, *J. Phys. A*, 1995, V.28, 5291–5304; Basarab-Horwath P., Fushchych W.I. and Barannyk L.F., Solutions of the relativistic nonlinear wave equation by solutions of the nonlinear Schrödinger equation, *Rep. Math. Phys.*, 1997, V.39, 353–374.

[12] Yehorchenko I.A., Differential invariants and construction of conditionally invariant equations, in Proceedings of Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–15 July, 2001, Kyiv), Editors A.G. Nikitin, V.M. Boyko and R.O. Popovych, Kyiv, Insitute of Mathematics, 2002, V.43, Part 1, 256–262; math-ph/0304029.

[13] Fushchych W.I. and Yegorchenko I.A., Second-order differential invariants of the rotation group $O(n)$ and of its extension $E(n)$, $P(l,n)$, *Acta Appl. Math.*, 1992, V.28, 69–92.

[14] Lahno V. and Magda O., The group classification of nonlinear wave equations, math-ph/0310049.

[15] Lisle I.G., Equivalence transformations for classes of differential equations, Thesis, University of British Columbia, 1992; http://www.ise.canberra.edu.au/mathstat/StaffPages/LisleDissertation.pdf;
Lisle I.G. and Reid G.J., Symmetry classification using invariant moving frames, ORCCA Technical Report TR-00-08, University of Western Ontario, http://www.orcca.on.ca/TechReports/2000/TR-00-08.html, *Foundations of Computational Mathematics*, submitted.