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The Connection of the Degasperis–Procesi Equation with the Vakhnenko Equation

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Travelling-wave solutions of the Degasperis–Procesi equation (DPE) are investigated. The solutions are characterized by two parameters. Hump-like, loop-like and coshoidal periodic-wave solutions are found; hump-like, loop-like and peakon solitary-wave solutions are obtained as well. Hone and Wang showed a connection between the DPE and the Vakhnenko equation (VE). Comparing the solutions of the DPE and the VE, we observe that, for both equations at interaction of waves, there are three kinds of phaseshift that depend on the ratio of wave amplitudes. In particular, there is a case when two interacted waves have phaseshifts in the positive direction.

1 Introduction

This report deals with the Degasperis–Procesi equation (DPE) [1–3]

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \qquad (1)$$

which is contained in the family of equations [1]

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \qquad (2)$$

(parameter b is constant). When $b = 2$, equation (2) reduces to the well-known Cammasa–Holm equation (CH) [4, 5]

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \qquad (3)$$

Mikhailov and Novikov developed a powerful extension of the symmetry classification method [6], and applying this to the equation (2) they found that only the cases $b = 2$, 3 could possess infinitely many commuting symmetries, and so only these two cases are integrable. In a recent paper Degasperis, Holm and Hone [2] constructed a Lax pair of the equation (1), and hence proved the integrability of the Degasperis–Procesi equation. Unfortunately, there are no solutions of the direct problem and the inverse problem of the spectral equation from this Lax pair.

Hone and Wang [7] have shown the connection of the DPE (1) with the integrable Vakhnenko equation (VE) [8–10]

$$(u_t + uu_x)_x + u = 0. \qquad (4)$$

Indeed, the transformation $x \to \varepsilon x - t/3\varepsilon$, $t \to \varepsilon t$, $u \to u - 1/3\varepsilon^2$ reduces the DPE to the VE when $\varepsilon \to 0$

$$[(u_t + uu_x)_x + u]_x = \varepsilon^2 (u_t + 4uu_x). \qquad (5)$$

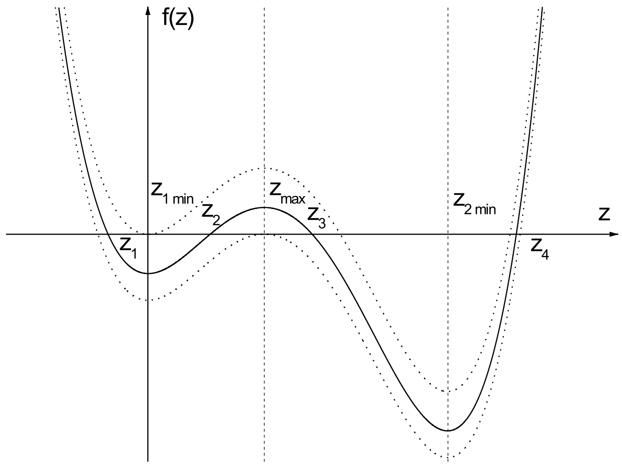

Figure 1. The polynomial $f(z)$. The interval of integration is between the roots z_2 and z_3.

Recently, we have investigated equation (4) by direct integration [8,9], by Hirota's method [11] as well as by the inverse scattering transform method [10]. In the context of this paper, it is of interest that the VE has two families of travelling-wave solutions (see Figs. 1, 2 in [8]); in particular, there is a soliton solution in loop-like form.

Comparing the properties of the solutions of the DPE (1) and the VE (4), we observe that for both equations under interaction of waves, there are three kinds of phaseshift that depend on the ratio of wave amplitudes [2, 12, 13]. In particular, there is a case when two interacted waves have phaseshifts in the positive direction.

However, it engages our attention that soliton solutions have not been observed for the DPE recently. If the VE has a loop-like solution, it follows that the DPE should admit a loop-like solution. In this paper we investigate the travelling-wave solutions of the DPE. The solutions are characterized by two parameters. Hump-like, loop-like and coshoidal periodic-wave solutions are found; hump-like, loop-like and peakon solitary-wave solutions are obtained as well.

2 Travelling-wave solutions

We consider the DPE (1). Restricting our attention to travelling waves, we introduce new variables

$$z = (u - v)/|v|, \qquad \eta = x - vt - x_0, \qquad \tau = |v|t, \tag{6}$$

where v and x_0 are arbitrary constants with $v \neq 0$. In this case the initial equation is reduced to the ODE

$$(zz_\eta)_{\eta\eta} = (4z + 3c)z_\eta \qquad \text{with} \qquad c := v/|v| = \pm 1. \tag{7}$$

After two integrations we have

$$(zz_\eta)^2 = f(z) \tag{8}$$

with

$$f(z) = z^4 + 2cz^3 + Az^2 + B = (z - z_1)(z - z_2)(z - z_3)(z - z_4),$$

where $f(z)$ is a polynomial of fourth order and is shown in Fig. 1.

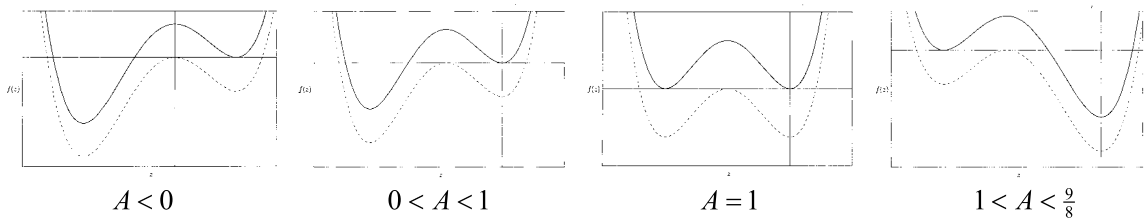

Figure 2. Four possible cases for the polynomial $f(z)$.

Analyzing equation (8), we observe that the roots of the polynomial $f(z)$ are real and the interval of integration is between the roots z_2 and z_3. We can write two forms of solution. The first form of solution is

$$z = \frac{z_2 - z_1 \operatorname{sn}^2(w|m)}{1 - n \operatorname{sn}^2(w|m)} \qquad \text{with} \qquad n = \frac{z_3 - z_2}{z_3 - z_1}, \tag{9}$$

where

$$w = p\zeta, \qquad p = \tfrac{1}{2}\sqrt{(z_4 - z_2)(z_3 - z_1)}, \qquad m = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_4 - z_2)(z_3 - z_1)}$$

and

$$\eta = [wz_1 + (z_2 - z_1)\Pi(n; w|m)]/p. \tag{10}$$

Here $\operatorname{sn}(w|m)$ is a Jacobian elliptic function and $\Pi(n; w|m)$ is the elliptic integral of the third kind. The second form of solution is

$$z = \frac{z_3 - z_4 \operatorname{sn}^2(w|m)}{1 - n \operatorname{sn}^2(w|m)} \qquad \text{with} \qquad n = \frac{z_3 - z_2}{z_4 - z_2} \tag{11}$$

and

$$\eta = [wz_4 + (z_4 - z_3)\Pi(n; w|m)]/p. \tag{12}$$

Hence the solution is given in parametric form with w as a parameter. The solutions are characterized by the two parameters A and B, or equivalently A and m.

3 Classification of solutions

The equation (7) is invariant under the transformation $z \to -z$, $c \to -c$; this corresponds to the transformation $u \to -u$, $v \to -v$. Hence we need to consider only the case $c = 1$ (i.e. $v > 0$).

Four cases are possible for the polynomial $f(z)$ corresponding to different ranges of values of A (see Fig. 2). Examples of the corresponding solutions are illustrated in Fig. 3. In the first column are periodic solutions with $m \neq 1$. In the second column are the corresponding limiting cases with $m = 1$. These are solitary wave solutions, except for the figure in the second line which is a periodic peakon solution.

4 Conclusion

Travelling-wave solutions of the Degasperis–Procesi equation are investigated. The solutions are characterized by two parameters. Hump-like, loop-like and coshoidal periodic-wave solutions are found; hump-like, loop-like and peakon solitary-wave solutions are obtained as well.

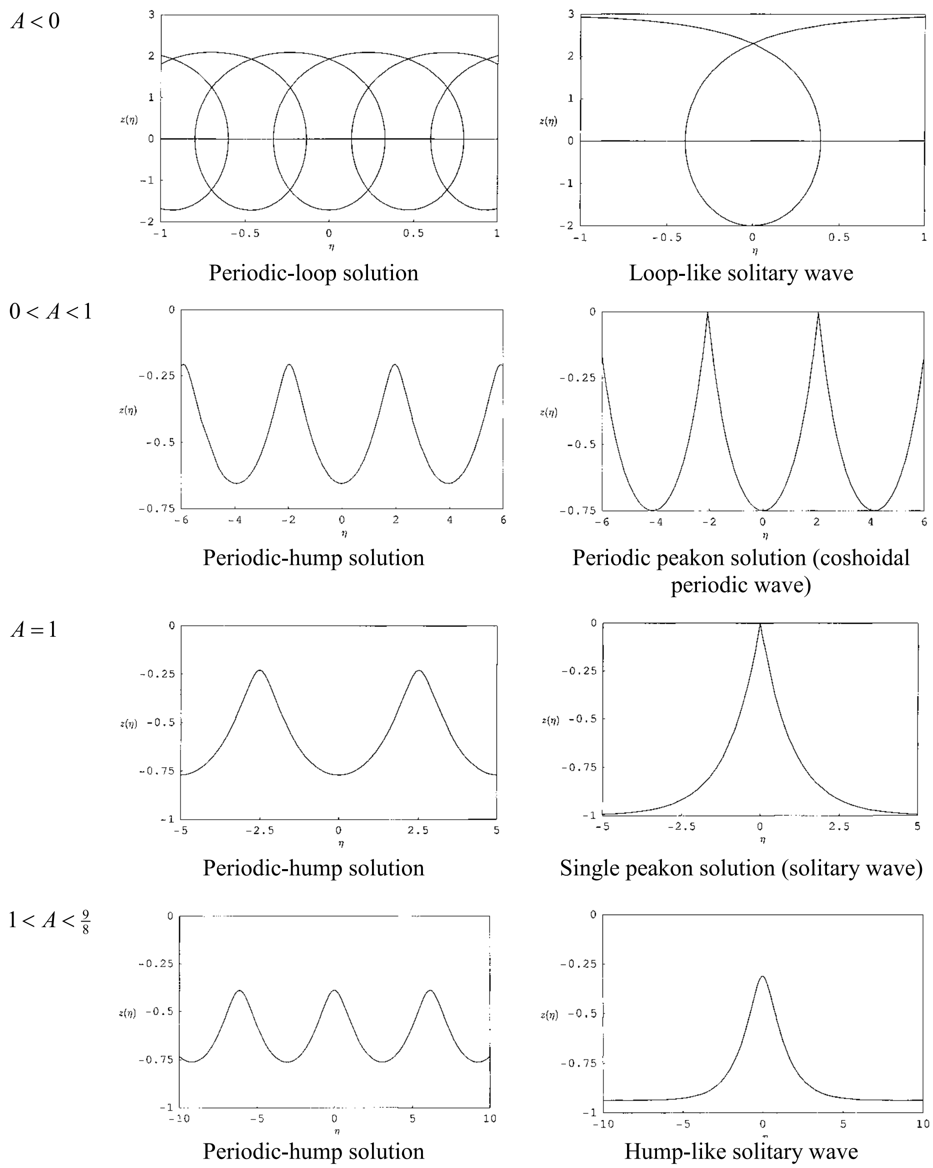

Figure 3. Travelling-wave solutions.

Acknowledgements

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