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Symmetries and Supersymmetries in Trapped Ion Hamiltonian Models

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The implementation of a vibronic coupling in the context of trapped ions, and its diagonalization are reported. The specific spin-boson-like Hamiltonian model is studied exploiting its symmetries and supersymmetries. Applications are discussed.

1 Introduction

Trapped ions provide an interesting physical scenario wherein fundamental aspects of quantum mechanics may be *tested* and useful applications in *quantum computation* may be realized [1].

Generally speaking, a time dependent quadrupolar electric field is responsible for a charged particle dynamics which may be *effectively described* as the motion of a point into a quadratic potential [2]. When the harmonically confined particle is an ion, the system we obtain possesses both bosonic and fermionic degrees of freedom, the first corresponding to the ion center of mass motion, the second describing the atomic internal state, i.e. the electronic configuration. In most of the cases, in trapped ion dynamics only two atomic levels are involved, making the Pauli operator two-level system description effective.

Once the ion has been confined, acting upon it via laser field it is possible to implement vibronic couplings (i.e. involving both bosonic and fermionic degrees of freedom) practically at will [3]. Unfortunately, sometimes, the vibronic coupling problem is not solvable even if implementable.

The study of symmetries plays a central role in the analysis of trapped ion dynamical problems. Indeed, in most of the cases, symmetries provide an elegant and effective way for finding Hamiltonian eigensolutions.

In this paper we describe a trapped ion physical situation described by a Hamiltonian model possessing interesting and useful symmetries. In detail, conservation of total excitation number and rotational invariance are found. Moreover a Second Order Hidden Supersymmetry is discovered. Symmetry based diagonalization provides the possibility of writing down the time evolution concerning simple initial conditions. Interesting nonclassical behaviors like the generation of GHZ [7] states are predicted.

2 The physical system

The physical system on which we focus our attention consists in a two-level ion confined into a three dimensional isotropic trap, described by the unperturbed Hamiltonian

$$\hat{H}_0 = \frac{\hbar\omega_A}{2}\hat{\sigma}_3 + \hbar\omega_T \sum_{j=x,y,z} \hat{a}_j^\dagger \hat{a}_j,$$

where ω_A is the relevant atomic Bohr frequency, $\hat{\sigma}_3$, third component of the Pauli vector, describes the two level atomic system, ω_T is the common frequency of the degenerate trap whose annihilation (creation) operators are $\hat{a}_j$ ($\hat{a}_j^\dagger$). The total system of Bohr frequencies involving atomic transitions are given by

$$\omega = \omega_A + (m_x + m_y + m_z)\,\omega_T,$$

where m_x, m_y, m_z are relative numbers. Considering a one-dimensional dynamics, the frequency $\omega = \omega_A + m\omega_T$ is referred to as the *carrier*, *a red sideband* or a *blue sideband* depending on $m = 0$, $m < 0$ or $m > 0$ respectively.

2.1 Implementation of the Hamiltonian model

Acting upon the system via a laser field of amplitude $\vec{E}_{0,x}$, propagating along x and tuned to the second red sideband, an interaction term is added to the Hamiltonian:

$$\hat{H}_{\text{int}}^{(S)} = -\vec{d} \cdot \vec{E}(\vec{r}),$$

where $\vec{d} = \vec{d}_\pm \hat{\sigma}_+ + \vec{d}_\pm^* \hat{\sigma}_-$ is the two-level restriction of the atomic dipole operator and $\vec{E}(\hat{\vec{r}}) = \vec{E}_{0,x} e^{i\left(k_x \hat{X} - \omega_L t\right)} + \vec{E}_{0,x}^* e^{-i\left(k_x \hat{X} - \omega_L t\right)}$, $\hat{X}$ being the trapped ion center of mass position operator, describes the field.

Passing to the interaction picture and expanding the exponentials using the Baker–Campbell–Hausdorff formula, the interaction Hamiltonian in the Rotating Wave approximation is

$$\hat{H}_{\text{int}} = -\hbar g_x \left[e^{-\frac{\eta_L^2}{2}} \sum_{s,j=0}^{\infty} \frac{(-i\eta_x)^{s+j}}{s!j!} (\hat{a}_x^\dagger)^j \, \hat{a}_x^s e^{i(j-s)\omega_T t} e^{-i(\omega_A - \omega_L)t} \hat{\sigma}_- \right] + \text{h.c.}, \tag{1}$$

where $g_x \equiv \vec{d}_\pm \cdot \vec{E}_{0,x}$, $\eta_x \equiv \frac{k_x}{2\pi}\sqrt{\frac{\hbar}{\mu\omega_T}}$, μ being the mass of the ion is the so called Lamb–Dicke parameter η_x expressing the ratio between the spatial dimensions of the bosonic ground state wave function and the laser field wavelength, and hence governing the influence of the e.m. field gradient to the laser-driven trapped ion dynamics.

Retaining only stationary terms in equation (1), and observing that

$$e^{i(j-s)\omega_T t} e^{-i(\omega_A - \omega_L)t} = e^{-i(j-s-2)\omega_T t}$$

one obtains

$$\hat{H}_{\text{int}} \approx -\hbar g_x \left[e^{-\frac{\eta_x^2}{2}} \sum_{j=0}^{\infty} \frac{(-i\eta_x)^{2(j+1)}}{j!(j+2)!} (\hat{a}_x^\dagger)^{j+2} \, \hat{a}_x^j \right] \hat{\sigma}_- + \text{h.c.} \tag{2}$$

In the so-called Lamb–Dicke limit, meaning that $\eta_x \ll 1$, the Hamiltonian in equation (2) may be cast in the form

$$\hat{H}_{\text{int}} \approx \hbar\gamma_x \left[(\hat{a}_x^\dagger)^2 \hat{\sigma}_- + (\hat{a}_x)^2 \hat{\sigma}_+ \right] \qquad \text{with} \quad \gamma_x \equiv g_x e^{-\frac{\eta_x^2}{2}} \frac{\eta_x^2}{2}.$$

Simultaneously acting with three laser fields tuned to the second red sideband and directed along the three x, y and z axis directions one obtains

$$\hat{H}_{\text{int}} \approx \hbar \left[\gamma_x \hat{a}_x^2 + \gamma_y \hat{a}_y^2 + \gamma_z \hat{a}_z^2 \right] \hat{\sigma}_+ + \text{h.c.}$$

Suitably adjusting the laser field amplitudes and the Lamb–Dicke parameters one can set equal the three γ-parameters, obtaining

$$\hat{H}_{\text{int}} \approx \hbar\gamma \left(\hat{a}_x^2 + \hat{a}_y^2 + \hat{a}_z^2 \right) \hat{\sigma}_+ + \text{h.c.}, \tag{3}$$

where $\gamma \equiv \gamma_x = \gamma_y = \gamma_z$.

2.2 Symmetries

The interaction picture Hamiltonian model in equation (3) describing two boson trimodal Jaynes–Cummings-like transitions, possesses interesting symmetries useful in view of $\hat{H}_{\rm int}$ diagonalization.

First of all observe that the total excitation number operator is conserved during the dynamics. Indeed, the operator

$$\hat{N} = \sum_{j=x,y,z} \hat{a}_j^\dagger \hat{a}_j + \hat{\sigma}_z + 1$$

is a constant of motion, meaning that the *phase changes*

$$\hat{a}_j \to e^{i\phi}\hat{a}_j, \qquad \hat{\sigma}_\pm \to e^{\mp i2\phi}\hat{\sigma}_\pm$$

leave unchanged the Hamiltonian operator.

Rotational invariance is the second important symmetry. In fact the center of mass angular momentum does commute with $\hat{H}_{\rm int}$:

$$[\hat{H}_{\rm int}, \hat{\vec{L}}] = \vec{0},$$

where the three components of $\hat{\vec{L}}$ are given by

$$\hat{L}_k = i(\hat{a}_j^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_j) \qquad (j,l,k = x,y,z \text{ and cyclic}).$$

The operators $\hat{N}$, $\hat{L}^2$, $\hat{L}_z$ all commute each other and with $\hat{H}_{\rm int}$. It is worth noting that such three operators also commute with the unperturbed Hamiltonian $\hat{H}_0$, so that they really are constants of motion, in the sense that no explicit time dependence has been introduced in the passage to the interaction picture.

3 Hamiltonian model eigensolutions

To solve the interaction picture Hamiltonian eigensolution problem

$$\hat{H}_{\rm int}|\psi\rangle = E_\psi|\psi\rangle$$

observe first of all that

$$\hat{H}_{\rm int}^2 = (\hbar\gamma)^2 \left[\hat{N}^2 + \hat{N} - \hat{L}^2\right]. \tag{4}$$

Such a formula determines, up to a sign, the eigenvalues of $\hat{H}_{\rm int}$ as

$$|e_{n,l}| = \hbar\gamma\sqrt{[n(n+1) - l(l+1)]},$$

$n = 2N + L + \frac{5}{2} + \hat{\sigma}_z$ and l being the quantum numbers related to the excitation number operator and to the square of the angular momentum, $N, l = 0, 1, \ldots$, $\hat{\sigma}_z = \pm 1$.

In addition to the individualization of the eigenvalue structure, equation (4) strongly suggests the existence of a Foldy–Wouthuysen-like transformation able to diagonalize the Hamiltonian model [4,5]. Such a circumstance is related to a *Second Order Hidden Supersymmetry* the system possesses.

Generally speaking, a Hamiltonian $\hat{H}_S$ is said to be supersymmetric if two operators $\hat{Q}$ and $\hat{Q}^\dagger$ exist satisfying the following algebra [6]

$$[\hat{H}_S, \hat{Q}] = [\hat{H}_S, \hat{Q}^\dagger] = 0, \qquad \{\hat{Q}, \hat{Q}^\dagger\} = \hat{H}_S, \qquad \{\hat{Q}, \hat{Q}\} = \{\hat{Q}^\dagger, \hat{Q}^\dagger\} = 0. \tag{5}$$

Starting from this, it is possible to introduce the Hermitian supercharges

$$\hat{Q}_+ = \hat{Q} + \hat{Q}^\dagger, \qquad \hat{Q}_- = i\big(\hat{Q} - \hat{Q}^\dagger\big) \tag{6}$$

satisfying the relation

$$\hat{Q}_-^2 = \hat{Q}_+^2 = \hat{H}_S. \tag{7}$$

Usually, supersymmetry is used to solve the $\hat{H}_S$ Hamiltonian eigenvalue problem or at least to predict the specific degeneration of the Hamiltonian spectrum. Nevertheless, if our attention is focused on one of the supercharges, the knowledge of relations in equation (5) and (7) proves to be useful. More in detail, a unitary transformation having $\hat{Q}_-$ as generator is able to *diagonalize* $\hat{Q}_+$ (*partially* or *completely* depending on the specific problem) and vice versa.

Since in our case we find an operator $\hat{K}_- = i\left\{\left[\hat{a}_x^2 + \hat{a}_y^2 + \hat{a}_z^2\right]\hat{\sigma}_+ - \text{h.c.}\right\}$ such that $\hat{Q}_-^2 \equiv \big(\hbar\gamma\hat{K}\big)^2 = \hat{H}_{\text{int}}^2$ we can set $\hat{Q}_+ = \hat{H}_{\text{int}}$, $\hat{Q}_- = \hbar\gamma\hat{K}$.

The unitary operator

$$\hat{U} = \exp\left(-\frac{\pi}{4}\frac{\hat{Q}_-}{|\hat{Q}_-|}\right) = \exp\left(-\frac{\pi\hat{K}}{4\left(\hat{N}^2 + \hat{N} - \hat{L}^2\right)^{1/2}}\right). \tag{8}$$

transforms $\hat{H}_{\text{int}}$ as follows:

$$\hat{U}^\dagger\hat{H}_{\text{int}}\hat{U} = \hbar\gamma\big(\hat{N}^2 + \hat{N} - \hat{L}^2\big)^{1/2}\hat{\sigma}_3. \tag{9}$$

Observe the accordance of equation (9) with equation (4) due to $\hat{\sigma}_3^2 = 1$, taking in mind that the unitary operator in equation (8) does not transform the square of $\hat{H}_{\text{int}}$ despite $\hat{H}_{\text{int}}$ itself is modified. Hence squaring equation (9) gives exactly equation (4), which is not sensible to the anti-transformation induced by $\hat{U}^\dagger$.

Standard algebraic manipulations leading to the expansion of the angular momentum eigenfunctions in the Fock basis complete the diagonalization procedure.

4 Applications and conclusive remarks

The investigated features of the Hamiltonian Model $\hat{H}_{\text{int}}$ lead to the possibility of inducing quantum dynamics satisfying certain symmetry conditions, for instance, rotational invariance. As an example assume the system prepared in the vacuum vibrational state and in the excited atomic level:

$$|\psi(t=0)\rangle = |0,0,0,+\rangle, \tag{10}$$

where the first three numbers refer to the x, y, and z vibrational excitation numbers, while the sign indicates the atomic ground ($-$) or excited ($+$) state.

In the light of the previous results, it is straightforward to find that once the laser interaction is turned on the system evolves in the following way:

$$|\psi(t)\rangle = \cos\left(\sqrt{6}\gamma t\right)|0,0,0,+\rangle - \frac{i}{\sqrt{3}}\sin\left(\sqrt{6}\gamma t\right)\left[|2,0,0,+\rangle + |0,2,0,+\rangle + |0,0,2,+\rangle\right]. \tag{11}$$

After a $\frac{\pi}{2}$-pulse, i.e. at the instant of time t_0 such that $\sqrt{6}\gamma t_0 = \frac{\pi}{2}$, up to a global phase factor, the system is found to be in the GHZ state [7]

$$|\psi(t_0)\rangle = \frac{1}{\sqrt{3}}|2,0,0,+\rangle + |0,2,0,+\rangle + |0,0,2,+\rangle.$$

The importance of such a state relies in its applications in the study of the Bell's inequality violations [7]. In passing we observe that, as expected, starting from the spherically symmetric wavefunction given in equation (10), the dynamics preserves such a symmetry at any instant of time. As another example consider an non-spherically symmetric initial state which distinguishes the z-direction from the other two. Let be

$$|\psi(t=0)\rangle = |0,0,2,-\rangle.$$

The relevant time evolution is given by

$$|\psi(t)\rangle = \frac{1}{3}\big[\cos\big(\sqrt{6}\gamma t\big)+2\big]|0,0,2,-\rangle + \frac{1}{3}\big[\cos\big(\sqrt{6}\gamma t\big)-1\big]\,[|2,0,0,-\rangle + |0,2,0,-\rangle] - \frac{i}{\sqrt{3}}\sin\big(\sqrt{6}\gamma t\big)|0,0,0,+\rangle. \qquad (12)$$

Due to the fact that $\{|2,0,0,-\rangle, |0,2,0,-\rangle, |0,0,2,-\rangle, |0,0,0,+\rangle\}$ is an excitation number of invariant subspace, the states involved in the two considered dynamics, the ones in equation (11) and equation (12), are the same. Nevertheless, the weights are significantly different. In particular, as well and easily seen, the initial symmetry breaking is preserved by the time evolution. Indeed, for instance, despite $|0,0,0,+\rangle$ appears, no instant of time exists when the state $|\psi(t)\rangle$ coincides with it.

Summarizing, in this paper we have described the trapped ion physical scenario and focused our attention to a specific experimental configuration (three lasers suitably tuned and so on) corresponding to a second order vibronic Jaynes–Cummings like interaction model.

The Hamiltonian exhibits interesting and useful symmetry properties: total excitation number conservation, spherical invariance and a hidden supersymmetry. Exploiting all such features we succeed in finding Hamiltonian Model eigensolutions and, hence, in describing the dynamics it is responsible for, with respect to some simple initial conditions.

As expected, the degree of spherical symmetry is found to be preserved during all the time evolution. Moreover, in correspondence to a vacuum vibrational state initial state, a GHZ state is obtained.

[1] Leibfried D., Blatt R., Monroe C. and Wineland D., Quantum dynamics of single trapped ions, *Rev. Mod. Phys.*, 2003, V.75, 281–324.

[2] Ghosh P.K., Ion traps, Oxford, Clarendon Press, 1995.

[3] Vogel W. and Wallentowitz S., Manipulation of the quantum state of a trapped ion, in Coherence and statistics of photons and atoms, Editor J. Perina, New York, Wiley, 2001.

[4] Nikitin A.G., On exact Foldy–Wouthuysen transformation, *J. Phys. A: Math. Gen.*, 1998, V.31, 3297–3300.

[5] Tsai W. and Yildiz A., Motion of charged particles in a homogeneous magnetic field, *Phys. Rev. D*, 1971, V.4, N 12, 3643–3648.

[6] Cooper F., Khare A. and Sukhatme U., Supersymmetry in quantum mechanics, Singapore, World Scientific, 2001.

[7] Greenberger D.M., Horne M.A., Shimony A. and Zeilinger A., Bell's theorem without inequalities, *Amer. J. Phys.*, 1990, V.58, 1131–1143;
Hardy L., Spooky action at a distance in quantum mechanics, *Contemporary Physics*, 1998, V.39, N 6, 419–429.