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Infinite Order Symmetries for Two-Dimensional Separable Schrödinger Equations

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Consider a non-relativistic Hamiltonian operator H in 2 dimensions consisting of a kinetic energy term plus a potential. We show that if the associated Schrödinger eigenvalue equation admits an orthogonal separation of variables, there is a calculus to describe the (in general) infinite-order differential operator symmetries of the Schrödinger equation. The calculus is formal but can be made rigorous when all functions in the eigenvaue equation are analytic. The infinite-order calculus exhibits structure that is not apparent when one studies only finite-order symmetries. The search for finite-order symmetries can then be reposed as one of looking for solutions of a coupled system of PDEs that are polynomial in certain parameters. We go further and extend the calculus to the situation where the Schrödinger equation admits a second-order symmetry operator, not necessarily associated with orthogonal separable coordinates.

1 Introduction

Consider a Schrödinger operator

$$H \equiv L_1 = \Delta_2 + V(\boldsymbol{x}),$$

where Δ_2 is the Laplace–Beltrami operator on a real or complex two-dimensional Riemannian space and $\boldsymbol{x} = (x_1, x_2)$ are orthogonal coordinates on that space such that the Schrödinger equation

$$H\Psi = E\Psi$$

separates multiplicatively in these coordinates. We here consider locally analytic solutions of this equation; application of boundary conditions is a separate step. We will introduce a calculus to describe the differential symmetry operators for this system, even those of infinite order. The question of finding finite order symmetries is reposed as a question of finding solutions of finite systems of PDEs that are polynomial in certain parameters. (See [1] where we introduced infinite-order conformal symmetry operators for the time-dependent Schrödinger equation in one space variable.) We give a number of examples where this method yields finite order differential symmetry operators. These questions as to when a system with two second-order constants of the motion, classical or quantum, (generated by an orthogonal separation of variables) admits additional polynomial constants of the motion are closely related to the concept of superintegrability [2–7]. We also consider all cases where the Schrödinger equation admits a second-order symmetry, not necessarily associated with orthogonal separable coordinates, or even separable coordinates at all. In each instance we construct additional symmetries.

2 Two-dimensional separable systems

Consider the case of orthogonal separable coordinates in a general Riemannian space. In the local separable coordinates $\{x, y\}$ the Schrödinger operator has the form

$$H = L_1 = \frac{1}{f_1(x) + f_2(y)} \left(\partial_x^2 + \partial_y^2 + v_1(x) + v_2(y) \right). \tag{1}$$

and, due to the separability, there is the invariant

$$L_2 = \frac{f_2(y)}{f_1(x) + f_2(y)} \left(\partial_x^2 + v_1(x) \right) - \frac{f_1(x)}{f_1(x) + f_2(y)} \left(\partial_y^2 + v_2(y) \right),$$

i.e., $[L_2, H] = 0$, [9, 10]. We have the operator identities

$$f_1(x)H + L_2 = \partial_x^2 + v_1(x), \qquad f_2(y)H - L_2 = \partial_y^2 + v_2(y). \tag{2}$$

We look for a partial differential operator $\tilde{L}(H, L_2, x, y)$ that satisfies

$$[H, \tilde{L}] = 0. \tag{3}$$

and $[L_2, \tilde{L}] \neq 0$. We require that the invariant take the standard form

$$\tilde{L}(H, L_2, x, y) = \sum_{j,k} \left(A_{j,k}(x, y)\partial_{xy} + B_{j,k}(x, y)\partial_x + C_{j,k}(x, y)\partial_y + D_{j,k}(x, y) \right) H^j L_2^k. \tag{4}$$

Note that if the formal operators (4) contained partial derivatives in x and y of orders ≥ 2 we could use the identities (2), recursively, and rearrange terms to achieve the unique standard form (4).

Using the operator identities

$$\partial_x H = H\partial_x - \frac{f_1'}{f_1 + f_2} H + \frac{v_1'}{f_1 + f_2}, \qquad \partial_y H = H\partial_y - \frac{f_2'}{f_1 + f_2} H + \frac{v_2'}{f_1 + f_2},$$
$$\partial_x L_2 = L_2\partial_x - \frac{f_1' f_2}{f_1 + f_2} H + \frac{f_2 v_1'}{f_1 + f_2}, \qquad \partial_y L_2 = L_2\partial_y + \frac{f_1 f_2'}{f_1 + f_2} H - \frac{f_1 v_2'}{f_1 + f_2},$$

we see that

$$\begin{aligned}
&(f_1(x) + f_2(y))[H, A(x, y)\partial_{xy} + B(x, y)\partial_x + C(x, y)\partial_y + D(x, y)] \\
&\quad = (A_{xx} + A_{yy} + 2B_y + 2C_x)\partial_{xy} + (B_{xx} + B_{yy} - 2A_y v_2 + 2D_x - Av_2')\partial_x \\
&\quad + (2A_y f_2 + Af_2')\partial_x H - 2A_y \partial_x L_2 + (C_{xx} + C_{yy} - 2A_x v_1 + 2D_y - Av_1')\partial_y \\
&\quad + (2A_x f_1 + Af_1')\partial_y H + 2A_x \partial_y L_2 + (D_{xx} + D_{yy} - 2B_x v_1 - 2C_y v_2 - Bv_1' - Cv_2') \\
&\quad + (2B_x f_1 + 2C_y f_2 + Bf_1' + Cf_2')H + (2B_x - 2C_y)L_2.
\end{aligned}$$

Thus the condition (3) for the symmetries of (1) is equivalent to the system of equations

$$\partial_{xx} A_{j,k} + \partial_{yy} A_{j,k} + 2\partial_y B_{j,k} + 2\partial_x C_{j,k} = 0, \tag{5}$$
$$\begin{aligned}
&\partial_{xx} B_{j,k} + \partial_{yy} B_{j,k} - 2\partial_y A_{j,k} v_2 + 2\partial_x D_{j,k} - A_{j,k} v_2' \\
&\quad + (2\partial_y A_{j-1,k} f_2 + A_{j-1,k} f_2') - 2\partial_y A_{j,k-1} = 0,
\end{aligned} \tag{6}$$
$$\begin{aligned}
&\partial_{xx} C_{j,k} + \partial_{yy} C_{j,k} - 2\partial_x A_{j,k} v_1 + 2\partial_y D_{j,k} - A_{j,k} v_1' \\
&\quad + (2\partial_x A_{j-1,k} f_1 + A_{j-1,k} f_1') + 2\partial_x A_{j,k-1} = 0,
\end{aligned} \tag{7}$$
$$\partial_{xx} D_{j,k} + \partial_{yy} D_{j,k} - 2\partial_x B_{j,k} v_1 - 2\partial_y C_{j,k} v_2 - B_{j,k} v_1' - C_{j,k} v_2' \tag{8}$$

$$+ (2\partial_x B_{j-1,k} f_1 + 2\partial_y C_{j-1,k} f_2 + B_{j-1,k} f_1' + C_{j-1,k} f_2') + (2\partial_x B_{j,k-1} - 2\partial_y C_{j,k-1}) = 0.$$

Note that condition (4) makes sense for infinite order differential equations, in domains where the functions f_j, v_j are analytic. Indeed, one can consider H, L_2 as parameters in these equations. Then once $\tilde{L}$ is expanded as a convergent power series in these parameters, the terms are reordered so that the powers of the parameters are on the right, before they are replaced by explicit differential operators. Alternatively one can consider the operator $\tilde{L}$ as acting on a simultaneous eigenbasis of the commuting operators H and L_2, in which case the parameters are the eigenvalues. In this view we can write

$$\tilde{L}(H, L_2, x, y) = A(x, y, H, L_2)\partial_{xy} + B(x, y, H, L_2)\partial_x + C(x, y, H, L_2)\partial_y + D(x, y, H, L_2),$$

and consider $\tilde{L}$ as an at most second-order order differential operator in x, y that is analytic in the parameters H, L_2. Then the above system (5), (6), (7), (8) can be written in the more compact form

$$A_{xx} + A_{yy} + 2B_y + 2C_x = 0, \tag{9}$$

$$B_{xx} + B_{yy} - 2A_y v_2 + 2D_x - Av_2' + (2A_y f_2 + Af_2')H - 2A_y L_2 = 0, \tag{10}$$

$$C_{xx} + C_{yy} - 2A_x v_1 + 2D_y - Av_1' + (2A_x f_1 + Af_1')H + 2A_x L_2 = 0, \tag{11}$$

$$D_{xx} + D_{yy} - 2B_x v_1 - 2C_y v_2 - Bv_1' - Cv_2' + (2B_x f_1 + 2C_y f_2 + Bf_1' + Cf_2')H + (2B_x - 2C_y)L_2 = 0. \tag{12}$$

and this system has many solutions.

Remark 1. Although our derivation has implicitly assumed analyticity of A, B, C, D in H, L_2 in a neighborhood of $(0, 0)$. It is easy to show that the same equations (9)–(12) arise if these functions are analytic in the neighborhood of any point (H^0, L_2^0).

We start with a very special case

$$A \equiv 0, \quad B = X(x, H, L_2), \quad C = Y(y, H, L_2), \quad D = \tilde{X}(x, H, L_2) + \tilde{Y}(y, H, L_2). \tag{13}$$

Then the above PDEs uncouple into ODEs for X and Y, whose structure we can easily analyse. The separated equation for X can be written in the compact form

$$X''' + 4(v_1 - f_1 H - L_2)X' + 2(v_1' - f_1' H)X = -2P(H, L_2), \qquad \tilde{X} = -\frac{1}{2}X', \tag{14}$$

where the separation constant $P(H, L_2)$ is a given analytic function of its arguments. (We can take it to be a polynomial.) The first equation (14) always has solutions for any f_1, v_1, say continuously differentiable. Thus we can always construct M and it will be analytic in the parameters H, L_2. Further we have the result

Lemma 1. *Let $\Psi_1(H, L_2, x)$, $\Psi_2(H, L_2, x)$ be a basis of solutions for the equation*

$$\left(\frac{d^2}{dx^2} + v_1(x) - f_1(x)H - L_2\right)\Psi(x) = 0.$$

Then $S_1(x) = \Psi_1^2$, $S_2(x) = \Psi_1\Psi_2$, $S_3(x) = \Psi_2^2$ is a basis of solutions for the homogeneous equation

$$S''' + 4(v_1 - f_1 H - L_2)S' + 2(v_1' - f_1' H)S = 0. \tag{15}$$

Thus the general solution of the first equation (14) is a particular solution of this equation plus an arbitrary linear combination of S_1, S_2, S_3.

Similarly, the separation equation for $C = Y$ is

$$Y''' + 4(v_2 - f_2 H + L_2)Y' + 2(v_2' - f_2' H)Y = 2P(H, L_2), \qquad \tilde{Y} = -\frac{1}{2}Y'.$$

Once we have obtained X and Y, then we see that the corresponding operator L_3 commutes with H. Thus we can view L_3 as an infinite order differential symmetry operator for H. In special cases this will be a finite order operator.

It is important to note that (for $P \neq 0$) L_3 is not just a function of H and L_2. Indeed, a straightforward computation yields $[L_2, L_3] = P(H, L_2) \neq 0$. In fact, an analogous constructions to the above but based on the operator L_2 yields an operator L_4 such that $H = L_1$, L_2, L_3, L_4 satisfy the commutation relations

$$[L_1, L_2] = [L_1, L_3] = [L_2, L_4] = [L_3, L_4] = 0, \qquad [L_2, L_3] = [L_1, L_4] = P(L_1, L_2).$$

If we choose $P(L_1, L_2) = I$, the identity operator, then these are the canonical commutation relations.

Example 1. We consider Cartesian coordinates in flat space and assume that the potential is separable in these coordinates. Thus we have $f_1(x) = f_2(y) = \frac{1}{2}$ and

$$H = \partial_x^2 + \partial_y^2 + v_1(x) + v_2(y), \qquad L_2 = \frac{1}{2}(\partial_x^2 + v_1(x)) - \frac{1}{2}(\partial_y^2 + v_2(y)).$$

We look for a 3rd order constant of the motion L_3. Thus we have

$$P(H, L_2) = \sum_{j+k\leq 2} \alpha_{jk} H^j L_2^k$$

and the x-dependent part of the symmetry operator must take the form

$$M = (X_{10}\partial_x + \tilde{X}_{10})H + (X_{01}\partial_x + \tilde{X}_{01})L_2 + (X_{00}\partial_x + \tilde{X}_{00}), \qquad \tilde{X}_{jk} = -\frac{1}{2}X'_{jk}.$$

This leads to the system of equations (labeled by the powers (j, k) of $H^j L_2^k$)

$$\begin{aligned}
&(2,0) \quad X'_{10} = \alpha_{20},\\
&(1,1) \quad X'_{01} + 2X'_{10} = \alpha_{11},\\
&(0,2) \quad 2X'_{01} = \alpha_{02},\\
&(1,0) \quad -\frac{1}{2}X'''_{10} - v_1 X_{10} - 2v_1 X'_{10} + X'_{00} = \alpha_{10},\\
&(0,1) \quad -\frac{1}{2}X'''_{01} - v_1 X_{01} - 2v_1 X'_{01} + 2X'_{00} = \alpha_{01},\\
&(0,0) \quad -\frac{1}{2}X'''_{00} - v_1 X_{00} - 2v_1 X'_{00} = \alpha_{00}.
\end{aligned} \tag{16}$$

These equations are equivalent to

$$\begin{aligned}
&X_{10} = \alpha_{20}x + c_{10}, \qquad X_{01} = \frac{1}{2}\alpha_{02}x + c_{01}, \qquad \alpha_{02} + 4\alpha_{20} = 2\alpha_{11},\\
&X'_{00} = (\alpha_{20}x + c_{10})v_1' + 2\alpha_{20}v_1 + \alpha_{10},\\
&2X'_{00} = \left(\frac{1}{2}\alpha_{02}x + c_{01}\right)v_1' + \alpha_{02}v_1 + \alpha_{01}, \qquad -\frac{1}{2}X'''_{00} - v_1 X_{00} - 2v_1 X'_{00} = \alpha_{00}.
\end{aligned}$$

A very similar computation, with the same polynomial $P(H, L_2)$, yields the possibilities for $v_2(y)$, and the construction of the y-dependent part of the symmetry operator L_3. Then L_3 is a third-order quantum constant of the motion.

3 The general case

Up to now we have only considered the special case (13): $A = 0, B = X(x)$, $C = Y(y)$, $D = \tilde{X}(x) + \tilde{Y}(y)$ of conditions (9)–(12). Let us now consider the case such that $A \equiv 0$, but, otherwise, B, C, D are arbitrary. Then there is a function $G(x, y, H, L_2)$ such that

$$B = -\partial_x G, \qquad C = \partial_y G,$$

and the determining conditions simplify to

$$\begin{aligned}
&1)\ \ G_{xxxy} + G_{xyyy} = 0,\\
&2)\ \ \frac{1}{2}G_{xxxx} + 2G_{xx}v_1 + G_x v_1' - (2G_{xx}f_1 + G_x f_1')H - 2G_{xx}L_2\\
&\qquad = \frac{1}{2}G_{yyyy} + 2G_{yy}v_2 + G_y v_2' - (2G_{yy}f_2 + G_y f_2')H + 2G_{yy}L_2.
\end{aligned}$$

The first determining equation means that $G(x, y) = K(x, y) + F(x) + J(y)$ where F, J are arbitrary and K is harmonic: $K_{xx} + K_{yy} = 0$. This representation is unique in K, F, J, up to the addition of the harmonic separable function $\tilde{K}(x, y) = \frac{a}{2}(x^2 - y^2) + bx + cy + d$. Alternatively, we can write

$$G(x, y) = z_1(x + iy) + z_2(x - iy) + F(x) + J(y),$$

where z_1, z_2 are arbitrary analytic functions. Then only condition 2) remains to be satisfied.

Example 2. If we make the ansatz $G = X(x, H, L_2)Y(y, H, L_2)$ then, in addition to the well known angular momentum invariant, we find the following two polynomial invariants:

1. $X = \left(\frac{1}{4} + L_2\right)\cos x + s(1 + \beta H), \qquad Y = \left(\frac{1}{4} + L_2\right)\cosh y + t(1 + \xi H),$

   $v_1(x) = 2s\frac{\sin x}{\cos^2 x} + \frac{a_1}{\cos^2 x}, \qquad f_1(x) = -2s\beta\frac{\sin x}{\cos^2 x} + \frac{a_2}{\cos^2 x},$

   $v_2(y) = 2t\frac{\sinh y}{\cosh^2 y} + \frac{b_1}{\cosh^2 y}, \qquad f_2(y) = -2t\xi\frac{\sinh y}{\cosh^2 y} + \frac{b_2}{\cosh^2 y},$

   $D = -\frac{1}{2}(\frac{1}{4} + L_2)\left(t\cos x(1 + \xi H) + s\cosh y(1 + \beta H)\right).$

2. $\tilde{L} = -2x(y^2 + 4L_2)\partial_x + 2y(x^2 - 4L_2)\partial_y + x^2 - y^2,$

   $v_1(x) = \frac{1}{8}x^2 + \frac{a_1}{x^2}, \qquad f_1(x) = \frac{a_2}{x^2}, \qquad v_2(y) = \frac{1}{8}y^2 + \frac{b_1}{y^2}, \qquad f_2(y) = \frac{b_2}{y^2}.$

Example 3. Again we consider the special case of conditions (9)–(12) such that $A \equiv 0$. Now we require

$$G(x, y) = -2\log(X(x) + Y(y)) + \mathcal{F}(x) + \mathcal{J}(y) = K(x, y) + F(x) + J(y),$$

where F, J are arbitrary and K is harmonic. Then the harmonic requirement on K implies that

$$K = -2\log(X + Y) + \tilde{F}(x) + \tilde{J}(y),$$

where

$$\begin{aligned}
(X')^2 &= \frac{\alpha}{12}X^4 + \frac{\beta}{3}X^3 + \gamma X^2 + 2\delta X + \phi,\\
(Y')^2 &= -\frac{\alpha}{12}Y^4 + \frac{\beta}{3}Y^3 - \gamma Y^2 + 2\delta Y - \phi,
\end{aligned}$$

$$X'' = \frac{\alpha}{6}X^3 + \frac{\beta}{2}X^2 + \gamma X + \delta, \qquad Y'' = -\frac{\alpha}{6}Y^3 + \frac{\beta}{2}Y^2 - \gamma Y + \delta.$$

Further,

$$\tilde F(x) = \frac{1}{3}\frac{X'''}{X'}, \qquad \tilde J(y) = \frac{1}{3}\frac{Y'''}{Y'},$$

and the metric and potential terms have the solution

$$v_1 - f_1 H = \frac{-\frac{a}{12}X^4 - \frac{b}{3}X^3 + \frac{b_1}{2}X^2 + \eta_1 X + \eta_2}{24(X')^2},$$
$$v_2 - f_2 H = \frac{\frac{a}{12}Y^4 - \frac{b}{3}Y^3 - \frac{b_1}{2}Y^2 + \eta_1 Y - \eta_2}{24(Y')^2}.$$

Here, α, β, γ, δ, ϕ and

$$a = a^{(1)} + a^{(2)}H, \qquad b = b^{(1)} + b^{(2)}H, \qquad b_1 = b_1^{(1)} + b_1^{(2)}H,$$
$$\eta_1 = \eta_1^{(1)} + \eta_1^{(2)}H, \qquad \eta_2 = \eta_2^{(1)} + \eta_2^{(2)}H$$

are parameters. The remaining condition is

$$\begin{aligned}&\frac{1}{2}F'''' + 2F''(v_1 - f_1H - L_2) + F'(v_1 - f_1'H) - \frac{1}{2}J'''' - 2J''(v_2 - f_2H - L_2) - J'(v_2' - f_2'H)\\ &\quad = \frac{1}{36}\left(\frac{a}{2}X^2 + bX - \frac{a}{2}Y^2 + bY\right) + \frac{2}{3}\left(\frac{X'''}{X'}(v_1 - f_1H) - \frac{Y'''}{Y'}(v_2 - f_2H)\right)\\ &\quad + \tilde F'(v_1' - f_1'H) - \tilde J'(v_2' - f_2'H).\end{aligned}$$

The simplest family of solutions is obtained by setting $F \equiv \tilde F$, $J \equiv \tilde J$ and $\alpha = \beta = a = b = 0$.

Now we consider the general case of conditions (9)–(12). Then there are two functions $F(x, y, H, L_2)$, $G(x, y, H, L_2)$ such that

$$A = \partial_{xy}F, \qquad B = -\frac{1}{2}\partial_{xyy}F - \partial_x G, \qquad C = -\frac{1}{2}\partial_{xxy}F + \partial_y G,$$

and the determining conditions simplify to

$$\begin{aligned}1)\ \ &2G_{xyyy} + \frac{1}{2}F_{xyyyyy} + 2F_{xyyy}(v_2 - f_2H + L_2) + 3F_{xyy}(v_2' - f_2'H) + F_{xy}(v_2'' - f_2''H)\\ &= -2G_{xxxy} + \frac{1}{2}F_{xxxxxy} + 2F_{xxxy}(v_1 - f_1H - L_2)\\ &+ 3F_{xxy}(v_1' - f_1'H) + F_{xy}(v_1'' - f_1''H),\\ 2)\ \ &\frac{1}{2}F_{xxxxyy} + 2F_{xxyy}(v_1 - f_1H) + F_{xxy}(v_2' - f_2'H) + \frac{1}{2}G_{xxxx}\\ &+ 2G_{xx}(v_1 - f_1H - L_2) + G_x(v_1' - f_1'H)\\ &= -\frac{1}{2}F_{xxyyyy} - 2F_{xxyy}(v_2 - f_2H) - F_{xyy}(v_1' - f_1'H) + \frac{1}{2}G_{yyyy}\\ &+ 2G_{yy}(v_2 - f_2H + L_2) + G_y(v_2' - f_2'H).\end{aligned}$$

Example 4. Consider the problem of finding all third-order symmetry operators corresponding to potentials that separate in Cartesian coordinates in Euclidean space. Then $f_1 = f_2 = \frac{1}{2}$. Third-order symmetry operators correspond to $F = F(x, y)$ and $G = g(x, y)L_1 + h(x, y)L_2 + k(x, y)$. Now let $f(x, y) = F_{xy}$, and substitute these expressions into conditions 1) and 2).

Equating coefficients of L_1^2, L_2^2, L_1L_2 in 1) and coefficients of L_1, L_2 in 2), we obtain easily the expressions

$$\begin{aligned}
f(x,y) &= e_{00} + e_{10}x + e_{01}y + \frac{c}{2}(x^2 - y^2) + e_{11}xy, \\
g(x,y) &= c_{00} + c_{10}x + c_{01}y + c_{20}(x^2 + y^2) + c_{11}xy + c_{30}(x^3 + 3xy^2) \\
&\quad + c_{21}\left(\frac{1}{3}y^3 + x^2y\right) - \frac{c}{12}\left(x^3y + xy^3\right), \\
h(x,y) &= d_{00} + d_{10}x + d_{01}y - 2c_{20}\left(x^2 - y^2\right) + d_{11}xy - 2c_{30}\left(x^3 - 3xy^2\right) \\
&\quad - 2c_{21}\left(x^2y - \frac{1}{3}y^3\right) + \frac{c}{6}(x^3y - xy^3),
\end{aligned}$$

where e_{ij}, c_{ij}, d_{ij}, c are constants. Equating coefficients of L_1 and L_2 in 1) and integrating, we can get expressions for k_x and k_y. The integrability condition $\partial_x k_y = \partial_y k_x$, together with equating coefficients of the constant term in 2) leads to a system of ordinary differential equations for the potential terms:

$$\begin{aligned}
&\left(\frac{1}{2}d_{11} - c_{11} - 2c_{21}x + \frac{1}{2}cx^2\right)v_1'' + (-12c_{21} + 3cx)v_1' + 3cv_1 = -\ell_5 - \ell_4 x + \frac{1}{2}\ell_1 x^2, \\
&\left(\frac{1}{2}d_{10} - c_{10} - 4c_{20}x - 6c_{30}x^2\right)v_1'' - (12c_{20} + 36c_{30}x)v_1' - 36c_{30}v_1 = -m_2 - m_1 x + \frac{1}{2}\ell_2 x^2, \\
&(12c_{30} - e_{01} + e_{11}x)v_1'' - 3e_{11}v_1' = -q_1 x + q_3, \\
&(d_{11} - e_{00} - e_{10}x - 4c_{21}x)v_1'' + (-3e_{10} - 4c_{21})v_1' = -q_2 x + q_4,
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{1}{2}d_{11} + c_{11} + 12c_{30}y - \frac{1}{6}cy^2)v_2'' + (36c_{30} - \frac{7}{3}cy\right)v_2' - 3cv_2 = -\ell_3 - \ell_2 y - \frac{1}{2}\ell_1 y^2, \\
&\left(\frac{1}{2}d_{01} + c_{01} - 4c_{20}y - 2c_{21}y^2\right)v_2'' + (12c_{20} + 12c_{21}y)v_2' + 12c_{21}v_2 = -m_3 + m_1 y + \frac{1}{2}\ell_4 y^2, \\
&(4c_{21} + e_{10} + e_{11}y)v_2'' + 3e_{11}v_2' = q_1 y + q_2, \\
&\left(e_{00} + e_{01}y - 12c_{30}y - \frac{1}{3}cy^2\right)v_2'' + \left(3e_{01} - 36c_{30} - \frac{2}{3}cy\right)v_2' = -q_3 y - q_4.
\end{aligned}$$

Here, ℓ_j, m_j, q_j are constants. The strategy is now clear. The f, g, h expressions simply define the higher order (potential-independent) terms in the Euclidean symmetry operators and are easy to understand. For a true third-order symmetry h_x, h_y, g_x, g_y cannot all vanish. For each such choice of terms we can solve explicitly the ODEs for v_1 and v_2. Since the same v_1, v_2 have to satisfy all these equations, this puts restrictions on the constants. Then we verify that each common solution v_1, v_2 actually corresponds to a 3rd order symmetry. A tedious case-by-case procedure leads to all solutions, provided not all of the ODEs for either v_1 or v_2 are vacuous. If, say, the ODEs for v_1 are vacuous, then we must make use of the remaining integrability condition. Equating coefficients of the constant term in 1) leads to a coupled partial differential equation for the potential terms v_1, v_2

$$\begin{aligned}
0 = \Bigg[& - 2(4c_{20} + 12c_{30}x + 4c_{21}y - cxy)v_2 \\
& + \left(\frac{1}{2}d_{10} + c_{10} + \frac{1}{2}d_{11}y + c_{11}y + 6c_{30}y^2 - \frac{1}{6}cy^3\right)v_1'
\end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{1}{2}d_{01} + c_{01} + 4c_{20}y + \frac{1}{2}d_{11}x + c_{11}x + 12c_{30}xy + 2c_{21}y^2 - \frac{1}{6}cxy^2\right) v_2'\Bigg] v_1 \\
&+ \Bigg[- \left(4c_{20}x + 6c_{30}x^2 + 4c_{21}xy - \frac{1}{2}cx^2y\right) v_2 \\
&+ \frac{1}{2}\left(\frac{1}{2}d_{10} + c_{10} + \frac{1}{2}d_{11}y + c_{11}y + 6c_{30}y^2 - \frac{1}{6}cy^3\right) v_1 \\
&- \frac{1}{2}\left(\frac{1}{2}d_{01}x + c_{01}x + 4c_{20}xy + \frac{1}{4}d_{11}x^2 + \frac{1}{2}c_{11}x^2 + 6c_{30}x^2y + 2c_{21}xy^2 - \frac{1}{12}cx^2y^2\right) v_2' \\
&- \frac{1}{4}m_2y^2 - \frac{1}{12}y^3\ell_5 - \frac{1}{2}\left(\frac{1}{2}d_{11} - c_{11}\right) V_2 + \frac{1}{2}n_2y + \frac{1}{2}p_2 + e_{01} - cy\Bigg] v_1' \\
&+ \Bigg[2(4c_{20} + 12c_{30}x + 4c_{21}y - cxy)v_1 \\
&+ \left(\frac{1}{2}d_{01} - c_{01} + \frac{1}{2}d_{11}x - c_{11}x - 2c_{21}x^2 + \frac{1}{6}cx^3\right) v_2' \\
&- \left(\frac{1}{2}d_{10} - c_{10} - 4c_{20}x + \frac{1}{2}d_{11}y - c_{11}y - 6c_{30}x^2 - 4c_{21}xy + \frac{1}{2}cx^2y\right) v_1'\Bigg] v_2 \\
&+ \Bigg[\left(4c_{20}y + 12c_{30}xy + 2c_{21}y^2 - \frac{1}{2}cxy^2\right) v_1 \\
&+ \frac{1}{2}\left(\frac{1}{2}d_{01} - c_{01} + \frac{1}{2}d_{11}x - c_{11}x - 2c_{21}x^2 + \frac{1}{6}cx^3\right) v_2 \\
&- \frac{1}{2}\left(\frac{1}{2}d_{10}y - c_{10}y - 4c_{20}xy + \frac{1}{4}d_{11}y^2 - \frac{1}{2}c_{11}y^2 - 6c_{30}x^2y - 2c_{21}xy^2 + \frac{1}{4}cx^2y^2\right) v_1' \\
&- \frac{1}{4}m_3x^2 - \frac{1}{12}x^3\ell_3 - \frac{1}{2}\left(\frac{1}{2}d_{11} + c_{11}\right) V_1 - \frac{1}{2}n_1x - \frac{1}{2}p_1 + e_{10} + cx\Bigg] v_2' \\
&+ \frac{1}{4}\left(\frac{1}{2}d_{10} + c_{10} + \frac{1}{2}d_{11}y + c_{11}y + 6c_{30}y^2 - \frac{1}{6}cy^3\right) v_1''' \\
&+ \frac{1}{4}\left(\frac{1}{2}d_{01} - c_{01} + \frac{1}{2}d_{11}x - c_{11}x - 2c_{21}x^2 + \frac{1}{6}cx^3\right) v_2'''. \qquad (17)
\end{aligned}
$$

Here, $v_j = V_j'$. This equation is complicated, but if the ODEs for v_1 are vacuous, then v_2 is a second-order polynomial in y and all constants in the above equation must be zero, except, d_{11}, d_{01}, d_{10}, ℓ_3, m_3, n_1, n_2, p_1, p_2. Thus v_1 must satisfy a simple nonlinear ODE and all solutions can be obtained. The details can be found in [12], with a different method. (That paper is focused on 3rd order invariants alone, rather than considering them as a special case of infinite-order invariants as is done here.)

Factorized solutions for the general conditions 1), 2), exist for all spaces and separable potentials.

Theorem 1. *For any v_1, v_2, f_1, f_2 there are always solutions for equations 1), 2) in which $A \not\equiv 0$, $G \equiv 0$ and F factors as $F = \mathcal{X}(x, H, L_2)\mathcal{Y}(y, H, L_2)$, where $\mathcal{X}'\mathcal{Y}' \neq 0$.*

Proof. Making the indicated substitutions, we see that the equations reduce to

$$
1)\quad \frac{1}{2}\mathcal{X}^{(1)}\mathcal{Y}^{(5)} + 2\mathcal{X}^{(1)}\mathcal{Y}^{(3)}(v_2 - f_2H + L_2) + 3\mathcal{X}^{(1)}\mathcal{Y}^{(2)}(v_2' - f_2'H) + \mathcal{X}^{(1)}\mathcal{Y}^{(1)}(v_2'' - f_2''H)
$$

$$= \frac{1}{2}\mathcal{X}^{(5)}\mathcal{Y}^{(1)} + 2\mathcal{X}^{(3)}\mathcal{Y}^{(1)}(v_1 - f_1H - L_2) + 3\mathcal{X}^{(2)}\mathcal{Y}^{(1)}(v_1' - f_1'H)$$
$$+ \mathcal{X}^{(1)}\mathcal{Y}^{(1)}(v_1'' - f_1''H),$$
$$2)\quad \frac{1}{2}\mathcal{X}^{(4)}\mathcal{Y}^{(2)} + \frac{1}{2}\mathcal{X}^{(2)}\mathcal{Y}^{(4)} + 2\mathcal{X}^{(2)}\mathcal{Y}^{(2)}(v_1 + v_2 - f_1H - f_2H)$$
$$+ \mathcal{X}^{(2)}\mathcal{Y}^{(1)}(v_2' - f_2'H) + \mathcal{X}^{(1)}\mathcal{Y}^{(2)}(v_1' - f_1'H) = 0.$$

Variables separate in these equations and we have

$$1a)\quad \frac{1}{2}\mathcal{X}^{(5)} + 2\mathcal{X}^{(3)}(v_1 - f_1H - L_2) + 3\mathcal{X}^{(2)}(v_1' - f_1'H) + \mathcal{X}^{(1)}(v_1'' - f_1''H) = \alpha\mathcal{X}^{(1)},$$
$$1b)\quad \frac{1}{2}\mathcal{Y}^{(5)} + 2\mathcal{Y}^{(3)}(v_2 - f_2H + L_2) + 3\mathcal{Y}^{(2)}(v_2' - f_2'H) + \mathcal{Y}^{(1)}(v_2'' - f_2''H) = \alpha\mathcal{Y}^{(1)},$$

where α is a constant. Similarly, if $\mathcal{X}^{(2)}\mathcal{Y}^{(2)} \neq 0$ we have the separation equations

$$2a)\quad \frac{1}{2}\mathcal{X}^{(4)} + 2\mathcal{X}^{(2)}(v_1 - f_1H - L_2) + \mathcal{X}^{(1)}(v_1' - f_1'H) = \beta\mathcal{X}^{(2)},$$
$$2b)\quad \frac{1}{2}\mathcal{Y}^{(4)} + 2\mathcal{Y}^{(2)}(v_2 - f_2H + L_2) + \mathcal{Y}^{(1)}(v_2' - f_2'H) = -\beta\mathcal{Y}^{(2)},$$

These equations are consistent if

$$A)\ \alpha = \beta = 0 \qquad \text{or} \qquad B)\ \alpha\mathcal{X}' = \beta\mathcal{X}''', \quad \alpha\mathcal{Y}' = -\beta\mathcal{Y}'''.$$

Now set $X = \mathcal{X}'$, $Y = \mathcal{Y}'$. Considering case A) first, we see that we have a solution of equations (9)–(12) whenever $X'Y' \neq 0$ and X and Y satisfy the ordinary differential equations

$$X''' + 4X'(v_1 - f_1H - L_2) + 2X(v_1' - f_1'H) = 0,$$
$$Y''' + 4Y'(v_2 - f_2H + L_2) + 2Y(v_2' - f_2'H) = 0,$$

essentially the same as the third order homogeneous ordinary differential equation (15).

Now suppose that $X'Y' \equiv 0$, but $X' \neq 0$. Then we have $v_2' - f_2'H = 0$ so v_2', f_2' are constants. Further, X satisfies the ordinary differential equation

$$\frac{1}{2}X'''' + 2X''(v_1 - f_1L - L_2) + 3X'(v_1' - f_1'H) + X'(v_1'' - f_1''H) = 0.$$

Finally, if $X = Y = 1$ we have $v_2'' - f_2''H = v_1'' - f_1''H = \alpha(H)$, corresponding to oscillator potentials.

For case B) we can assume $\beta = 1$. Then we find, with $\alpha = a^2$,

$$X(x) = c_1\sinh(ax) + c_2\cosh(ax), \quad f_1(x) = \frac{c_3}{X(x)^2}, \quad v_1(x) = \frac{4L_2 + 2 - a^2}{4} + \frac{c_4}{X(x)^2},$$
$$Y(y) = d_1\sin(ax) + d_2\cos(ax), \quad f_2(y) = \frac{d_3}{Y(y)^2}, \quad v_2(y) = -\frac{4L_2 + 2 - a^2}{4} + \frac{d_4}{Y(y)^2}. \quad \blacksquare$$

Remark 2. The underlying structure of the solutions of the general equations (9)–(12) is fairly simple. Let $u_1(x, L_2) = u_1[L_2]$, $u_2(x, L_2) = u_2[L_2]$ be a basis of solutions of the separated equation

$$\left(\frac{d^2}{dx^2} + v_1(x) - f_1(x)H - L_2\right)u = 0,$$

and let $w_1(y, L_2)$, $w_2(y, L_2)$ be a basis of solutions of the separated equation

$$\left(\frac{d^2}{dy^2} + v_2(y) - f_2(y)H - L_2\right) w = 0.$$

Then for any parameter $\hat{L}_2$ the operator

$$S(\hat{L}_2) = w_2[\hat{L}_2]u_2[\hat{L}_2]\left(w_1[L_2]u_1[L_2]\partial_{xy} - w_1'[L_2]u_1[L_2]\partial_x - w_1[L_2]u_1'[L_2]\partial_y + w_1'[L_2]u_1'[L_2]\right)$$

is a symmetry operator of L_1 that maps any eigenspace of L_2 into another (generally different) eigenspace. It is not hard to characterize the space spanned by all linear combinations of functions $w_2[\hat{L}_2]u_2[\hat{L}_2]w_1[L_2]u_1[L_2]$ and this gives the equations for A. Similarly we can characterize B, C, and D. Also the solutions of (9)–(12) in the special case $A \equiv 0$ can be obtained by taking limits of the above general solutions. Indeed

$$\begin{aligned}\tilde{S}(\hat{L}_2) = {} & (\partial_L w_2[\hat{L}_2]u_2[\hat{L}_2] + w_2[\hat{L}_2]\partial_l u_2[\hat{L}_2]) \\ & \times \left(-w_1'[L_2]u_1[L_2]\partial_x - w_1[L_2]u_1'[L_2]\partial_y + w_1'[L_2]u_1'[L_2]\right) \\ & + (\partial_L w_1[\hat{L}_2]u_1[\hat{L}_2] + w_1[\hat{L}_2]\partial_l u_1[\hat{L}_2]) \\ & \times \left(-w_2'[L_2]u_2[L_2]\partial_x - w_2[L_2]u_2'[L_2]\partial_y + w_2'[L_2]u_2'[L_2]\right) \\ & - (\partial_L w_2[\hat{L}_2]u_1[\hat{L}_2] + w_2[\hat{L}_2]\partial_l u_1[\hat{L}_2]) \\ & \times \left(-w_1'[L_2]u_2[L_2]\partial_x - w_1[L_2]u_2'[L_2]\partial_y + w_1'[L_2]u_2'[L_2]\right) \\ & - (\partial_L w_1[\hat{L}_2]u_2[\hat{L}_2] + w_1[\hat{L}_2]\partial_l u_2[\hat{L}_2]) \\ & \times \left(-w_2'[L_2]u_1[L_2]\partial_x - w_2[L_2]u_1'[L_2]\partial_y + w_2'[L_2]u_1'[L_2]\right)\end{aligned}$$

is a symmetry with the right properties.

4 Lie form and nonorthogonal separation in two dimensions

We know that if a Hamiltonian $H = \sum_{i,j=1}^{2} g^{ij}p_ip_j$ admits a constant of the motion L that is quadratic in the momenta

$$L = \sum_{i,j=1}^{2} a^{ij}p_ip_j, \qquad \{H, L\} = 0$$

and if the roots of the determinant $|a^{ij} - \lambda g^{ij}|$ are distinct, then the eigenforms define new (separable) variables x_1, x_2 and the associated Schrödinger operator can be written in Liouville form

$$H = \frac{1}{f_1(x_1) + f_2(x_2)}\left(\partial_{x_1x_1} + \partial_{x_2x_2} + v_1(x_1) + v_2(x_2)\right).$$

However, it may be that the roots of this determinant are equal. In this case H cannot be put into Liouville form, but rather Lie form, which for a suitable choice of variables (non-separable) is

$$H = \frac{1}{x + B(y)}\left(\partial_{xy} + \frac{1}{2}K(y)\right) + \frac{1}{2}U'(y).$$

The associated quantum constant is $L = \partial_{xx} - 2yH + U(y)$.

How can we calculate a third symmetry? We look for a quantum constant of the form

$$L^o = M(H, L, y)\partial_y + N(H, L, y).$$

Applying the condition $[H, L^o] = 0$, we see that the functions M and N are of the form

$$M = \frac{1}{2H - U'(y)},$$
$$N = -\frac{1}{2}\int \frac{U''K - 2K'H + K'U' + 4B'H^2 + 4B'U'H + B'U'^2}{\sqrt{L + 2yH - U}(2H - U')^2} dy.$$

According to our operational calculus, these relations make sense when the operators are applied to a simultaneous eigenspace of H and L. Note in particular that if we consider the free Hamiltonian then L^o has the particularly simple form

$$L^o = H \int \frac{B'(y)}{\sqrt{L + 2yH}} dy - \partial_y.$$

While this appears to be only a formal expression it makes good sense for a spectral resolution analytic in a neighborhood of $(H_0, L_0) \neq (0, 0)$. Indeed, the following explicit example is informative.

Example 5. Consider the zero potential case $K = U \equiv 0$, $B(y) = y^2$. We can formally evaluate the integral in the expression for L^o by integrating by parts, and then multiply by $3H$ to obtain

$$L^i = 2(L + 2yH)^{1/2}(-L + yH) - 3H\partial_y.$$

This expression can immediately be interpreted as the differential operator

$$\hat{L} = 2\partial_x(-L + yH) - 3\partial_y H = -2\partial_x^3 + \frac{1}{x + y^2}(6y\partial_x^2\partial_y - 3\partial_y^2\partial_x)$$

which can be verified to commute with H. Indeed, if $B(y)$ is a polynomial then, through integration by parts, we can always uncover a symmetry operator of finite order. In this particular example the Hamiltonian also admits a second order symmetry operator

$$\hat{N} = x\partial_x^2 + \frac{3}{4}\partial_y^2 - \frac{3xy + \frac{1}{3}y^3}{x + y^2}\partial_x\partial_y,$$

and $[\hat{N}, L] = \hat{L}$. However, for general polynomial $B(y)$ the corresponding invariant $\hat{L}$ cannot be obtained as a commutator of other finite differential invariants.

It is clear from the method of the example that if one takes $U(y)$ as a constant and $B(y)$ and $K(y)$ as polynomials then, as before, we can generate explicit finite order differential operators that commute with H.

There is one remaining possibility for a quadratic constant of the motion in two dimensions: the constant may be associated with *nonorthogonal* separation of variables. In two dimensions there is only one case: separation in light cone (null) coordinates, [11]. For this case the Hamiltonian takes the form

$$H = \partial_z\partial_{\bar{z}} + f(\bar{z})$$

and there is a first-order symmetry operator ∂_z, so ∂_{zz} is a second-order constant of the motion. In addition there is a quadratic constant

$$L = M\partial_z + \frac{i}{2}\int \bar{z}\frac{df}{d\bar{z}}d\bar{z}.$$

where $M = i(z\partial_z - \bar{z}\partial_{\bar{z}})$.

5 Further work

A particular feature of our approach is that we can replace the partial differential operators L_1, L_2 (that do not commute with the underlying variables x, y) by scalar parameters that commute with all variables. It provides a relatively simple tool to uncover invariants. Encouraged by the theorem above, we expect to be able to characterize the structure of the algebra of infinite dimensional symmetries of L_1. Of particular interest would be conditions that would guarantee that the symmetry conditions had solutions that were polynomials in the parameters.

In future papers we will show that there is a similar calculus to express differential symmetries of the time-independent Schrödinger equation on an n-dimensional Riemannian manifold, when the equation admits R-separation in an orthogonal coordinate system. Further there is a calculus to express differential conformal symmetries of the time-dependent Schrödinger equation on an n-dimensional Riemannian manifold, when the equation admits R-separation. The time-dependent case is particularly interesting because some of its conformal symmetries can be considered as energy-shifting operators for the time-independent equations. These energy-shifting operators are usually of infinite order, but appear first order in our canonical form. They correspond to raising and lowering operators in the equations of mathematical physics.

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