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Nonlocal Symmetry and Integrable Classes of Abel Equation

Vyacheslav M. BOYKO

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *boyko@imath.kiev.ua*

We suggest an approach for description of integrable cases of the Abel equations using the procedure of increasing the order and equivalence transformations for the induced second-order equations.

A diversity of methods were developed to date for finding solutions of nonlinear ordinary differential equations (ODE). Everybody who encounters integration of a particular ODE uses, as a rule, the accumulated databases (or reference books) of the classes of ODE and methods for their integration (e.g. [12, 19]). But if an ODE does not belong to any of the described classes then it does not mean that there is no approaches for finding solutions of this ODE in the closed form.

The symmetry approach is one of the most algorithmic approaches for integration and lowering of the order of ODE that admit a certain nontrivial symmetry (see e.g. Lie's book [13], the books [10, 17, 18] and review papers [10, 25]). In the framework of the symmetry approach (and its modifications) it is possible to obtain many of the known classes of integrable ODE. However, the needs of the applications stimulate new research into development of new methods for construction of ODE solutions in the closed form. The papers [2–10, 14–16, 18–25] may give an idea of current developments and directions of research in the field of symmetry (algebraic) methods for investigation of ODE.

In this paper we study Abel equations of the first and the second kind [1, 12, 19]

$$\dot{p} = p^3 f_4(y) + p^2 f_3(y) + p f_2(y) + f_1(y), \tag{1}$$

$$\dot{p}(p + f_0(y)) = p^3 f_4(y) + p^2 f_3(y) + p f_2(y) + f_1(y), \tag{2}$$

where $p = p(y)$, $\dot{p} = \frac{dp}{dy}$, f_i, $i = 0, \ldots, 4$, are arbitrary smooth functions (with f_1, f_2, f_3, f_4 not identically vanishing simultaneously). Equations (1), (2) along with the Riccati equation are among the "simplest" nonlinear first-order ODE that have extensive applications. At the same time the problem of description of integrable classes of these equations stays within the focus of current research, and was previously considered in many papers (see e.g. [5–8, 16, 19, 20, 22–24].

Note that the Abel equations of the first and the second kind (1), (2) are related with each other by a local change of variables (namely, the equation (2) can be reduced to the form (1) by means of the change of variables $p = 1/v(y) - f_0$). Besides, the well-known Riccati equation is a partial case of equation (1).

The problem of finding Lie symmetries for the first-order ODE is equivalent to finding solutions for these equations, and for this reason the direct application of the Lie method is complicated in the general case. On of the well-known approaches in the cases when for a given ODE it is not feasible (or not effective) to apply the Lie method directly, is increasing of the order of the ODE under consideration (in particular, to obtain a second-order ODE related to the respective ODE by a change of variables). For examples of utilisation of such approach we can refer to papers [2–6, 9, 14–16]. In such cases, if the "induced" equation of a higher order admits a non-trivial Lie symmetry (that generated a non-local symmetry for the initial equation), we can speak of so-called hidden symmetries for an initial equation (for more details see [2–4]).

Further we will consider the following second-order ODE

$$\ddot{y} = \dot{y}^4 f_4(y) + \dot{y}^3 f_3(y) + \dot{y}^2 f_2(y) + \dot{y} f_1(y), \tag{3}$$

$$\ddot{y}(\dot{y} + f_0(y)) = \dot{y}^4 f_4(y) + \dot{y}^3 f_3(y) + \dot{y}^2 f_2(y) + \dot{y} f_1(y), \tag{4}$$

where $y = y(x)$, $\dot{y} = \frac{dy}{dx}$, $\ddot{y} = \frac{d^2y}{dx^2}$, related to the Abel equations (1) and (2).

The substitution $\dot{y} = p(y)$ reduces equations (3) and (4) respectively to the Abel equations (1) and (2) (reduction of the order for equations (3) and (4)). Such reduction is induced by the Lie operator $X_1 = \partial_x$ (that corresponds to invariance of equations (3) and (4) with respect to translations by the variable x). This is exactly the fact that explains why we consider equations (3) and (4).

In the case when (3) or (4) are invariant with respect to another operator (that is when (3) or (4) admit two-dimensional Lie algebras), then equations (3) and (4) are integrable in the framework of the Lie approach. And in this way we can obtain exact solutions of the equations (1) and (2) respectively.

Further we will consider only the equation (4) (since equations (1)–(4) are interconnected – see Remark 3). Let (4) admit a two-dimensional Lie algebra

$$L = \langle X_1, X_2 \rangle, \qquad X_1 = \partial_x, \qquad X_2 = \xi(x,y)\partial_x + \eta(x,y)\partial_y. \tag{5}$$

We will consider a problem of description of inequivalent equations (4) that are invariant with respect to two-dimensional Lie algebras of the form (5) (non-equivalent realisations of the operator X_2 in the algebra (5) will determine canonical representatives for equation (4)).

It is well-known that any two-dimensional Lie algebra in the general case, by means of choosing the basis operators X_1 and X_2 in an appropriate manner, may be reduced to four nonequivalent cases (see e.g. [13,17]). In the framework of our problem additional cases arise as we have fixed the form of the operator X_1.

So, it is quite straightforward to show that equation (4) may admit a two-dimensional Lie algebra (5) only of one of the following types:

1. $[X_1, X_2] = 0, \quad \operatorname{rank} L = 1;$
2. $[X_1, X_2] = 0, \quad \operatorname{rank} L = 2;$
3. $[X_1, X_2] = X_1, \quad \operatorname{rank} L = 1;$
4. $[X_1, X_2] = X_1, \quad \operatorname{rank} L = 2;$
5. $[X_1, X_2] = X_2, \quad \operatorname{rank} L = 1;$
6. $[X_1, X_2] = X_2, \quad \operatorname{rank} L = 2.$ (6)

Further, utilising classification of two-dimensional algebras (6), we obtain that equation (4) may admit only the following realisations of two-dimensional Lie algebras (5):

1. $X_1 = \partial_x, \qquad X_2 = \xi(y)\partial_x, \qquad \xi(y) \not\equiv \text{const};$
2. $X_1 = \partial_x, \qquad X_2 = \xi(y)\partial_x + \eta(y)\partial_y,$
   $\xi(y) \not\equiv \text{const or } \xi(y) \equiv 0, \quad \eta(y) \neq 0;$
3. $X_1 = \partial_x, \qquad X_2 = (x + \xi(y))\partial_x, \qquad \xi(y) \not\equiv \text{const or } \xi(y) \equiv 0;$
4. $X_1 = \partial_x, \qquad X_2 = (x + \xi(y))\partial_x + \eta(y)\partial_y,$
   $\xi(y) \not\equiv \text{const or } \xi(y) \equiv 0, \quad \eta(y) \neq 0;$
5. $X_1 = \partial_x, \qquad X_2 = e^x \xi(y)\partial_x, \qquad \xi(y) \not\equiv 0;$
6. $X_1 = \partial_x, \qquad X_2 = e^x(\xi(y)\partial_x + \eta(y)\partial_y), \qquad \eta(y) \neq 0.$ (7)

It is clear that using these realisations we can describe equations of the form (4) that are invariant with respect to two-dimensional Lie algebras (similarly as we have discussed in [22]). However, this way is too cumbersome, and thus obtained types of equations (4) will be quite complicated (functions f_i, $i = 0, \ldots, 4$ in (4) will be expressed through coefficients of the operator X_2 from realisations (7)).

It is straightforward to show that the most general transformations that preserve the form of the operator X_1 we look as follows:

$$t = x + \omega(y), \qquad u = g(y), \tag{8}$$

where $\omega(y)$, $g(y)$ are arbitrary smooth functions, $g(y) \not\equiv \mathrm{const}$.

After substitution (8) equation (4) takes the form

$$\begin{aligned} \ddot{u}((1-\omega' f_0)\dot{u} + f_0 g')g'^2 &= \big(f_4 - \omega''(1-\omega' f_0) - \omega' f_3 + \omega'^2 f_2 - \omega'^3 f_1\big)\dot{u}^4 \\ &+ \big(g' f_3 - \omega'' g' f_0 + g''(1-\omega' f_0) - 2\omega' g' f_2 + 3\omega'^2 g' f_1\big)\dot{u}^3 \\ &+ \big(g'^2 f_2 + g'' g'^2 f_0 - 3\omega' g'^2 f_1\big)\dot{u}^2 + f_1 g'^3 \dot{u}, \end{aligned} \tag{9}$$

where $\omega' = \frac{d\omega}{dy}$, $\omega'' = \frac{d^2\omega}{dy^2}$, $g' = \frac{dg}{dy}$, $g'' = \frac{d^2 g}{dy^2}$ (in addition in (9) all functions of the variable y should be expressed as functions of the variable u).

With $(1 - \omega' f_0) \not\equiv 0$ equation (9) belongs again to the class of equations (4).

Remark 1. With $(1 - \omega' f_0) \equiv 0$ after the substitution (8), equation (4) is transformed to the equation (3), that is reduced to the Abel equation of the first kind (1).

Remark 2. It is possible to regard that $(1 - \omega' f_0) \not\equiv 0$ for the equation (4) as a result of the substitution (8) (we attain that by combination of transformations (8)).

Thus (8) are equivalence transformations for (4), and, besides, these transformations preserve the form of the operator $X_1 = \partial_x$ in the algebra (5).

Remark 3. So, the transformations (8) are equivalence transformations for the class of equations (3)–(4). Moreover, if we prolongate these transformations for $\dot{u} = p$ then these transformations form an equivalence transformation group for (1)–(2).

Thus, by means of transformations (8), realisations (7) of the algebra (5) may be reduced to the simplest canonical form. The transformations (8) in that process will not take us out of the class of equations (4).

By means of transformations (8) the realisations (7) of two-dimensional Lie algebras (5) admitted for equation (4) are reduced to the following canonical realisations:

$$\begin{aligned} &1.\ X_1 = \partial_x, \qquad X_2 = y\partial_x; \\ &2.\ X_1 = \partial_x, \qquad X_2 = \partial_y; \\ &3.\ X_1 = \partial_x, \qquad X_2 = x\partial_x; \\ &4.\ X_1 = \partial_x, \qquad X_2 = x\partial_x + y\partial_y; \\ &5.\ X_1 = \partial_x, \qquad X_2 = e^x\partial_x; \\ &6.\ X_1 = \partial_x, \qquad X_2 = e^x(\partial_x + \partial_y). \end{aligned} \tag{10}$$

In accordance to (10) we obtain the following integrable cases for equation (4) that are non-equivalent with respect to (8):

1. $\ddot{y} = \alpha(y)\dot{y}^3$;

2. $\ddot{y}(\dot{y}+e) = d\dot{y}^4 + c\dot{y}^3 + b\dot{y}^2 + a\dot{y}$;
3. $\ddot{y} = \alpha(y)\dot{y}^2$;
4. $y\ddot{y}(\dot{y}+e) = d\dot{y}^4 + c\dot{y}^3 + b\dot{y}^2 + a\dot{y}$;
5. $\ddot{y}(\dot{y}+\beta(y)) = \alpha(y)\dot{y}^3 + (1-\alpha(y)\beta(y))\dot{y}^2 - \beta(y)\dot{y}$;
6. *a*) $f_0 = 0:$

$$\ddot{y} = de^y\dot{y}^3 + (-3de^y + c)\dot{y}^2 + (-de^y - (2c+1) + be^{-y})\dot{y} + (-de^y + (c+1) - be^{-y} + ae^{-2y});$$

b) $f_0 \neq 0:$

$$\ddot{y}(\dot{y}+\alpha(y)) = -\dot{y}^3 + (1-\alpha(y))\dot{y}^2 + \alpha(y)\dot{y}, \tag{11}$$

where $\alpha(y)$, $\beta(y)$ are arbitrary smooth functions, a, b, c, d, e are constants.

The case 6a in (11) may be simplified by means of the substitution $t = x$, $u = e^y$ (see (8) and (9)).

Equations (11) determine non-equivalent cases of the form (4) that admit two-dimensional algebras (10) up to equivalence transformations (8).

Thus, summarising the above, we come to the following scheme for integration of the Abel equation (2):

- we increase the order of equation (2), considering a second-order equation (4);
- if a corresponding equation (4) admits a two-dimensional Lie algebra, then we reduce this algebra to one of the canonical forms (10), and thus the equation is reduced to the respective canonical forms (11);
- we integrate the canonical form (11);
- making reverse changes of variables we obtain the solution of the Abel equation (2).

It is obvious from the above that there is an alternative way for generation of new integrable cases of the Abel equation based on utilisation of the relation between the Abel equations of the first and the second kind, and relation between the equations (3) and (4) by means of the transformations (8). Thus, starting from some integrable Abel equation (that is of such equation for which the solution is known) it is possible to obtain new integrable cases of the Abel equations (solutions of these equations will be related through transformations (8)). It would be possible to use for this purpose even the well-known Riccati equation that is a partial case of the equation (1) (for generation of integrable Riccati equations an approach that is proposed in [20] may be used).

We hope that new results for classification of integrable classes of ODE may be obtained also using our classification of inequivalent realizations of real low-dimensional Lie algebras [21].

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[1] Abel N.H., Oeuvres Complétes II, Editors S. Lie and L. Sylow, Christiana, 1839.

[2] Abraham-Shrauner B., Hidden symmetries and non-local group generators for ordinary differential equations, *IMA J. Appl. Math.*, 1996, V.56, 235–252.

[3] Abraham-Shrauner B., Hidden symmetries, first integrals and reduction of order of nonlinear ordinary differential equations, *J. Nonlinear Math. Phys.*, 2002, V.9, Suppl. 2, 1–9.

[4] Abraham-Shrauner B. and Guo A., Hidden symmetries associated with the projective group of nonlinear first-order ordinary differential equations, *J. Phys. A: Math. Gen.*, 1992, V.25, 5597–5508.

[5] Adam A.A. and Mahomed F.M., Non-local symmetries of first-order equations, *IMA J. Appl. Math.*, 1998, V.60, 187–198.

[6] Adam A.A., Mahomed F.M. and Kara A.H., New solutions of the Abel equation, *Lie Groups Appl.*, 1994, V.1, 1–10.

[7] Cheb-Terrab E.S. and Roche A.D., Abel ODEs: equivalence and integrable classes, *Comp. Phys. Comm.*, 2000, V.130, 204–231.

[8] Cheb-Terrab E.S. and Roche A.D., An Abel ordinary differential equation class generalizing known integrable classes, *European J. Appl. Math.*, 2003, V.14, 217–229.

[9] Edelstein R.M., Govinder K.S. and Mahomed F.M., Solution of ordinary differential equation via nonlocal transformations, *J. Phys. A: Math. Gen.*, 2001, V.34, 1141–1152.

[10] Ibragimov N.H., Group analysis of ordinary differentail equations and the invariance principle in mathematical physics, *Russian Math. Surveys*, 1992, V.47, N 4, 89–156.

[11] Ibragimov N.H., Elementary Lie group analysis and ordinary differential equations, Chichester, John Wiley & Sons Ltd., 1999.

[12] Kamke E., Differentialgleichungen: Lösungsmethoden und Lösungen, New York, Chelsea Publishing, 1959.

[13] Lie S., Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen, Leipzig, Teubner, 1891.

[14] Mahomed F.M. and Leach P.G.L., Lie algebras associated with scalar second-order ordinary differential equations, *J. Math. Phys.*, 1989, V.30, 2770–2777.

[15] Mahomed F.M. and Leach P.G.L., Symmetry Lie algebras on nth order ordinary differential equations, *J. Math. Anal. Appl.*, 1990, V.151, 80–107.

[16] Mellin C.M., Mahomed F.M. and Leach P.G.L., Solution of generalized Emden–Fowler equations with two symmetries, *Internat. J. Non-Linear Mech.*, 1994, V.29, 529–538.

[17] Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986.

[18] Olver P., Equivalence, invariants, and symmetry, Cambridge University Press, 1995.

[19] Polyanin A.D. and Zaitsev V.F., Handbook of exact solutions for ordinary differential equations, Boca Raton, CRC Press, 1995.

[20] Popovych R.O. and Boyko V.M., Differential invariants and application to Riccati-type systems, in Proceedings of Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–15 July, 2001, Kyiv), Editors A.G. Nikitin, V.M. Boyko and R.O. Popovych, Kyiv, Institute of Mathematics, 2002, V.43, Part 1, 184–193; math-ph/0112057.

[21] Popovych R.O., Boyko V.M., Nesterenko M.O. and Lutfullin M.W., Realizations of real low-dimensional Lie algebras, *J. Phys. A: Math. Gen.*, 2003, V.36, 7337–7360; math-ph/0301029.

[22] Revenko I.V. and Boyko V.M., Symmetry and some integrable cases of Abel's equation of the second kind, *Dopovidi of NAS of Ukraine*, 1995, N 7, 16–18.

[23] Schwarz F., Symmetry analysis of Abel's equation, *Studies in Appl. Math.*, 1998, V.100, 269–294.

[24] Schwarz F., Algorithmic solution of Abel's equation, *Computing*, 1998, V.61, 39–46.

[25] Schwarz F., Solving second-order differential equations with Lie symmetry, *Acta Appl. Math.*, 2000, V.60, 39–113.