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A Note on Pliability and the Openness of the Multiexponential Map in Carnot Groups

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Abstract. In recent years, several notions of non-rigidity of horizontal vectors in Carnot groups have been proposed, motivated, in particular, by the characterization of monotone sets and Whitney extension properties. In this note we compare some of these notions.

Key words: pliability; rigidity; multiexponential map; Carnot group; endpoint map

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1 Introduction

Consider a Carnot group $\mathbb{G}$ with stratified Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ (see Definition 2.1). The endpoint map $E\colon L^\infty([0,1],\mathfrak{g}_1) \to \mathbb{G}$ (see Definition 2.2) plays a crucial role in studying metric and topological properties of $\mathbb{G}$. The goal of this paper is to explore and clarify the connections among several openness and submersivity properties of the endpoint map and of the multiexponential map

$$\Gamma^{(p)}\colon\ (\mathfrak{g}_1)^p \to \mathbb{G}, \qquad \Gamma^{(p)}(Y_1,\dots,Y_p) := \exp(Y_1)\cdots\exp(Y_p),$$

where $p \in \mathbb{N}$ and $\exp\colon \mathfrak{g}_1 \to \mathbb{G}$ is the exponential map, i.e., the restriction of E to $\mathfrak{g}_1$, seen as the subspace of $L^\infty([0,1],\mathfrak{g}_1)$ made of constant functions. In particular, given a vector $X \in \mathfrak{g}_1$, we are interested in the following properties of the multiexponential map and of $E_X(\cdot) := E(X+\cdot)$, the endpoint map based at X:

(H) there exists $p \in \mathbb{N}$ such that the map $\Gamma^{(p)}$ is open at $(X,\dots,X) \in (\mathfrak{g}_1)^p$;

(SbH) there exists $p \in \mathbb{N}$ such that the map $\Gamma^{(p)}$ is a submersion at $(X,\dots,X) \in (\mathfrak{g}_1)^p$;

(P) the map E is open at X (i.e., E_X is open at 0).

One of the earliest appearances of these conditions in the literature is the observation by Cheeger and Kleiner in [3] that in the three-dimensional Heisenberg group every $X \in \mathfrak{g}_1$ satisfies condition (SbH) with $p = 2$. As a consequence, they deduced the following *cone property*: if a horizontally convex set C contains a point p in its closure and an open ball B, then it also contains the cone over B with vertex at p.

In the paper [2], Arena, Caruso, and Monti generalized the submersivity of the map $(Y_1,Y_2) \mapsto \exp(Y_2)\exp(Y_1)$ to the case where $\mathbb{G}$ is of step two of Métivier type and Morbidelli proved in [8] that the cone property holds for all Carnot groups of step two.

Montanari and Morbidelli later observed in [7] that the cone property can be obtained under the weaker assumption that every $X \in \mathfrak{g}_1$ satisfies condition (H) (which is clearly implied by (SbH)). In addition, they showed that if a vector $X \in \mathfrak{g}_1$ satisfies condition (SbH), then the Carnot–Carathéodory distance from the unit element of $\mathbb{G}$ is Pansu differentiable at $\exp(X)$. (They also showed that (SbH) implies the related notion of *deformable direction* introduced by Pinamonti and Speight in [9]).

Condition (H) appears again in the paper [10] by Rigot and it is used to characterize precisely monotone subsets of $\mathbb{G}$. We adapt the terminology introduced in [10], by distinguishing between the (H)-*condition* and the *submersive* (H)-*condition* (SbH) introduced above.

On the other hand, condition (P), called *pliability*, was introduced in [4], where Juillet and the second author proved the C^1 Whitney extension property for horizontal curves in Carnot groups where every vector in $\mathfrak{g}_1$ satisfies the property (P) (see also the recent preprint [12] by Speight and Zimmerman for a generalization to horizontal curves tangent to pliable directions). The definition of pliability proposed in [4] is actually given in terms of the openness of the restriction of the endpoint map to a suitable proper subspace of $L^\infty([0,1],\mathfrak{g}_1)$. The equivalence between (P) and pliability as defined in [4] is one of the contributions of this paper. For a precise discussion on this point, see Remark 3.3.

The first question addressed in this paper concerns the relationship between pliability and the (H)-conditions. Later, Sacchelli and the second author introduced in [11] the notion of *strong pliability* (SP), and they improved the results of [4] for horizontal curves in sub-Riemannian manifolds where every vector in $\mathfrak{g}_1$ of the tangent Carnot group satisfies (SP). Condition (SP) and the analogous notion of *p-free* (H)-*condition* (FH) are defined as follows:

(SP) for all $\eta > 0$ there exists $Y \in L^\infty([0,1],\mathfrak{g}_1)$ such that $\|Y\|_\infty < \eta$, $E_X(Y) = E_X(0)$, and the map E_X is a submersion at Y;

(FH) for all $\eta > 0$ there exist an integer $p \geq 2$ and $W_1,\dots,W_p \in \mathfrak{g}_1$ such that $\|W_i - X\| < \eta$ for $i = 1,\dots,p$, $\Gamma^{(p)}(W_1,\dots,W_p) = \Gamma^{(p)}(X,\dots,X)$, and $\Gamma^{(p)}$ is a submersion at $(W_1,\dots,W_p)$.

Finally, note that (SP) holds true in particular for $X \in \mathfrak{g}_1$ that are regular points of the endpoint map (i.e., those X for which the straight line $t \mapsto \exp(tX)$ is not abnormal). This property motivates a last condition:

(Reg) the map E is a submersion at X (i.e., E_X is a submersion at 0).

Our main result is the following.

Theorem 1.1. *The following hold:*

- *pliability, strong pliability, and the* (FH)-*condition are equivalent;*
- *regularity* (Reg) *and the submersive* (H)-*condition are equivalent and imply the* (H)-*condition, which, in turn, implies the preceding three conditions.*

We note that the equivalence between the (H)-condition and the submersive (H)-condition does not hold; see Remark 3.13. Theorem 1.1 is proved in Section 3 in a slightly more general version, see Theorem 3.5.

2 Preliminaries

Let us recall some basic facts about Carnot groups, endpoint, and exponential maps.

Definition 2.1. A Carnot group is a nilpotent, connected, and simply connected Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ admits a stratification, i.e., a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s,$$

such that $[\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_{k+1}$ for $k = 1,\dots,s-1$ and $[\mathfrak{g}_1, \mathfrak{g}_s] = 0$.

The *left translation* by $g \in \mathbb{G}$ is the map $L_g\colon \mathbb{G} \to \mathbb{G}$ defined by $L_g(x) := gx$. Through this map, we have the identification

$$(L_g)_*\colon\ \mathfrak{g} \to T_g\mathbb{G}, \qquad \mathfrak{g} \ni X \mapsto (L_g)_*X \in T_g\mathbb{G}.$$

Every vector $X \in \mathfrak{g}$ defines the left-invariant vector field $X(g) := (L_g)_*X$. By abuse of notation, we identify the vector and the induced left-invariant vector field, in particular $X = X(1_{\mathbb{G}})$.

From now on, $I := [0,1] \subset \mathbb{R}$ denotes the unit interval. An absolutely continuous curve $\gamma\colon I \to \mathbb{G}$ is said to be *horizontal* if $\dot\gamma(t) \in (L_{\gamma(t)})_*\mathfrak{g}_1$ for a.e. $t \in I$, that is,

$$\left(L^{-1}_{\gamma(t)}\right)_*\dot\gamma(t) = Y(t) \qquad \text{a.e.},$$

for some function $Y \in L^\infty(I, \mathfrak{g}_1)$, called the *control* of γ.

Definition 2.2. The *endpoint map* is the mapping

$$E\colon\ L^\infty(I, \mathfrak{g}_1) \to \mathbb{G}, \qquad E(Y) := \gamma(1),$$

where γ is the unique solution to the Cauchy problem

$$\begin{aligned}&\left(L^{-1}_{\gamma(t)}\right)_*\dot\gamma(t) = Y(t), \qquad t \in I,\\ &\gamma(0) = 1_{\mathbb{G}}.\end{aligned}$$

Given $X \in \mathfrak{g}_1$, the *endpoint map based at* X is the mapping $E_X\colon L^\infty(I, \mathfrak{g}_1) \to \mathbb{G}$ defined by $E_X(\cdot) := E(X + \cdot)$.

In the following, we need the following well known result, which is proved, for instance, in [1, Section 8].

Lemma 2.3. *For every $X \in \mathfrak{g}_1$, the endpoint map $E_X\colon L^\infty(I, \mathfrak{g}_1) \to \mathbb{G}$ is of class C^1 as soon as $L^\infty(I, \mathfrak{g}_1)$ is endowed with the L^2 topology.*

If $X \in \mathfrak{g}$ and $\gamma\colon I \to \mathbb{G}$ is the integral curve of (the left-invariant vector field identified with) X with $\gamma(0) = 1_{\mathbb{G}}$, then the *exponential map* is defined by

$$\exp\colon\ \mathfrak{g} \to \mathbb{G}, \qquad \exp(X) := \gamma(1).$$

Notice that $\gamma(t) = \exp(tX)$. Such a curve is called a *straight line*.

We denote the flow of a left-invariant vector field X by $\mathrm{e}^{tX}\colon \mathbb{G} \to \mathbb{G}$. We have that

$$\mathrm{e}^{tX}(g) = g\exp(tX).$$

For every $\lambda \in \mathbb{R}$, we define the dilation on the Lie algebra $\delta_\lambda\colon \mathfrak{g} \to \mathfrak{g}$ as the linear map such that

$$\delta_\lambda(X) = \lambda^i X \qquad \text{for all } X \in \mathfrak{g}_i.$$

By abuse of notation, we use the same symbol to denote the dilation on the group $\delta_\lambda\colon \mathbb{G} \to \mathbb{G}$ defined by

$$\delta_\lambda(\exp(X)) = \exp(\delta_\lambda(X)) \tag{2.1}$$

for every $X \in \mathfrak{g}$, so that $(\delta_\lambda)_* = \delta_\lambda$. In particular, for every $X \in \mathfrak{g}_1$ and $Y \in L^\infty(I, \mathfrak{g}_1)$, one has

$$\delta_\lambda(E_X(Y)) = E_{\lambda X}(\lambda Y). \tag{2.2}$$

3 Proof of main result

Let us give a slightly more general form of the definitions of (P) and (SP).

Definition 3.1. Let V be a vector subspace of $L^\infty(I,\mathfrak{g}_1)$. A vector $X\in\mathfrak{g}_1$ is *V-pliable* if the map E_X restricted to V is open at 0. $L^\infty(I,\mathfrak{g}_1)$-pliability coincides with pliability as defined by condition (P).

Definition 3.2. Let V be a vector subspace of $L^\infty(I,\mathfrak{g}_1)$. The vector $X\in\mathfrak{g}_1$ is said to be *strongly V-pliable* if for all $\eta>0$ there exists $Y\in V$ such that

(i) $\|Y\|_\infty<\eta$;

(ii) $E_X(Y)=E_X(0)$;

(iii) $E_X|_V$ is a submersion at Y (i.e., the differential of $E_X|_V$ at Y is surjective).

$L^\infty(I,\mathfrak{g}_1)$-strong pliability coincides with strong pliability as defined by condition (SP).

Remark 3.3. In [4, Definition 3.4] and [11, Definition 4.1], the pliability and the strong pliability of a vector X are defined in terms of the openness and the submersion property of the extended map $\mathcal{E}_X(Y):=(E_X(Y),Y(1))$ restricted to the space

$$\mathcal{C}_0:=\big\{Y\in C^0(I,\mathfrak{g}_1)\mid Y(0)=0\big\}.$$

However, in [4, Proposition 3.7] the openness of $\mathcal{E}_X|_{\mathcal{C}_0}$ is shown to be equivalent to the $\mathcal{C}_0$-pliability (in the sense of Definition 3.1), while in [11, Lemma 4.2] the strong pliability as defined in [11, Definition 4.1] is shown to be equivalent to that introduced in Definition 3.2. As we will see in Theorem 3.5 (considering both $V=\mathcal{C}_0$ and $V=L^\infty(I,\mathfrak{g}_1)$), it turns out that X is $\mathcal{C}_0$-pliable if and only if it is pliable, since both properties are equivalent to the p-free (H)-condition.

Before stating Theorem 1.1 in a slightly generalized version, let us introduce a useful notion.

Definition 3.4. We say that a subspace V of $L^\infty(I,\mathfrak{g}_1)$ is *L^∞-stably L^2-dense in* $L^\infty(I,\mathfrak{g}_1)$ if there exists a constant $C>0$ such that, for every $Y\in L^\infty(I,\mathfrak{g}_1)$, there exists a sequence $(Z_k)_{k\in\mathbb{N}}$ in V converging to Y in $L^2(I,\mathfrak{g}_1)$ and such that

$$\|Z_k\|_\infty\le C\|Y\|_\infty \qquad \text{for every } k\in\mathbb{N}.$$

Examples of L^∞-stably L^2-dense subspaces of $L^\infty(I,\mathfrak{g}_1)$ include $\mathcal{C}_0$ and the subspace of $L^\infty(I,\mathfrak{g}_1)$ made of piecewise constant functions.

Theorem 1.1 is a corollary of the following result.

Theorem 3.5. *Let $\mathbb{G}$ be a Carnot group with Lie algebra $\mathfrak{g}$. Let V be an L^∞-stably L^2-dense subspace of $L^\infty(I,\mathfrak{g}_1)$. For every $X\in\mathfrak{g}_1$, the following conditions are equivalent:*

V-(P) *X is V-pliable;*

V-(SP) *X is strongly V-pliable;*

(FH) *X satisfies the p-free* (H)*-condition.*

Moreover, X satisfies the submersive (H)*-condition* (SbH) *if and only if X is a regular point of E* ((Reg)*-condition), and both conditions imply*

(H) *X satisfies the* (H)*-condition;*

which, in turn, implies the three equivalent conditions V-(P), V-(SP), *and* (FH).

The structure of the proof of Theorem 3.5 is summarized in the following diagram:

$$\begin{tikzcd}
\text{(Reg)} \arrow[rr, Rightarrow] \arrow[d, Leftrightarrow, dashed] & & \text{(SP)} \arrow[d, Rightarrow] & & \text{(FH)} \arrow[ll, Rightarrow, dashed] \\
\text{(SbH)} \arrow[r, Rightarrow] & \text{(H)} \arrow[r, Rightarrow, dashed] & \text{(P)} \arrow[r, Rightarrow] & V\text{-(P)} \arrow[r, Rightarrow, dashed] & V\text{-(SP)} \arrow[u, Rightarrow, dashed]
\end{tikzcd}$$

Here solid arrows denote implications that follow immediately from the definitions, while dashed arrows are proved via different lemmas.

The implication V-(P) $\dashrightarrow$ V-(SP) is a consequence of Lemmas 3.6 and 3.8 (as explained in Remark 3.9). The implications (H) $\dashrightarrow$ (P) and (FH) $\dashrightarrow$ (SP) are proved in Lemma 3.10. The implication V-(SP) $\dashrightarrow$ (FH) is proved in Lemma 3.11. The equivalence (Reg) $\Leftrightarrow$ (SbH) is proved in Lemma 3.12. Finally, we notice that the implications (Reg) $\Longrightarrow$ (SP) and (SbH) $\Longrightarrow$ (FH) cannot be reversed, see Remark 3.13.

Lemma 3.6. *A vector $X \in \mathfrak{g}_1$ is pliable if and only if it is strongly pliable.*

Proof. The fact that strong pliability implies pliability is trivial. The converse can be proved in the following way.

Assume that $X \in \mathfrak{g}_1$ is pliable. It results from the homogeneity property (2.2) of E_X that $\frac{X}{2}$ and $-\frac{X}{2}$ are also pliable. Let $g_1 = \exp(X)$ and $g_{\frac{1}{2}} = \exp\left(\frac{X}{2}\right)$. Fix $\eta > 0$ and denote by B_η be the ball of radius η in $L^\infty(I, \mathfrak{g}_1)$ centered at 0. By pliability of the vector $-\frac{X}{2}$, the set $\mathcal{U}_- = g_1 E_{-\frac{X}{2}}(B_\eta)$ is a neighborhood of $g_{\frac{1}{2}}$. Now we can fix $0 < \eta' < \eta$ such that $\mathcal{U} := E_{\frac{X}{2}}(B_{\eta'}) \subset \mathcal{U}_-$. Since $\frac{X}{2}$ is pliable, $\mathcal{U}$ is a neighborhood of $g_{\frac{1}{2}}$.

Consider

$$\left(L_{\gamma(s)}^{-1}\right)_* \dot\gamma(s) = \frac{X(\gamma(s))}{2} + Y_s(\gamma(s)),$$

which can be seen as a control system with control $s \mapsto Y_s$ taking values in the ball of center 0 and radius η' in $\mathfrak{g}_1$. By a well-known result in geometric control theory (see, e.g., [5, Section 3, Theorem 1]), there exist $k \in \mathbb{N}$, $Y_1, \dots, Y_k \in \mathfrak{g}_1$ with $\|Y_i\| < \eta'$ for $i = 1, \dots, k$, and $t_1, \dots, t_k > 0$ such that $(s_1, \dots, s_k) \mapsto \mathrm{e}^{s_k(\frac{X}{2}+Y_k)} \circ \dots \circ \mathrm{e}^{s_1(\frac{X}{2}+Y_1)}(1_{\mathbb{G}})$ is a submersion at $(t_1, \dots, t_k)$, with $\sum_{i=1}^k t_i$ arbitrary small. Since t_k can be increased without affecting the submersion property, we can assume without loss of generality that $\sum_{i=1}^k t_i = 1$.

Set, for every $\varepsilon \in \mathbb{R}$ and $j \in \{1, \dots, k\}$,

$$\tilde Y_j(\varepsilon) = Y_j + \frac{\varepsilon}{t_j}\left(\frac{X}{2} + Y_j\right),$$

and notice that

$$t_j\left(\frac{X}{2} + \tilde Y_j(\varepsilon)\right) = (t_j + \varepsilon)\left(\frac{X}{2} + Y_j\right), \qquad j \in \{1, \dots, k\},\ \varepsilon \in \mathbb{R}.$$

As a consequence, the map

$$(\varepsilon_1, \dots, \varepsilon_k) \mapsto \mathrm{e}^{t_k(\frac{X}{2}+\tilde Y_k(\varepsilon_k))} \circ \dots \circ \mathrm{e}^{t_1(\frac{X}{2}+\tilde Y_1(\varepsilon_1))}(1_{\mathbb{G}})$$

is a submersion at $(\varepsilon_1, \dots, \varepsilon_k) = (0, \dots, 0)$. In particular, letting $Y' \in L^\infty(I, \mathfrak{g}_1)$ be equal to Y_1 on $[0, t_1)$, then to Y_2 on $[t_1, t_1 + t_2)$, and so on, we have that $Y' \in B_{\eta'}$ and $E_{\frac{X}{2}}$ is a submersion at Y'.

Letting $g'_{\frac{1}{2}} = E_{\frac{X}{2}}(Y') \in \mathcal{U}$, there exists $Y'' \in B_\eta$ such that $g_1 E_{-\frac{X}{2}}(Y'') = g'_{\frac{1}{2}}$. We set

$$Y(t) = \begin{cases} 2Y'(2t), & t \in \left[0, \frac{1}{2}\right], \\ -2Y''(2(1-t)), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, we have $E_X(Y) = g_1$ and, since $\mathrm{d}_{Y'}E_{\frac{X}{2}}$ is surjective, also the differential at $\mathrm{d}_Y E_X$ is surjective. Since $\|Y\|_\infty < 2\eta$ and η is arbitrarily small, the strong pliability of X is proved. ■

Remark 3.7. Let us stress that Lemma 3.6 relies essentially on the special properties of the endpoint map, in the sense that for the general case of a smooth open map F at a point X, it is not true that there exist points arbitrarily close to X whose image through F coincide with $F(X)$ and at which F is a submersion. Consider, for instance, $F\colon L^\infty(I,\mathbb{R}) \ni X \mapsto \left(\int_I X(t)\mathrm{d}t\right)^3 \in \mathbb{R}$. The map F is open at 0, but for every $X \in L^\infty(I,\mathbb{R})$ with $F(X) = 0$ the differential of F at X is the null map.

Lemma 3.8. *Assume that V is an L^∞-stably L^2-dense subspace of $L^\infty(I,\mathfrak{g}_1)$. Then, a vector $X \in \mathfrak{g}_1$ is strongly pliable if and only if X is strongly V-pliable.*

Proof. Since one implication is trivial, let us assume that X is strongly pliable and let us prove that it is strongly V-pliable. Fix $\eta > 0$ and $C > 0$ as in Definition 3.4. Then there exists $Y \in L^\infty(I,\mathfrak{g}_1)$ with $\|Y\|_\infty < \frac{\eta}{2C}$ such that $E_X(Y) = E_X(0)$ and $\mathrm{d}E_X(Y)$ is surjective. Pick $Y_1,\dots,Y_d \in L^\infty(I,\mathfrak{g}_1)$ such that $\mathrm{d}E_X(Y)Y_1,\dots,\mathrm{d}E_X(Y)Y_d$ is a basis of $T_{E_X(0)}\mathbb{G}$, where d is the dimension of $\mathbb{G}$. Since V is dense in $L^\infty(I,\mathfrak{g}_1)$ for the L^2-topology, and the map $Z \mapsto \mathrm{d}E_X(Y)Z$ is continuous in $L^2(I,\mathfrak{g}_1)$, we can assume that $Y_1,\dots,Y_d \in V$ by a standard approximation argument.

Consider the function $\mathbb{R}^d \ni (\alpha_1,\dots,\alpha_d) \mapsto E_X(Y + \alpha_1 Y_1 + \cdots + \alpha_d Y_d)$. By the inverse function theorem, for $r > 0$ sufficiently small, such a function diffeomorphically maps $D = \{(\alpha_1,\dots,\alpha_d) \in \mathbb{R}^d \mid |\alpha_1|,\dots,|\alpha_d| < r\}$ into $E_X(D)$. Without loss of generality,

$$r < \frac{\eta}{2(\|Y_1\|_\infty + \cdots + \|Y_d\|_\infty)}.$$

Let $Z \in V$ be such that $\|Z - Y\|_2 < \varepsilon$, for some $\varepsilon > 0$ to be fixed later. Since V is L^∞-stably L^2-dense in $L^\infty(I,\mathfrak{g}_1)$, we can assume that $\|Z\|_\infty \le C\|Y\|_\infty < \frac{\eta}{2}$. Notice that $E_X(Z + \alpha_1 Y_1 + \cdots + \alpha_d Y_d)$ is arbitrarily close to $E_X(Y + \alpha_1 Y_1 + \cdots + \alpha_d Y_d)$ as ε goes to zero, uniformly with respect to $(\alpha_1,\dots,\alpha_d) \in D$ in any fixed system of coordinates. Using a classical degree argument similar to the one in [11, Lemma 5.5], and choosing $\varepsilon > 0$ small enough, we have that $E_X(0) = E_X(w)$ for some $w \in \{Z + \alpha_1 Y_1 + \cdots + \alpha_d Y_d \mid |\alpha_1|,\dots,|\alpha_d| < r\} \subset V$ and $\mathrm{d}E_X(w)Y_1,\dots,\mathrm{d}E_X(w)Y_d$ is a basis of $T_{E_X(0)}\mathbb{G}$.

Notice that, up to further reducing ε if necessary,

$$\|w\|_\infty \le \|Z\|_\infty + r(\|Y_1\|_\infty + \cdots + \|Y_d\|_\infty) < \eta.$$

Therefore, Properties (i), (ii), and (iii) in Definition 3.2 are satisfied by $w \in V$. ■

Remark 3.9. Let V be an L^∞-stably L^2-dense subspace of $L^\infty(I,\mathfrak{g}_1)$. Then, by the two previous results, it follows that

$$\text{str. } V\text{-pliable} \implies V\text{-pliable} \implies \text{pliable} \overset{\text{Lemma 3.6}}{\implies} \text{str. pliable} \overset{\text{Lemma 3.8}}{\implies} \text{str. } V\text{-pliable}.$$

Lemma 3.10. *If a vector $X \in \mathfrak{g}_1$ satisfies the* (H)*-condition* (*respectively, the* (FH)*-condition*), *then it is pliable* (*respectively, strongly pliable*).

Proof. If $X \in \mathfrak{g}_1$ satisfies the (H)-condition, there exists $p \in \mathbb{N}$ such that the map

$$(Y_1,\dots,Y_p) \mapsto \exp(Y_1)\cdots\exp(Y_p)$$

is open at $(X,\dots,X)$. Hence, given any $\eta > 0$, $\left\{\mathrm{e}^{Y_p}\circ\cdots\circ\mathrm{e}^{Y_1}(1_\mathbb{G}) \mid \|Y_i - X\| < \eta \text{ for } i = 1,\dots,p\right\}$ is a neighborhood of $\mathrm{e}^{pX}(1_\mathbb{G})$. Let $\delta_{\frac{1}{p}}$ be the dilation in $\mathbb{G}$ of parameter $1/p$. Then, thanks to (2.1), for every $(Y_1,\dots,Y_p) \in (\mathfrak{g}_1)^p$ we have

$$\mathrm{e}^{\frac{1}{p}Y_p} \circ \cdots \circ \mathrm{e}^{\frac{1}{p}Y_1}(1_\mathbb{G}) = \delta_{\frac{1}{p}}\left(\mathrm{e}^{Y_p} \circ \cdots \circ \mathrm{e}^{Y_1}(1_\mathbb{G})\right).$$

Since, moreover, $e^{\frac{1}{p}Y_p} \circ \cdots \circ e^{\frac{1}{p}Y_1}(1_{\mathbb{G}})$ is the image by E_X of the control constantly equal to Y_i on $((i-1)/p, i/p)$ for $i = 1, \dots, p$, it turns out that the image by E_X of the set

$$\left\{Y \in L^\infty(I, \mathfrak{g}_1) \mid \|Y\|_\infty < \eta,\ Y|_{(i/p,(i+1)/p)} \text{ constant } \forall i = 0, \dots, p-1\right\}$$

is a neighborhood of $e^X(1_{\mathbb{G}})$, which implies that X is pliable.

The proof in the case of the (FH)-condition is similar. ∎

Lemma 3.11. *If $X \in \mathfrak{g}_1$ is strongly pliable, then it satisfies the* (FH)*-condition.*

Proof. Let $X \in \mathfrak{g}_1$ be strongly pliable, and denote by $\mathrm{PC} = \mathrm{PC}(I, \mathfrak{g}_1)$ the subspace of $L^\infty(I, \mathfrak{g}_1)$ made of piecewise constant functions with rational discontinuity points. By Remark 3.9, X is strongly PC-pliable.

Fix $\eta > 0$. By the strong PC-pliability, there exist $Z, Y_1, \dots, Y_d \in \mathrm{PC}$, with $\|Z\|_\infty < \eta$, such that $E_X(Z) = E_X(0)$ and $\mathrm{d}E_X(Z)Y_1, \dots, \mathrm{d}E_X(Z)Y_d$ is a basis of $T_{E_X(0)}\mathbb{G}$. By definition of the space PC, there exists an integer $p \geq 2$ such that $Z|_{((i-1)/p,i/p)}$, $Y_1|_{((i-1)/p,i/p)}, \dots, Y_d|_{((i-1)/p,i/p)}$ are constant for every $i = 1, \dots, p$.

Denote by Z_i and $Y_{j,i}$ the values of Z and Y_j on the interval $\left(\frac{i-1}{p}, \frac{i}{p}\right)$. Then the map $\Gamma^{(p)} \colon (\mathfrak{g}_1)^p \to \mathbb{G}$ defined by

$$\Gamma^{(p)}(W_1, \dots, W_p) := e^{W_p} \circ \cdots \circ e^{W_1}(1_{\mathbb{G}}),$$

satisfies $\Gamma^{(p)}(X+Z_1, \dots, X+Z_p) = \Gamma^{(p)}(X, \dots, X)$. It remains to show that $\Gamma^{(p)}$ is a submersion at $(X + Z_1, \dots, X + Z_p)$.

We have that $Y_j = \sum_{i=1}^p Y_{j,i}\chi_{((i-1)/p,i/p)}$ and so

$$\mathrm{d}E_X(Z)Y_j = \sum_{i=1}^p \mathrm{d}E_X(Z)Y_{j,i}\chi_{(\frac{i-1}{p},\frac{i}{p})}.$$

Moreover,

$$\begin{aligned}
&\mathrm{d}E_X(Z)Y_{j,i}\chi_{(\frac{i-1}{p},\frac{i}{p})}\\
&\quad= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\left(e^{\frac{X+Z_p}{p}} \circ \cdots \circ e^{\frac{X+Z_{i+1}}{p}} \circ e^{\frac{X+Z_i+\varepsilon Y_{j,i}}{p}} \circ e^{\frac{X+Z_{i-1}}{p}} \circ \cdots \circ e^{\frac{X+Z_1}{p}}(1_{\mathbb{G}})\right)\\
&\quad\overset{(2.1)}{=} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \delta_{\frac{1}{p}}\left(e^{X+Z_p} \circ \cdots \circ e^{X+Z_{i+1}} \circ e^{X+Z_i+\varepsilon Y_{j,i}} \circ e^{X+Z_{i-1}} \circ \cdots \circ e^{X+Z_1}(1_{\mathbb{G}})\right)\\
&\quad= \left(\delta_{\frac{1}{p}}\right)_* \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\left(e^{X+Z_p} \circ \cdots \circ e^{X+Z_{i+1}} \circ e^{X+Z_i+\varepsilon Y_{j,i}} \circ e^{X+Z_{i-1}} \circ \cdots \circ e^{X+Z_1}(1_{\mathbb{G}})\right)\\
&\quad= \left(\delta_{\frac{1}{p}}\right)_* \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \Gamma^{(p)}(X + Z_1, \dots, X + Z_{i-1}, X + Z_i + \varepsilon Y_{j,i}, X + Z_{i+1}, \dots, X + Z_p)\\
&\quad= \left(\delta_{\frac{1}{p}}\right)_* \mathrm{d}\Gamma^{(p)}(X + Z_1, \dots, X + Z_p) \cdot (0, \dots, 0, Y_{j,i}, 0, \dots, 0).
\end{aligned} \tag{3.1}$$

So the image of the differential of $\Gamma^{(p)}$ at $(X+Z_1, \dots, X+Z_p)$ contains the image by $(\delta_p)_* = \left(\delta_{\frac{1}{p}}\right)_*^{-1}$ of the span of $\mathrm{d}E_X(Z)Y_1, \dots, \mathrm{d}E_X(Z)Y_d$. Hence $\Gamma^{(p)}$ is a submersion at $(X+Z_1, \dots, X+Z_p)$. ∎

Lemma 3.12. *A vector $X \in \mathfrak{g}_1$ satisfies the submersive* (H)*-condition* (SbH) *if and only if it is a regular point of the endpoint map E.*

Proof. On the one hand, if $X \in \mathfrak{g}_1$ satisfies (SbH), then a computation analogous to (3.1) with $Z = 0$ shows that $\mathrm{d}E_X(0)$ is onto. On the other hand, note that the argument of Lemma 3.8 shows that, if E_X is a submersion at 0, then E_X restricted to PC is also a submersion at 0. Then the argument used in the proof of Lemma 3.11 shows that X satisfies the submersive (H)-condition. ∎

Remark 3.13. The (H)-condition is strictly weaker than the submersive (H)-condition (SbH). In particular, also pliability, strong pliability, and the (FH)-condition are strictly weaker than (SbH). Indeed, Carnot groups of step two are known to satisfy the (H)-condition [8, Theorem 2.1], while non-Métivier step two Carnot groups admit abnormal curves of the form $t \mapsto \exp(tX)$ (see [6, Proposition 3.6]) and, hence, vectors X that do not satisfy the submersive (H)-condition.

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References

[1] Agrachev A., Barilari D., Boscain U., A comprehensive introduction to sub-Riemannian geometry, *Cambridge Stud. Adv. Math.*, Vol. 181, Cambridge University Press, Cambridge, 2020.

[2] Arena G., Caruso A.O., Monti R., Regularity properties of H-convex sets, *J. Geom. Anal.* **22** (2012), 583–602.

[3] Cheeger J., Kleiner B., Metric differentiation, monotonicity and maps to L^1, *Invent. Math.* **182** (2010), 335–370, arXiv:0907.3295.

[4] Juillet N., Sigalotti M., Pliability, or the Whitney extension theorem for curves in Carnot groups, *Anal. PDE* **10** (2017), 1637–1661, arXiv:1603.02639.

[5] Jurdjevic V., Geometric control theory, *Cambridge Stud. Adv. Math.*, Vol. 52, Cambridge University Press, Cambridge, 1997.

[6] Montanari A., Morbidelli D., On the lack of semiconcavity of the subRiemannian distance in a class of Carnot groups, *J. Math. Anal. Appl.* **444** (2016), 1652–1674, arXiv:1508.00997.

[7] Montanari A., Morbidelli D., Multiexponential maps in Carnot groups with applications to convexity and differentiability, *Ann. Mat. Pura Appl.* **200** (2021), 253–272, arXiv:1908.01124.

[8] Morbidelli D., On the inner cone property for convex sets in two-step Carnot groups, with applications to monotone sets, *Publ. Mat.* **64** (2020), 391–421, arXiv:1808.06513.

[9] Pinamonti A., Speight G., Universal differentiability sets in Carnot groups of arbitrarily high step, *Israel J. Math.* **240** (2020), 445–502, arXiv:1711.11433.

[10] Rigot S., Monotone sets and local minimizers for the perimeter in Carnot groups, *Nonlinear Anal.* **237** (2023), 113369, 16 pages, arXiv:2206.03094.

[11] Sacchelli L., Sigalotti M., On the Whitney extension property for continuously differentiable horizontal curves in sub-Riemannian manifolds, *Calc. Var. Partial Differential Equations* **57** (2018), 59, 34 pages, arXiv:1708.02795.

[12] Speight G., Zimmerman S., Directional pliability, Whitney extension, and Lusin approximation for curves in Carnot groups, *Ann. Fenn. Math.* **50** (2025), 665–684, arXiv:2505.14678.