DYNAMICAL TOPOLOGY:
Slovak Spaces and Dynamical Compactness

SERGIY KOLYADA

Institute of Mathematics, NAS of Ukraine, Kyiv

www.imath.kiev.ua/∼skolyada

Based on joint works with Wen HUANG, Danylo KHILKO, Alfred PERIS, Julia SEMIKINA and Guo Hua ZHANG
and a work by Tomasz DOWNAROWICZ, Lubomir SNOHA and Dariusz TYWONIUK

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I. Slovak Spaces

By a (*topological*) *dynamical system* (X, T) we mean a (compact) metric space X with a continuous self-map $T : X \to X$.

The area of dynamical systems where one investigates dynamical properties that can be described in topological terms is called ***Topological Dynamics***.

Investigating the topological properties of spaces, maps that can be described in dynamical terms is in a sense the opposite idea.

And we call this area

Dynamical Topology.

Let X be a compact metrizable space and $S(X)$ be the set of all *continuous* maps from X into X and $H(X)$ be the set all self-homeomorphisms of X.Obviously, $S(X)$ is a ***topological semigroup*** and $H(X)$ is a ***topological group*** under composition and the compact-open topology.

One of the main topics of my research is to study relations between some properties (structure) of the topological semigroup $S(X)$, group $H(X)$ and possible values of the topological entropy (and/or some other chaotic properties) of its elements (continuous maps and homeomorphisms, respectively).

In the following joint paper with Julia Semikina

On topological entropy: when positivity implies +infinity, Proceedings of the AMS, 143 (2015), 1545-1558.

in particular, we proved

Theorem A

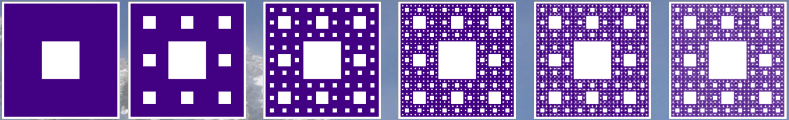
Let X be a compact metric space. If $S(X)$ is compact, then for any $f \in S(X)$ topological entropy of (X, f) and topological entropy the functional envelope $(S(X), F_f)$ is zero. If $H(X)$ is compact, then for any $f \in H(X)$ topological entropy of (X, f) is also zero.

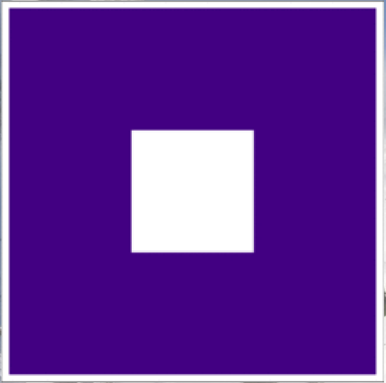
Now is not so much known about the topological structure of compact semigroup $S(X)$ and group $H(X)$. **The compactness** of the $S(X)$ and $H(X)$ is not a very strict condition and takes place sometimes, because both of them may be "very small".

Recall that a topological space X is said to be ***rigid***, if the full topological homeomorphism group $H(X)$ is the identity.

De Groot and Wille (in 1958) showed the existence of rigid spaces even as locally connected one-dimensional continua (*Peano curves*) of the plane.

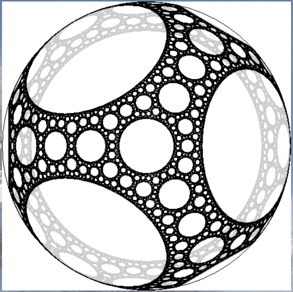
Sierpinski carpet is a square with interiors of a dense family of subsquares removed.

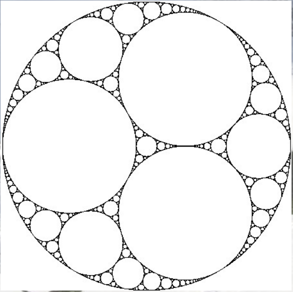


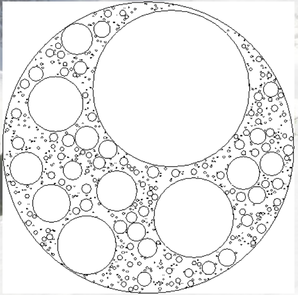


The first 6 steps in the construction of the Sierpinski carpet

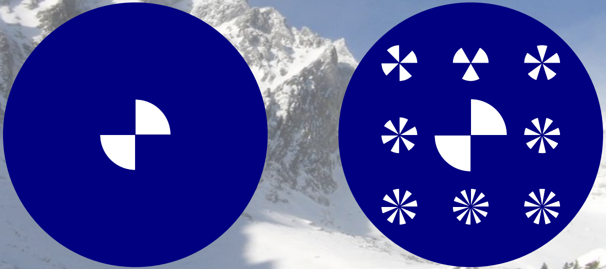
Sierpinski carpet, also known as the **Sierpinski universal curve**, is a one-dimensional planar Peano continuum.

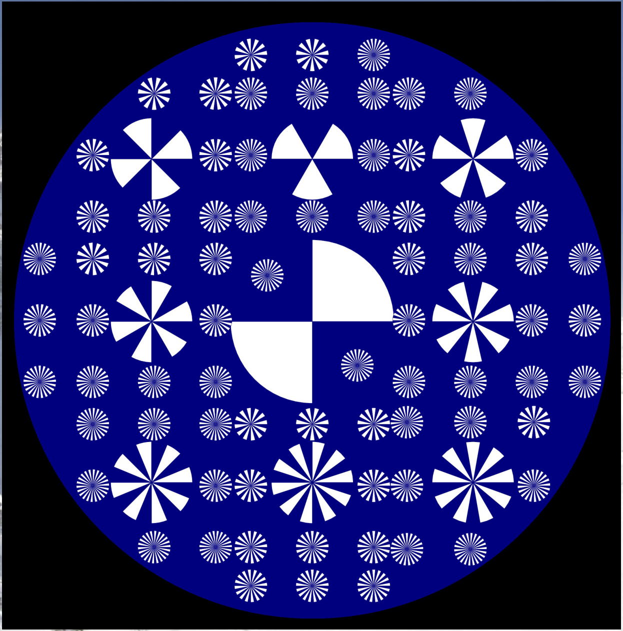






De Groot - Wille rigid plane locally connected one-dimensional continua is a disc with interiors of a dense family of propellers (with different numbers of blades) removed:





In fact, by using such kind of ideas, de Groot proved the following

Theorem 1 (Groups representation)

Let G be an arbitrary group. Then there exists a connected, locally connected,complete metric space X (or alternatively compact, Hausdorff) for which the group of all autohomeomorphisms $H(X)$ is isomorphic to G.

However, such a space need not exist in the class of compact metric spaces, because a compact metric space has cardinality at most c, while there are groups of arbitrary cardinalities. Nevertheless, as De Groot and Wille proved, if G is countable then X can be chosen to be a Peano continuum of any positive dimension.

Howard Cook (in 1967) constructed two metric continua:

- the space $S(X)$ consists only of the constant maps (sending all points from X to a fixed point of X) and the identity on X;
- the space $H(Y)$ is topologically equivalent to the Cantor set.

We still know nothing about the structure of the compact semigroup $S(X)$ in general. Also, if there is a connection between compactnesses of $S(X)$ and $H(X)$. Or, let $S(X)$ be a compact semigroup. Is $H(X)$ compact in that case?

It is known that a compact group $H(X)$ is always totally disconnected.

A dynamical system (X, T) gives rise to several "hyper-systems".

For example, T acts naturally on 2^X, the space of all compact subsets of X equipped with the Hausdorff metric. The induced map here is the continuous map $\tilde{T} : 2^X \mapsto 2^X$ given by $\tilde{T}(A) = T(A)$ for each $A \in 2^X$.

In the same spirit (although now the domains are not necessarily compact), 10 years ago jointly with J. Auslander and L. Snoha, we introduced and started to study the "hyper-system" $(S(X), F_T)$, where the transformation

$$F_T : S(X) \mapsto S(X) \text{ is defined by } F_T(\phi) = T \circ \phi \text{ for any } \phi \in S(X).$$

We called the system as the "***Functional Envelope***" of (X, T).

Why "functional envelope"? Because the system $(S(X), F)$ contains (properly, if $\operatorname{card} X \geq 2$) an isomorphic copy of (X, f). The map $\iota : X \to S(X)$ sending a point $a \in X$ to the constant map con_a, is an isometry (regardless of whether the uniform or the Hausdorff metric is used in $S(X)$)

and also a topological conjugacy, i.e., $\iota \circ f = F \circ \iota$.

The functional envelopes may differ in some dynamical properties which depend on the metric (for example the topological entropy in non-compact systems does). On the other hand, the essential dynamical properties of the functional envelopes do not depend on the metric applied on X.

Now suppose additionally that the map $T : X \mapsto X$ is a homeomorphism. Then the space $H(X)$ is invariant under F_T and $(H(X), F_T)$ becomes another kind of "functional envelope"

$$F_T : H(X) \mapsto H(X) \text{ is defined by } F_T(\phi) = T \circ \phi \text{ for any } \phi \in H(X).$$

Topological entropy: Bowen and Dinaburg definition

Topological entropy measures the evolution of distinct (distinguishable) orbits over time, thereby providing an idea of how complex the orbit structure of a system (X, T) is.

Assume for simplicity that X is compact metric. A set $E \subset X$ is said to be (n, ε)-*separated*, if for every $x \neq y \in E$ there is $i \in \{0, 1, \ldots, n-1\}$ such that $d(T^i(x), T^i(y)) \geq \varepsilon$.

Let $s(n, \varepsilon)$ be the maximal cardinality of an (n, ε)-separated set in X . By compactness, this number is always finite. One defines:

$$h_{top}(\varepsilon, T) = \limsup_{n \to \infty} \frac{\log s(n, \varepsilon)}{n}.$$

This limit gives the exponential growth of $s(n, \varepsilon)$ with a fixed resolutionas the length of orbits we consider tends to infinity.

Next, we want to decrease the resolution. The ***topological entropy*** is obtained as

$$h_{top}(T) = \sup_{\varepsilon > 0} h_{top}(\varepsilon, T) = \lim_{\varepsilon \to 0} h_{top}(\varepsilon, T).$$

Some basic properties of topological entropy

- $h_{top}(T) \geq h_{top}(S)$ if (Y, S) is a "topological factor" of (X, T) , i.e., $\phi \circ T = S \circ \phi$, where $\phi : X \to Y$ is a continuous surjection.
- $h_{top}(T) \geq h_{top}(T|_Y)$ if Y is a nonempty closed invariant subset of X.
- $h_{top}(T^n) = n \cdot h_{top}(T)$ for $n \geq 0$.
- $h_{top}(T \times S) = h_{top}(T) + h_{top}(S)$.
- $h_{top}(T \circ S) = h_{top}(S \circ T)$.

Let us gather some basic known facts and open problems concerning the entropy of the functional envelopes:

(1) The entropy of $(S(X), F_T)$ is always not smaller than that of (X, T); to see this it suffices to consider the subsemigroup of constant self-maps, which is conjugate to (X, T).

(2) On the other hand, it was not known whether the same inequality holds for $(H(X), F_T)$.

(3) Our with Semikina conjecture says that the functional envelope $(S(X), F_T)$ has entropy either zero or infinity. We proved that the conjecture holds true for all Peano continua and for all compact spaces with continuum many connected components. Otherwise the conjecture was open. The same problem can be posed for the functional envelope $(H(X), F_T)$ in case T is a homeomorphism.

Now I will speak about some nice results from the following recent paper:

Downarowicz, T., Snoha, L. and Tywoniuk, D. J Minimal spaces with cyclic group of homeomorphisms, Journal of Dynamics and Differential Equations, Online published on 03 April 2015, doi:10.1007/s10884-015-9433-2.

First of all they gave a positive answer for the question (2) for homogeneous spaces and extended the knowledge on the question (3). Namely,

Theorem B

Let $T : X \mapsto X$ be a homeomorphism of a homogeneous compact space. Then the entropy of the functional envelope $(H(X), F_T)$ is at least as large as that of (X, T).

Theorem C

Let $T : X \mapsto X$ be a self-homeomorphism of a compact zero-dimensional space. Then the entropies of $(S(X), F_T)$ and $(H(X), F_T)$ are either both zero or both infinite. They are equal to zero if and only if T is equicontinuous.

Nevertheless, they also showed that there exists a positive (even infinite) entropy homeomorphism T of a compact space X such that $(H(X), F_T)$ has entropy zero.

This proves that the general question (2) (the inequality between topological entopies of (X, T) and its functioanal envelope $(H(X), F_T)$) has a negative answer. It was solved by using a new class of spaces, which they have called ***Slovak spaces***, defined by the combination of two properties: the existence of a (minimal) homeomorphism, say T, and nonexistence of homeomorphisms other than the powers of T.

For such spaces they showed that the functional envelope $(H(X), F_T)$ always has entropy zero.

Existence of uniquely minimal spaces and applications

$T: X \mapsto X$ is called ***minimal*** if the orbit $\{x, T(x), T^2(x), ..., T^n(x), ...\}$ of any point $x \in X$ is dense in X.
For a compact metric space X there are two possibilities:

- X does not admit any minimal homeomorphism
- X admits a minimal homeomorphism (in this case, if X is infinite then in known examples usually (always?) X admits uncountably many homeomorphisms and even uncountably many of them are minimal)

Question

Is there a third possibility? That is, does there exist an infinite compact metric space X such that it admits, but only "a few minimal homeomorphisms?

Existence of uniquely minimal spaces and applications

Definition

An infinite compact metric space X is **Slovak** if it is **uniquely minimal** in the following sense: X admits a minimal homeomorphism T and $H(X) = \{T^n : n \in \mathbb{Z}\}$.

- The assumption that X is infinite eliminates two trivial examples: the one-point space and the two-point space.
- If X is Slovak then $\mathrm{card}\, X = \mathfrak{c}$, X has no isolated point and all iterates T^n, $n \in \mathbb{Z}$ are different, i.e. $\mathbf{H(X)} \approx \mathbb{Z}$. Moreover, **all iterates T^n, except identity, are minimal**.

Theorem

There exist Slovak spaces in the class of metric continua. (Moreover, the topological entropies of generating homeomorphisms T exhaust the interval $[0, \infty]$.)

Idea of a construction of uniquely minimal spaces

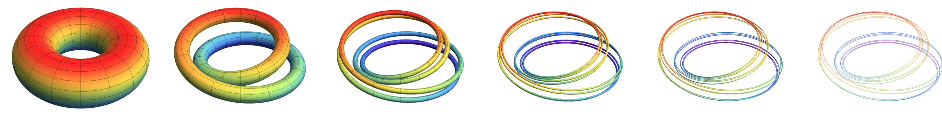
Every Slovak space is a non-degenerate continuum.

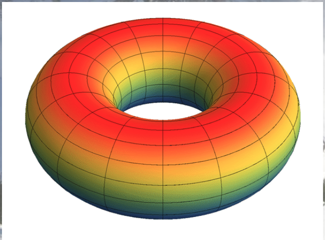
Step 1. $h : C \to C$ ... Cantor minimal homeo (arbitrary entropy)

Step 2. $SOL = C \times [0,1]/_\sim$ where $(x, 1) \sim (h(x), 0)$
(the generalized solenoid induced by (C, h))

Solenoids are among the simplest examples of indecomposable homogeneous continua. They are neither arcwise connected nor locally connected.

On the pictures below: a solid torus ($\mathbb{S}^1 \times D$) wrapped twice around inside another solid torus in $\mathbb{R}^3$. And each solenoid may be constructed as the intersection of a nested system of embedded solid tori in $\mathbb{R}^3$.





The first six steps in the construction of the Smale-Williams attractor

Φ = suspension flow over h (with ceiling function $\equiv 1$)

The suspension flow over h is the flow $\Phi_t, t \in \mathbb{R}$ defined on SOL by the formula

$$\Phi_t(y, s) = (h^{[t+s]}(y), \{t + s\}),$$

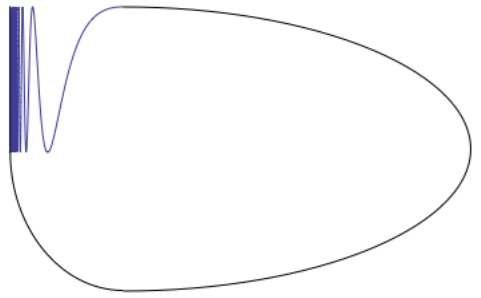
where $[\,\cdot\,]$ and $\{\cdot\}$ denote the integer and fractional parts of a real number, respectively.

Step 3. By B. Fayad (2000): $\exists t_0 \in \mathbb{R}$ such that

T = time t_0-map of Φ is a minimal homeo $SOL \to SOL$

Step 4 (technical step).
The Slovak space which we are constructing will be a subset of the cylinder $SOL \times [0,1]$. The main element of this construction is a topologist's sine curve.

Topologist's sine curve is a subset of the plane that is the union of the graph of the function $f(x) = \sin(1/x), 0 < x \leq 1$ with the segment $-1 \leq y \leq 1$ of the y-axis. The Warsaw circle is obtained by "closing up"the topologist's sine curve by an arc connecting points $(0,-1)$ and $(1, \sin(1))$.



- the continuum SOL has uncountably many composants (orbits of the flow Φ); choose a composant γ and a point $x_0 \in \gamma$.

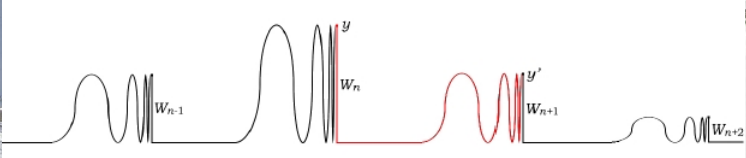
  The composant of a point p in a continuum A is the union of all proper subcontinua of A that contain p. If a continuum is indecomposable, then its composants are pairwise disjoint. The composants of a continuum are dense in that continuum.

- on a closed arc around x_0 (minus the point x_0 itself) and lying in γ, we define a function which looks like a one-sided topologist's sine curve (values in $[0,1]$, wiggles of height 1 in any left neighbourhood of x_0, constant value 0 to the right of x_0). It is continuous and not defined at x_0.
- extend it to a continuous function $f : SOL \setminus \{x_0\} \to [0,1]$

Idea of a construction of uniquely minimal spaces

- let $F = \sum_{n\in\mathbb{Z}} a_n f \circ T^n$, where the coefficients a_n are all strictly positive, $\sum_{n\in\mathbb{Z}} a_n = 1$ and satisfying some technical assumptions (F is defined on SOL minus the T-orbit of x_0).
- then one can show that both the mapping $(x, F(x)) \mapsto (Tx, F(Tx))$ and its inverse are uniformly continuous homeomorphisms of the graph of F. Therefore, the map $(x, F(x)) \mapsto (Tx, F(Tx))$ extends to a homeomorphism $\bar{T}$ (=notation) of $\bar{F}$ (=the closure of the graph of F).
- $\bar{F} \subseteq SOL \times [0, 1]$ is our **Slovak space**, looks as follows:

  the composant $\bar{\gamma}$ of $\bar{F}$ "above"γ has basicly this shape:



  the other composants of $\bar{F}$ are continuous bijective images of the real line

II. Dynamical compactness

One of the main idea of what we call ***Dynamical Compactness*** has a direct connection with the following.

A collection A of subsets of a set X is said to have the *finite intersection property* (FIP) if the intersection over any finite subcollection of A is nonempty.

The finite intersection property is useful in formulating an alternative definition of compactness:

A space is compact if and only if every collection of closed subsets satisfying the finite intersection property has nonempty intersection itself.

Furstenberg families

Denote by $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$ the set of all subsets of $\mathbb{Z}_+$. A subset $\mathcal{F}$ of $\mathcal{P}$ is called (*Furstenberg*) ***family***, if it is hereditary upward, that is, $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$.

For a family $\mathcal{F}$, the ***dual family*** of $\mathcal{F}$, denoted by $k\mathcal{F}$, is defined as

$$\{F \in \mathcal{P} : F \cap F' \neq \varnothing \text{ for any } F' \in \mathcal{F}\}.$$

Recall that a (Furstenberg) family $\mathcal{F}$ is a ***filter*** if 1) $\varnothing \notin \mathcal{F}$ and 2) $F_1, F_2 \in \mathcal{F}$ implies $F_1 \cap F_2 \in \mathcal{F}$. A (Furstenberg) family $\mathcal{F}$ is an ***ultrafilter*** if for any $F \in \mathcal{P}$ $F \in \mathcal{F}$ or in the complement of $\mathcal{F}$, i.e. $F \in \mathcal{P} \setminus \mathcal{F}$.

The ω-limit set of a point is a basic notion in theory of dynamical systems and means the collection of states which "attract" this point while going forward in time. More precise,

$$\omega_T(x) = \bigcap_{n \in \mathbb{N}} \overline{\{T^k(x) : k > n\}}.$$

By changing the time we introduce *$\omega_{\mathcal{F}}$-limit set with respect to a Furstenberg family $\mathcal{F}$*

$$\omega_{\mathcal{F}}(x) = \bigcap_{F \in \mathcal{F}} \overline{T^F x},$$

where $T^F x = \{T^n x : n \in F\}$ or, in other words,

$$\omega_{\mathcal{F}}(x) = \{z \in X : n_T(x, G) \in k\mathcal{F} \text{ for every neighborhood } G \text{ of } z\},$$

where $n_T(x, G) = \{n \in \mathbb{Z}_+ : T^n(x) \in G\}$.

Dynamical compactness

It is well known that if $\mathcal{F}$ is an ultrafilter, then for any point $x \in X$ its the $\omega_{\mathcal{F}}$-limit set is nonempty and is a singleton.

Fix a dynamical system (X, T), where X is a compact metric space , T is a continuos (surjective) map, and Furstenberg family $\mathcal{F}$. A dynamical system (X, T) is said to be ***dynamically compact*** *with respect to* $\mathcal{F}$ if $\omega_{\mathcal{F}}(x) \neq \varnothing$ for all $x \in X$.

Theorem

All dynamical systems are dynamically compact with respect to a family $\mathcal{F}$ if and only if $\mathcal{F}$ has the finite intersection property.

We will denote by $N_T(U,V)$ the visiting times of a set U to a set V in X:

$$N_T(U,V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n}V \neq \varnothing\}.$$

1. Let $\mathcal{N}_T$ be a (inclusion-minimal) Fustenberg family which contains all sets $N_T(U,V)$, where U, V are arbitrary opene subsets of X. (X,T) is called transitive compact if it is dynamically compact with respect to $\mathcal{N}_T$.
Let $\delta > 0$. For any opene set $U \subset X$ we define

$$S_T(U,\delta) = \{n \in \mathbb{Z}_+ : \text{ there are } x_1, x_2 \in U \text{ such that } d(T^n x_1, T^n x_2) > \delta\}.$$

2. Let $\mathcal{S}_T(\delta)$ be a (inclusion-minimal) Fustenberg family which contains all sets $S_T(U,\delta)$, where U is arbitrary opene subset of X. (X,T) is called multi-sensitive if the family $\mathcal{S}_T(\delta)$ has the finite intersection property.

Transitivity and sensitivity

(X, T) is called:

- (topologically) *transitive* if for any opene (open nonempty) sets U, V $N_T(U, V) \neq \varnothing$. Since T is surjective, the system (X, T) is transitive if and only if the set of all points with dense orbits is a dense G_δ subset of X.
- (*topologically*) *weakly mixing* if the product system $(X \times X, T \times T) : (x, y) \mapsto (T(x), T(y))$ is topologically transitive. Remark that Furstenberg showed (X, T) is weakly mixing if and only if the family $\mathcal{N}_T$ is a filter.

The following Figure presents a comparison between stronger forms of sensitivity for transitive systems.

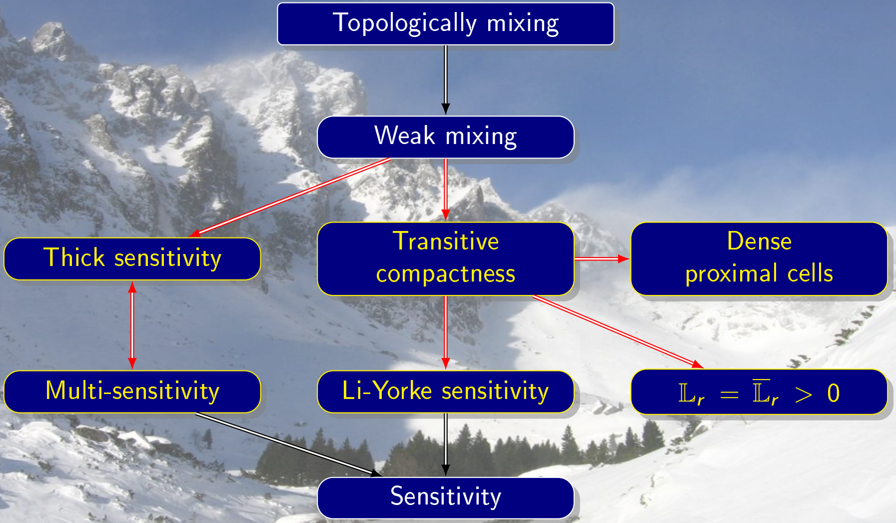


Figure “Topologically transitive systems”.

A pair of points (x, y) is called *proximal* if $\liminf_{n\to\infty} d(T^n x, T^n y) = 0$. (X, T) is called proximal if for any $x, y \in X$ the pair (x, y) is proximal. Note, it is known that (X, T) is proximal iff it has a unique minimal point which is fixed.

(X, T) is called *Li-Yorke sensitive* if there exists $\delta > 0$ such that for every $x \in X$ and every neighborhood U_x of x, there exist $y \in U_x$ such that (x, y) is a proximal pair while $\limsup_{n\to+\infty} d(T^n x, T^n y) > \delta$.

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ДЯКУЮ!

ĎAKUJEM!

THANK YOU VERY MUCH!