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Voss Surfaces in Sine-Gordon Hierarchies

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Abstract. We explore a method, initiated by Guichard in 1890, which allows to generate sequences of Voss surfaces, starting from an arbitrarily chosen pseudospherical surface and a seed solution of the Moutard equation, by means of two simple transformations. In this paper, we 1) identify the Guichard transformations with the well-known recursion operator for symmetries of the sine-Gordon equation and its inverse; 2) prove a lemma which allows us to derive the length of Guichard's sequences from the invariance properties of the initial sine-Gordon solution; 3) introduce an extended class of inverted operators, expanding the class of Voss surfaces obtainable by quadratures; 4) clarify relevant aspects of Guthrie's formalism, paving the way for the future employment of the entire division algebra of recursion operators. A number of Voss nets are presented explicitly.

Key words: Voss surface; Voss net; sine-Gordon equation; Moutard equation; recursion operator; Guichard transformation; pmKdV hierarchy; Khor'kova hierarchy; Guichard sequence; division algebra of recursion operators

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1 Introduction

In 1888, A. Voss studied conjugate nets formed by geodesics [61], uncovering their basic properties and their relation to pseudospherical surfaces, thereby also to the sine-Gordon equation

$$\phi_{xy} = \sin\phi. \tag{1.1}$$

Independently, C. Guichard [22] investigated the same class in the context of congruences, being the first to notice the relation to the Moutard equation

$$\Phi_{xy} = \Phi\cos\phi \tag{1.2}$$

and its transformations. Accounts of the first period of investigation were given by Gambier [21] and to some extent also by Salkowski [53]. The problem of finding explicit solutions was addressed by many authors, notably Eisenhart [13, 14, 15, 16, 17]. Razzaboni [49] sought minimal Voss surfaces. The most familiar explicit example is the right helicoid [61], carrying infinitely many Voss nets [18, 19]. For the Voss surfaces of revolution see Tachauer [58] and Gambier [20, 21], for explicit Voss surfaces associated with the Enneper surfaces see Diller [9].

Voss surfaces are also amenable to modern methods of integrable systems. Babich [3] obtained Voss surfaces from the ψ-function of the Lax pair for the sine-Gordon equation. In a different way, Lax integrability was established by Lin and Conte [37], who also revisited particular surfaces considered by Gambier.

Voss nets have seen a recent resurrection in connection with geometric modelling. In large-span free-form architecture, discrete Voss nets (Sauer and Graf [57], Sauer [56], Doliwa [12], Izmestiev et al. [28]) model elastically bent lath constructions with planar cladding (Montagne et al. [43, 44], Kilian et al. [30] and references therein). Smooth Voss nets are smooth limits of discrete Voss nets, revealing shapes achievable with this construction technology. In this vein, Rasoulzadeh [48] investigated deployable Voss nets.

In this paper, we draw on linking the classical results with the modern theory of integrable systems. Our starting point is the following classical construction, summarised in Section 2. In short, any solution ϕ to the sine-Gordon equation (1.1) determines a pseudospherical surface via the Gauss–Weingarten equations; any solution Φ to the Moutard equation (1.2) then determines a Voss surface algebraically, although possibly a degenerate one. This extends to linear combinations of solutions Φ by linearity.

In Section 3, we observe that Moutard's equation essentially matches the sine-Gordon symmetry condition. This is the way symmetries of the sine-Gordon equation give rise to Voss surfaces. We work through three simple explicit examples, summarised in Table 2.

In Section 4, we recall Olver's [45] recursion operator $\mathcal{R}$ for symmetries of the sine-Gordon equation. By definition, this is an integro-differential operator that sends symmetries to symmetries. To resolve multiple issues inherent in the integro-differential framework, Guthrie [24, 25] reinterpreted recursion operators essentially as Bäcklund auto-transformations for the linearised equation [40].[1]

Incidentally, the Guthrie form of $\mathcal{R}$ coincides with a transformation discovered by Guichard in 1890, the same being true for the inverse $\mathcal{R}^{-1}$. Since the treatment of $\mathcal{R}^{-1}$ appears to be somewhat problematic in recent literature (see footnote 6 below), we pay special attention to understanding of $\mathcal{R}^{-1}$.

When iterated, both $\mathcal{R}$ an $\mathcal{R}^{-1}$ generate two infinite families of symmetries, namely, the pmKdV and the Khor'kova hierarchies in two versions, differing by interchanging x and y. These are discussed in Section 5, along with some examples of the Guichard sequences of Voss nets, obtained from one and the same pseudospherical surface with the help of the recursion operator. Examples show that the sequences stop after a few steps, as observed already by Guichard.

The stopping problem (we call it the Guichard problem) is addressed in Section 6. With our Lemma 6.1, we link the number of independent nontrivial Voss surfaces in the Guichard sequence to the symmetry invariance properties of the initial sine-Gordon solution.

While Lemma 6.1 limits the number of Voss surfaces obtainable with the help of $\mathcal{R}$ and $\mathcal{R}^{-1}$, it imposes no restrictions on results obtained with the help of the general inversion $(\mathcal{R}+\lambda\,\mathrm{Id})^{-1}$. These are examined in Section 7 after reduction to $(\mathcal{R}+\mathrm{Id})^{-1}$. The main result is a formula for $(\mathcal{R}+\mathrm{Id})^{-1}\Phi$ given in Proposition 7.5. We explicitly obtain the first member of what we call the negative $\mathcal{R}+\mathrm{Id}$ hierarchy.

In Section 8, we demonstrate that the new hierarchy does indeed take us outside of the Guichard sequence. What is more, we show that Proposition 7.5 opens another path from a pseudospherical surface to a Voss net. A chance observation of this new route was the initial impetus for writing the present paper.

The paper is closed with a digression on the algebra of recursion operators. We prove that every element of the division algebra generated by $\mathcal{R}$ yields a true recursion operator for the sine-Gordon equation. This prepares the ground for the future use of the entire division algebra.

Smoothness is assumed throughout the paper.

2 Essentials about Voss nets and Voss surfaces

In this section, we briefly review known geometric facts that link Voss surfaces to pseudospherical surfaces and solutions of the Moutard equation.

[1]This point of view was actually pioneered in a series of papers by Papachristou, e.g., [46]. It is quite explicit in recent works by Habibullin et al., e.g., [26].

According to Dini [10], a surface is pseudospherical (i.e., has a constant negative Gaussian curvature) if and only if it admits a parameterisation such that the fundamental forms are

$$\begin{aligned} &\mathrm{I} = \mathrm{d}x^2 + 2\cos\phi\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}y^2, \qquad \mathrm{II} = 2\sin\phi\,\mathrm{d}x\,\mathrm{d}y, \\ &\mathrm{III} = \mathrm{d}x^2 - 2\cos\phi\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}y^2, \end{aligned} \tag{2.1}$$

where ϕ is the *intersection angle*, also known as the *Chebyshev angle*. The parameterisation (2.1) will be called the *Dini parameterisation*. The form of I (resp. II) implies that the Dini parameterisation is *Chebyshev* (resp. *asymptotic*) on the surface. The form of III conveys that Dini's parameterisation is Chebyshev on the Gaussian image.

The Gauss–Weingarten system corresponding to the fundamental forms (2.1) is

$$\begin{aligned} &\mathbf{r}_{xx} = \frac{\phi_x}{\tan\phi}\mathbf{r}_x - \frac{\phi_x}{\sin\phi}\mathbf{r}_y, \qquad \mathbf{r}_{xy} = \sin\phi\,\mathbf{n}, \qquad \mathbf{r}_{yy} = -\frac{\phi_y}{\sin\phi}\mathbf{r}_x + \frac{\phi_y}{\tan\phi}\mathbf{r}_y, \\ &\mathbf{n}_x = \frac{1}{\tan\phi}\mathbf{r}_x - \frac{1}{\sin\phi}\mathbf{r}_y, \qquad \mathbf{n}_y = \frac{1}{\tan\phi}\mathbf{r}_y - \frac{1}{\sin\phi}\mathbf{r}_x, \end{aligned} \tag{2.2}$$

while the Gauss–Mainardi–Codazzi system reduces to the sine-Gordon equation (1.1). Thus, the sine-Gordon equation determines both pseudospherical surfaces and Chebyshev nets on the sphere.

Definition 2.1 ([61])**.** *Voss nets* are conjugate nets formed by geodesics. *Voss surfaces* are surfaces capable of carrying a Voss net.

Recall that conjugate nets are defined by the diagonality of the second fundamental form, that is, $\mathrm{II}_{12} = 0$. As already observed by Voss [61], the Gaussian image of a Voss net is a Chebyshev net on the Gauss sphere, which corresponds to a solution of the sine-Gordon equation. The problem of reconstruction of the Voss net from its Gaussian image can be reduced to a second-order linear PDE, available in several distinct forms.[2] One of them is the Moutard equation (1.2), which lies at the roots of the classical relationship between infinitesimal deformations of pseudospherical surfaces and Voss nets in tangential coordinates [5, 14, 28]. This piece of classical theory is summarised in Proposition 2.2, which is a variation of [28, Theorem 2.25]. Explanatory references are given in Remark 2.3.

Proposition 2.2. *Let* $\mathbf{r}(x, y)$ *be a pseudospherical surface in Dini's parameterisation, let* ϕ *denote the Chebyshev angle. Let* Φ *be a solution of the Moutard equation* $\Phi_{xy} = \Phi\cos\phi$*.*

(*i*) *The net*

$$\mathbf{q} = \Phi\mathbf{n} - \frac{\Phi_y\mathbf{r}_x + \Phi_x\mathbf{r}_y}{\sin\phi}, \tag{2.3}$$

if non-singular, is a Voss net, possessing the same normal vector $\mathbf{n}$ *and satisfying* $\mathbf{q}\cdot\mathbf{n} = \Phi$*. Its fundamental forms are*

$$\mathrm{I} = \frac{X^2\,\mathrm{d}x^2 + 2XY\cos\phi\,\mathrm{d}x\,\mathrm{d}y + Y^2\,\mathrm{d}y^2}{\sin\phi^2}, \qquad \mathrm{II} = -X\,\mathrm{d}x^2 - Y\,\mathrm{d}y^2, \tag{2.4}$$

where

$$X = \Phi_{xx} + \Phi - \left(\frac{\Phi_x}{\tan\phi} + \frac{\Phi_y}{\sin\phi}\right)\phi_x, \qquad Y = \Phi_{yy} + \Phi - \left(\frac{\Phi_y}{\tan\phi} + \frac{\Phi_x}{\sin\phi}\right)\phi_y. \tag{2.5}$$

[2] For instance, Diller uses the linear equation $M_{xy}\sin\phi + \phi_x M_y + \phi_y M_x = 0$ and finds solutions M such that M_x and M_y are functions of ϕ.

The Gauss and mean curvatures are

$$K = \frac{\sin^2\phi}{XY}, \qquad H = -\frac{1}{2}\left(\frac{1}{X} + \frac{1}{Y}\right),$$

respectively.

(*ii*) *The singularity condition* $\det \mathrm{I} = 0$ *of the Voss net* (2.3) *is*

$$XY = 0. \tag{2.6}$$

(*iii*) *If one of functions X, Y vanishes everywhere, then so does the other. This happens if and only if the associated net degenerates to a point.*

(*iv*) *Every non-singular Voss net is obtainable in the way described in statement* (i) *from a solution ϕ of the sine-Gordon equation and a solution Φ of the Moutard equation.*

Proof. (i) The fundamental forms I and II of the net $\mathbf{q}$ given by equation (2.3) are obtained by straightforward computation. Since $\mathrm{II}_{12} = 0$, the parametric net is conjugate. Computation of geodesic curvatures shows that the net is also geodesic. The curvatures K, H are also straightforward to compute.

(ii) Clearly, $\det \mathrm{I} = XY/\sin^2\phi$, which proves the second statement.

(iii) It is easy to check that

$$X_y = -\frac{\phi_x}{\sin\phi} Y, \qquad Y_x = -\frac{\phi_y}{\sin\phi} X \tag{2.7}$$

in consequence of the sine-Gordon and Moutard equations. Hence, if $X = 0$, then $Y = 0$, and vice versa. Moreover,

$$\mathbf{q}_x = -\frac{X}{\sin\phi}\mathbf{r}_y, \qquad \mathbf{q}_y = -\frac{Y}{\sin\phi}\mathbf{r}_x, \tag{2.8}$$

which finishes the proof of the third statement.

(iv) Consider a conjugate net with the first fundamental form I given by formula (2.4) and the second fundamental form $\mathrm{II} = \xi X \, \mathrm{d}x^2 + \eta Y \, \mathrm{d}y^2$, where X, Y, ξ, η are arbitrary functions of x, y; this is generic for conjugate nets. The condition that the isoparametric curves are geodesics is equivalent to system (2.7). Assuming (2.7), the Gauss–Mainardi–Codazzi system reduces to $\xi_y = 0$, $\eta_x = 0$, $\phi_{xy} = \xi\eta\sin\phi$. Hence, ξ and η are functions of x and y, respectively. As such, they can be removed by separate transformations of x and y, preserving the net. We are left with the sine-Gordon equation for ϕ and system (2.7) for X, Y. These imply compatibility of the system

$$\Phi_{xx} + \Phi - \left(\frac{\Phi_x}{\tan\phi} + \frac{\Phi_y}{\sin\phi}\right)\phi_x = X, \qquad \Phi_{xy} - \Phi\cos\phi = 0,$$
$$\Phi_{yy} + \Phi - \left(\frac{\Phi_y}{\tan\phi} + \frac{\Phi_x}{\sin\phi}\right)\phi_y = Y$$

composed of the Moutard equation and equations (2.5) solved for Φ_{xx} and Φ_{yy}. Then the net $\mathbf{q}$ defined by formula (2.3) has the same normal vector $\mathbf{n}$, the same support function Φ, and the same fundamental forms (2.4). Therefore, ϕ and Φ are the solutions sought. ■

The Voss net (2.3) is said to be *associated* with the solution Φ and the initial pseudospherical surface $\mathbf{r}$.[3]

[3]This is consistent with the well-known general definition of associate nets [5, Section 225], [14, Section 155] via the relation $\mathrm{II}_{11}\mathrm{II}'_{22} - 2\mathrm{II}_{12}\mathrm{II}'_{12} + \mathrm{II}_{22}\mathrm{II}'_{11} = 0$.

Remark 2.3. We refer the reader to Bianchi [4, 5] or Eisenhart [14] or Izmestiev [28] for a broader geometric perspective. In particular, the geometric meaning contained in Proposition 2.2 can be explained as follows.

(1) According to [4, Section 81] or [14, Section 67], the *support function* of a surface $\mathbf{q}$ is the distance $\mathbf{q} \cdot \mathbf{n}$ of the tangent planes from the origin; the normal vector $\mathbf{n}$ and the support function $w = \mathbf{q} \cdot \mathbf{n}$ are referred to as the *tangential coordinates* of a surface.

(2) For a pseudospherical surface $\mathbf{r}$ in Dini's parameterisation, solutions Φ of the Moutard equation can be identified with the Weingarten characteristic functions of infinitesimal isometric deformations, see [5, Chapter XV, Section 226, case 1°] or [14, Chapter XI] or [28, Section 2.5]. According to Bianchi [5, Section 225], the associate net $\mathbf{q}$ can be obtained by taking Φ for the support function w; it is then known to possess the Voss property.

(3) Once Φ and $\mathbf{n}$ are known, the position vector $\mathbf{q}$ can be obtained by solving the linear system

$$\mathbf{q} \cdot \mathbf{n} = \Phi, \qquad \mathbf{q} \cdot \mathbf{n}_x = \Phi_x, \qquad \mathbf{q} \cdot \mathbf{n}_y = \Phi_y \tag{2.9}$$

(see [4, Section 81, equations (a) and (b)], [14, Section 67, equations (29) and (30)]). The explicit expression for the solution of system (2.9) is

$$\mathbf{q} = \mathrm{III}^{ij} \Phi_{,i} \mathbf{n}_{,j} + \Phi \mathbf{n}, \tag{2.10}$$

where III^{ij} denotes the inverse of the Gauss sphere's metric III_{ij}, cf. [4, Chapter V, Section 81, equation (34)] or [14, Section 67, equation (32)].

(4) Formula (2.10), which yields $\mathbf{q}$ in terms of Φ and $\mathbf{n}$, becomes [28, equation (24)], which can be reduced to our formula (2.3) by expressing $\mathbf{n}_x$ and $\mathbf{n}_y$ in terms of $\mathbf{r}_x$ and $\mathbf{r}_y$ via the Mainardi–Codazzi equations. This slight simplification makes the only difference between Proposition 2.2 and [28, Theorem 2.25].

The overall picture can be conveyed by the pullback diagram

$$\begin{tikzcd}
\boxed{\text{pseudospherical surface and its deformation}} \arrow[r, leftrightarrow, "\text{characteristic} \leftrightarrow \text{support}"] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \boxed{\text{Voss net}} \arrow[d, "\text{Gauss map}"] \\
\boxed{\text{pseudospherical surface}} \arrow[r, leftrightarrow, "\text{Gauss–Lelieuvre}"'] & \boxed{\text{Chebyshev net on the sphere}}
\end{tikzcd}$$

By the deformation we mean the infinitesimal isometric deformation [5, Chapter XV] or [14, Chapter XI].

Corollary 2.4. *Given a pseudospherical surface of Chebyshev angle ϕ, let Φ, Φ' be two solutions of the Moutard equation (1.2) and X, X' (resp. Y, Y') be the corresponding functions (2.5). Then the associated Voss nets*

$$\mathbf{q} = \Phi \mathbf{n} - \frac{\Phi_y \mathbf{r}_x + \Phi_x \mathbf{r}_y}{\sin \phi}, \qquad \mathbf{q}' = \Phi' \mathbf{n} - \frac{\Phi'_y \mathbf{r}_x + \Phi'_x \mathbf{r}_y}{\sin \phi},$$

differ by a translation if and only if any of the equivalent conditions $X = X'$ or $Y = Y'$ holds.

Proof. By Proposition 2.2 (iii) and linearity. ■

Given a solution ϕ of the sine-Gordon equation, a number of exact solutions Φ of the Moutard equation are known. Many have been known since the nineteenth century, see Guichard [22, 23]. They will be reviewed in the next section. However, not every pair ϕ, Φ produces a nontrivial Voss net because of Proposition 2.2 (ii) and Corollary 2.4.

Example 2.5. For all pseudospherical surfaces $\mathbf{r}$, the tangential coordinates $\Phi = \mathbf{c} \cdot \mathbf{n}$, where $\mathbf{c} = \text{const}$, and $\Phi = w = \mathbf{r} \cdot \mathbf{n}$ are solutions of the Moutard equation (1.2), see [22, p. 234]. Unfortunately, both satisfy the condition (2.6) and thus yield degenerate Voss nets. According to Corollary 2.4, these are not useful even in linear combinations with other solutions. For another derivation see Proposition 7.6 below.

3 Voss nets from symmetries

In this section, we exploit the link between the Moutard equation and symmetries of the sine-Gordon equation. We do so because the sine-Gordon theory is widely known, sufficiently developed, and well suited to our needs. For a background in the geometric theory of symmetries, recursion operators and coverings one can consult the paper [35] or books [6, 36].

All the mixed derivatives $\phi_{xy}, \phi_{xxy}, \phi_{xyy}, \dots$ can be expressed in terms of the pure derivatives $\phi, \phi_x, \phi_{xx}, \phi_{xxx}, \dots, \phi_y, \phi_{yy}, \phi_{yyy}, \dots$ via the sine-Gordon equation (1.1); the equation then determines a manifold, naturally coordinatised by $x, y, \phi, \phi_x, \phi_{xx}, \phi_{xxx}, \dots, \phi_y, \phi_{yy}, \phi_{yyy}, \dots$ and equipped with the *total derivatives, that is, the vector fields*

$$D_x = \frac{\partial}{\partial x} + \sum_\mu \phi_{x\mu} \frac{\partial}{\partial \phi_\mu}, \qquad D_y = \frac{\partial}{\partial y} + \sum_\mu \phi_{y\mu} \frac{\partial}{\partial \phi_\mu},$$

where $\mu = x^m y^n$ denotes an arbitrary monomial,[4] *$\phi_{xy\mu} = D_\mu \sin\phi$, and*

$$D_{x^m y^n} = D_x^m \circ D_y^n = D_y^n \circ D_x^m.$$

Assuming, as we do, that functions $\boldsymbol{\phi}(x, y, \phi, \phi_x, \phi_{xx}, \dots, \phi_y, \phi_{yy}, \dots)$ depend only on a finite number of arguments, all sums over μ in $D_x\boldsymbol{\phi}$, $D_y\boldsymbol{\phi}$ are finite.

A function $\boldsymbol{\phi}(x, y, \phi, \phi_x, \phi_{xx}, \dots, \phi_y, \phi_{yy}, \dots)$ is called a *symmetry* (more precisely, a *local symmetry*) of the sine-Gordon equation, if

$$D_{xy}\boldsymbol{\phi} = \boldsymbol{\phi} \cos\phi. \tag{3.1}$$

Equation (3.1) is called the *linearised sine-Gordon equation in total derivatives*. Examples of symmetries are provided by the members of the pmKdV hierarchy (4.8).

Denoting by t an auxiliary variable different from x, y (a "time"), the *flow* of a symmetry $\boldsymbol{\phi}$ is the evolution equation

$$\phi_t = \boldsymbol{\phi}(x, y, \phi, \phi_x, \phi_{xx}, \dots, \phi_y, \phi_{yy}, \dots). \tag{3.2}$$

Equation (3.1) is the compatibility condition between the flow (3.2) and the sine-Gordon equation (1.1). Indeed, the standard cross-differentiation test gives $D_{xy}\boldsymbol{\phi} = \phi_{txy} = \phi_{xyt} = D_t(\sin\phi)$ $= \phi_t \cos\phi = \boldsymbol{\phi}\cos\phi$.

A solution ϕ of the sine-Gordon equation is said to be *invariant* with respect to a symmetry $\boldsymbol{\phi}$ if $\boldsymbol{\phi}(x, y, \phi, \phi_x, \phi_{xx}, \dots, \phi_y, \phi_{yy}, \dots) = 0$. Thus, the invariant solutions are just the stationary solutions of the associated flows.

By *classical Lie symmetries* we mean symmetries that depend on x, y, ϕ and linearly on ϕ_x, ϕ_y at most. For the sine-Gordon equation, they are listed in Table 1 along with their basic properties. The scaling parameter γ is also known as the Bäcklund parameter.

[4] The so-called *Janet monomials* are an obvious alternative to the symmetric multiindices used in [6, 36].

symmetry $\boldsymbol{\phi}$	ϕ_x	ϕ_y	$x\phi_x - y\phi_y$
name	translation		scaling
1-parametric group	$x \mapsto x + \alpha$	$y \to y + \beta$	$x \mapsto \mathrm{e}^{\gamma} x,\ y \mapsto \mathrm{e}^{-\gamma} y$
asymptotic parameterisation	preserved		preserved
Chebyshev parameterisation	preserved		not preserved
pseudospherical surface	preserved		not preserved

Table 1. Classical Lie symmetries of the sine-Gordon equation.

Remark 3.1. As is well known [47], there are two types of sine-Gordon solutions invariant with respect to the classical symmetries. They are

(1) The *travelling-wave solutions*, which are solutions satisfying the invariant constraint $\phi_y = m\phi_x$, where $m = \text{const} \neq 0$. They are of the form $\phi = \phi(x + my)$. Generic solutions of this kind are periodic and expressible in terms of the Jacobi elliptic functions, see [47, Section 2.5.1] for details. For the corresponding pseudospherical surfaces see [62] and [47, Section 3.7.1]. For gypsum models visit the corresponding section of the Göttingen Collection of Mathematical Models and Instruments [51, Section E.V].

The only travelling-wave solutions expressible in terms of elementary functions are the kinks

$$4 \arctan\left(\mathrm{e}^{\mathrm{e}^{\gamma} x + \mathrm{e}^{-\gamma} y}\right) \tag{3.3}$$

of velocity $-\mathrm{e}^{2\gamma}$, for arbitrary $\gamma \in \mathbb{R}$. They correspond to the *Dini helicoids*

$$\begin{aligned}\mathbf{r} = \frac{1}{\cosh\gamma}\left[\frac{\cos(x-y)}{\cosh(\mathrm{e}^{\gamma}x + \mathrm{e}^{-\gamma}y)}, \frac{\sin(x-y)}{\cosh(\mathrm{e}^{\gamma}x + \mathrm{e}^{-\gamma}y)}, -\tanh(\mathrm{e}^{\gamma}x + \mathrm{e}^{-\gamma}y)\right]& \\ + [0, 0, x + y].&\end{aligned} \tag{3.4}$$

The corresponding spherical Chebyshev net is

$$\begin{aligned}\mathbf{n} = -\frac{\tanh(\mathrm{e}^{\gamma}x + \mathrm{e}^{-\gamma}y)}{\cosh\gamma}\left[\cos(x-y), \sin(x-y), \frac{1}{\sinh(\mathrm{e}^{\gamma}x + \mathrm{e}^{-\gamma}y)}\right]& \\ + \tanh\gamma\,[-\sin(x-y), \cos(x-y), 0]\,.&\end{aligned}$$

For $\gamma = 0$, the kink is

$$\phi = 4 \arctan \mathrm{e}^{x+y}, \tag{3.5}$$

while the Dini helicoid collapses to the Beltrami pseudosphere

$$\mathbf{r} = \left[\frac{\cos(x-y)}{\cosh(x+y)}, \frac{\sin(x-y)}{\cosh(x+y)}, x + y - \tanh(x+y)\right], \tag{3.6}$$

and the spherical Chebyshev net to

$$\mathbf{n} = -\tanh(x+y)\left[\cos(x-y), \sin(x-y), \frac{1}{\sinh(x+y)}\right]. \tag{3.7}$$

(2) The *scaling-invariant solutions*, that is, the solutions satisfying the invariant constraint $x\phi_x - y\phi_y = 0$. They are of the form $\phi = \phi(xy)$ and expressible in terms of the third Painlevé transcendent. For details see [47, Section 2.5.2]. The corresponding surfaces are known as the Amsler surfaces, see [47, Section 3.7.2] and references therein.

Obviously, solutions $\boldsymbol{\phi}$ of equation (3.1) can be viewed as differential operators (generally nonlinear) that map solutions ϕ of the sine-Gordon equation (1.1) to solutions

$$\Phi = \boldsymbol{\phi}(x, y, \phi, \phi_x, \phi_{xx}, \dots, \phi_y, \phi_{yy}, \dots)$$

of the Moutard equation (1.2). The resulting Voss nets are essentially those that arise along the path followed by Guichard [22, Sections V–XII]. We start with some important cases of degeneracy, understood in the sense of Proposition 2.2 (ii), that is, [28, Theorem 2.25].[5]

Proposition 3.2 (Guichard [22, Section XII])**.** *The Voss surface associated to $\Phi = \phi_x$ is degenerate if and only if the initial sine-Gordon solution satisfies the travelling wave ansatz $\phi = \phi(x + my)$, $m = \mathrm{const}$.*

Proof. "$\Rightarrow$": Assuming degeneracy, $\Phi = \phi_x$ satisfies $X = Y = 0$ by Proposition 2.2 (ii), which leads to

$$\phi_{yy} = \frac{\phi_y}{\phi_x} \sin\phi, \qquad \phi_{xy} = \sin\phi, \qquad \phi_{xx} = \frac{\phi_x}{\phi_y} \sin\phi.$$

The first equation is obtained by inserting $\Phi = \phi_x$ into Y given by formula (2.5), the second one is the sine-Gordon equation, while the third one is the compatibility condition between the first two. It follows that $(\phi_y/\phi_x)_x = 0 = (\phi_y/\phi_x)_y$, that is, $\phi_y/\phi_x = m = \mathrm{const}$. Hence, $\phi = \phi(x + my)$, which is the travelling wave ansatz.

"$\Leftarrow$": Let $\phi_y = m\phi_x$. Then $X = Y = 0$ by straightforward computation. ■

By linearity, Proposition 3.2 extends to all translational symmetries $a\phi_x + b\phi_y$. Thus, translational symmetries produce only degenerate Voss surfaces from both the Dini helicoids and the Beltrami pseudospheres. Still, one can obtain non-trivial results by using the scaling symmetry.

Example 3.3 (the *right helicoid*)**.** Applying the scaling symmetry $-\frac{1}{2}x\phi_x + \frac{1}{2}y\phi_y$ to the unit-velocity kink $\phi = 4\arctan(\mathrm{e}^{x+y})$, we get

$$\Phi = -\frac{x}{2}\phi_x + \frac{y}{2}\phi_y = -\frac{x - y}{\cosh(x + y)}.$$

To obtain the associated Voss surface, we insert Φ into formula (2.3) along with $\mathbf{r}$ and $\mathbf{n}$ given by equations (3.6) and (3.7), respectively. The resulting Voss net is

$$\mathbf{q} = \left[-\frac{\sin(x - y)}{\sinh(x + y)}, \frac{\cos(x - y)}{\sinh(x + y)}, x - y\right],$$

that is, the common right helicoid in one of its Voss parameterisations, see Figure 1.

Example 3.4 (*Kostin's helicoid*)**.** Applying the scaling symmetry to a kink of arbitrary speed and the corresponding Dini helicoid (3.4), we get

$$\mathbf{q} = \left[-\frac{\sin v}{\sinh u}, \frac{\cos v}{\sinh u}, v + (u - \coth u)\sinh\gamma\right], \tag{3.8}$$

where $u = \mathrm{e}^\gamma x + \mathrm{e}^{-\gamma} y$ and $v = x - y$; it collapses to the right helicoid when $\gamma = 0$. The curvatures are

$$K = -\frac{\tanh^4 u}{4\cosh^2\gamma}, \qquad H = -\frac{1}{2}\sinh u \tanh\gamma.$$

For pictures, see Figure 2.

[5] In Guichard [22], degeneracy is expressed by saying that "surfaces S are spheres."

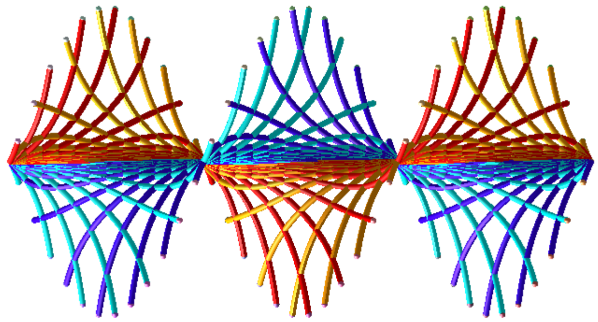

Figure 1. The Voss net on the right helicoid from the pseudosphere and the scaling symmetry.

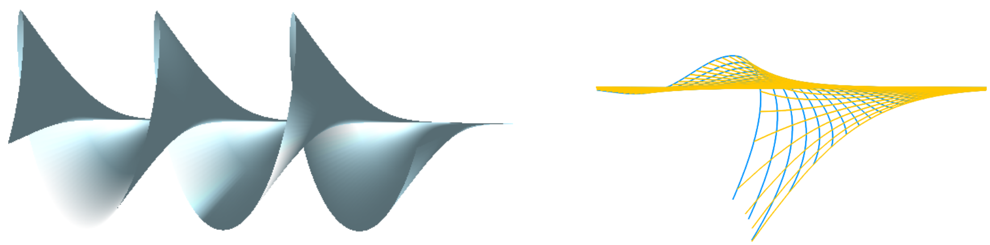

Figure 2. Kostin's helicoid (left); a part of its Voss net (right).

The surface turns out to be one of the Minkowskian analogues of the Dini helicoid, found by Kostin [32, Theorem 3]. The generating curve coincides with a (suitably reparameterised) Minkowskian analogue of the tractrix [32, equation (3)]. For every tangent line, the Lorentz distance between the point of tangency and the intersection with the η-axis is $\sqrt{-1}$, if the "speed of light" is $1/\sqrt{\sinh\gamma}$.

Turning back to the translational symmetries, they are still applicable to other pseudospherical surfaces. We provide only one simple example.

Example 3.5 (the *koru surface*)**.** Let us apply translational symmetries to the degenerate two-soliton sine-Gordon solution

$$\phi = 4\arctan\left(\frac{x-y}{\cosh(x+y)}\right), \tag{3.9}$$

see [47, Section 2.6.2] or [7, equation (2.12)]. The corresponding pseudospherical surface is the Kuen surface

$$\mathbf{r} = \frac{2\cosh u}{v^2+\cosh^2 u}[\cos v + v\sin v, \sin v - v\cos v, -\sinh u] + [0,0,u],$$

where $u = x+y$, $v = x-y$. The unit normal is

$$\mathbf{n} = \frac{2v}{v^2+\cosh^2 u}[-\cos v, -\sin v, \sinh u] + \frac{v^2-\cosh^2 u}{v^2+\cosh^2 u}[-\sin v, \cos v, 0].$$

For $\Phi = \phi_x + \phi_y$, the associated Voss surface (2.3) degenerates to a point. Therefore, we lose nothing if we restrict ourselves to the multiples of ϕ_x.

The translational symmetry $\frac{1}{4}\phi_x$ maps the Kuen surface to the Voss net

$$\left[\frac{v\sin v - \sinh^2 u\cos v}{v\cosh^3 u - v^3\cosh u}, -\frac{v\cos v + \sinh^2 u\sin v}{v\cosh^3 u - v^3\cosh u}, \frac{1}{2}\frac{\sinh^3 u - \left(v^2+1\right)\sinh u}{v\cosh^3 u - v^3\cosh u}\right].$$

The real singular points where $\det \mathrm{I} = 0$ (ridges) are determined by $2v = \pm \sinh 2u$, while the blow-up points where $\det \mathrm{I} = \infty$ (points at infinity) are determined by one of $v = 0$, $v = \pm \cosh u$. The curves intersect (see Figure 3), where the ridges extend to infinity.

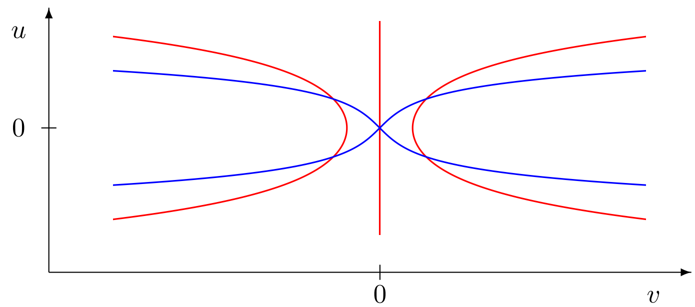

Figure 3. Singular and blow-up points of the koru surface ($\det \mathrm{I} = 0$ in blue, $\det \mathrm{I} = \infty$ in red).

To explain the name given to the surface, in Figure 4 we show part of the surface delimited by $|u| < 1$, $\cosh u < v$ next to the Hundertwasser koru flag of New Zealand [27].

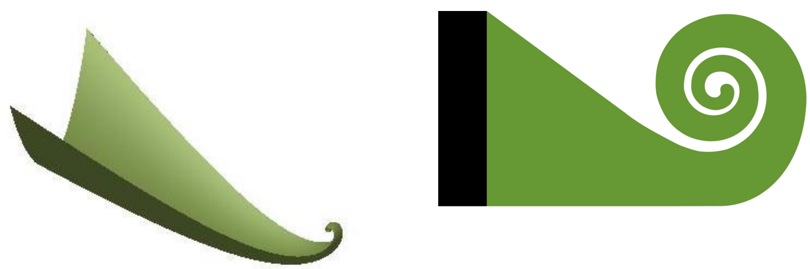

Figure 4. Part of the koru surface next to the Hundertwasser's koru flag.

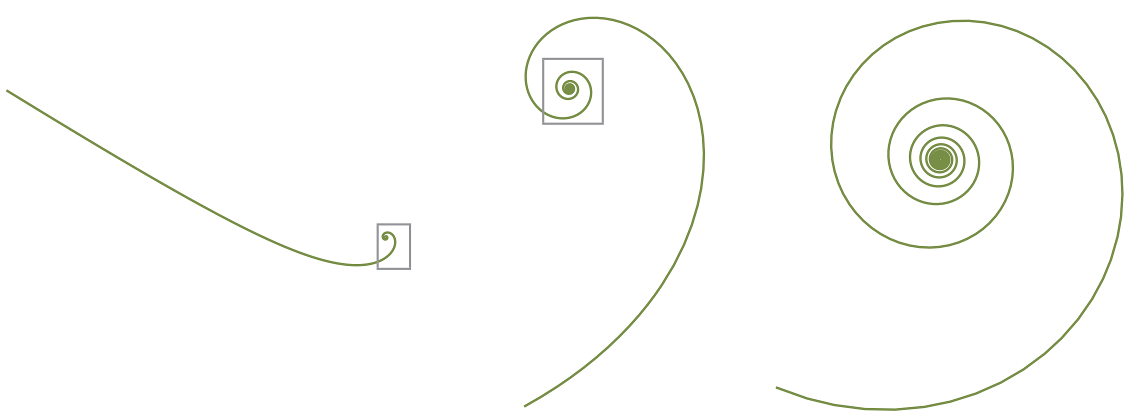

Figure 5. The infinitely curled spinal curve of the koru surface in three magnifications.

	translation	scaling
pseudosphere	degenerate (Proposition 3.2)	right helicoid (Example 3.3)
Dini helicoid	degenerate (Proposition 3.2)	Kostin's helicoid (Example 3.4)
Kuen surface	koru surface (Example 3.5)	eared screw (Example 3.6)

Table 2. Summary of examples.

Resembling an unfurling fern frond, the "spinal curve" $u = 0$ is the spiral $r = 1/\big(v^2 - 1\big)$, that is, the inversion of the Galileo spiral $r = v^2 - 1$ [52]. Here r and v mean the radial coordinates (radius and angle).

Example 3.6 (the *eared screw*)**.** Applying the scaling symmetry to the same Kuen surface as in Example 3.5, we get the Voss net

$$\Bigg[A \sin v + B \cos v, B \sin v - A\ \cos v,\\ \frac{v^4 \cosh u - v^2\big(\cosh^3 u + \cosh u + u \sinh u\big) + \big(1 - \sinh^2 u\big)\big(\cosh u - u \sinh u\big)}{v \cosh u\big(\cosh^2 u - v^2\big)}\Bigg],$$

$$A = \frac{2u + \sinh 2u}{\cosh u\big(\cosh^2 u - v^2\big)}, \qquad B = \frac{u - u \cosh 2u + \big(1 - v^2\big) \sinh 2u}{v \cosh u\big(\cosh^2 u - v^2\big)}.$$

The singularity condition $\det \mathrm{I} = 0$ is equivalent to

$$\big(v^2 - 1\big) \cosh 2u + (u \pm v) \sinh 2u + v^2 \mp 2uv - 1 = 0,$$

whereas $\det \mathrm{I} = \infty$ iff $v = \pm \cosh u$, see Figure 6. The intersections of blue and red lines lie at $v = \pm\sqrt{2}, \pm 1, 0$.

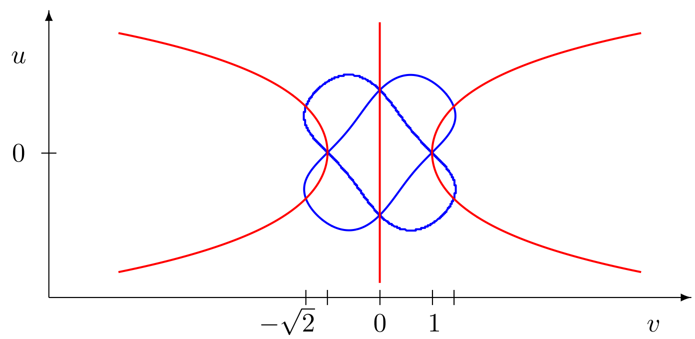

Figure 6. Singular points of the eared screw surface ($\det \mathrm{I} = 0$ in blue, $\det \mathrm{I} = \infty$ in red).

Although resembling the right helicoid, the surface is not minimal. In the area $v > \cosh u$, the surface looks like a screw with ears, see Figure 7. For the corresponding Voss net in a different parameter domain, see Figure 8.

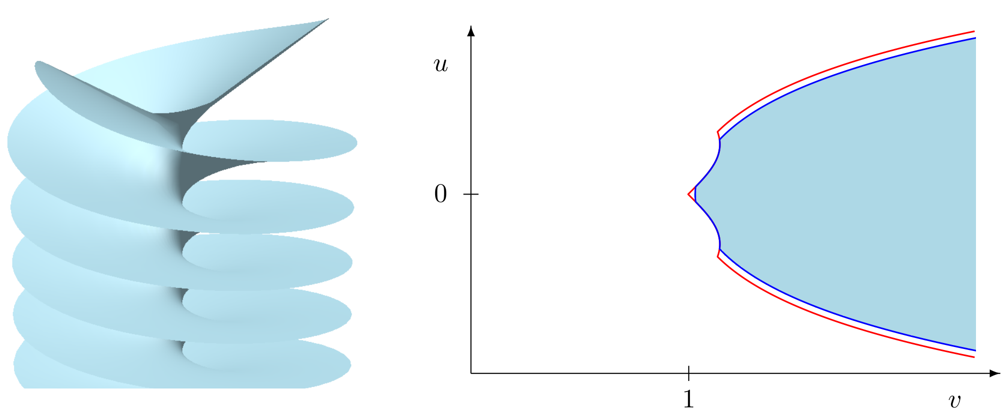

Figure 7. The eared screw surface (left) and its parameter domain (right).

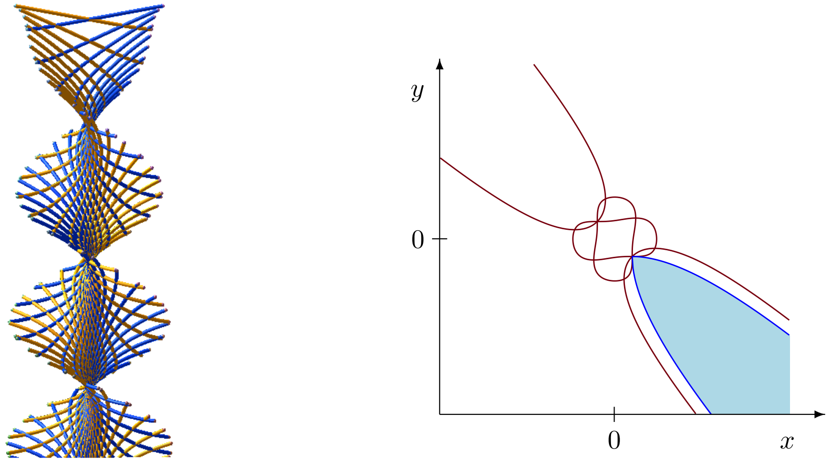

Figure 8. The Voss net of the eared screw surface (left) and its parameter domain (right).

4 Recursion operators of the sine-Gordon equation

The prevailing viewpoint is that recursion operators are linear operators, mapping symmetries to symmetries. They are available for many equations and have a rich and deep theory (see, e.g., [34, 36, 54, 63] and references therein) developed primarily in the integro-differential formalism, relying on formal inverses of total derivatives.

The integro-differential recursion operator due to Olver [45, equation (21)] acts on a sine-Gordon symmetry Φ by the formula

$$\mathcal{R}\Phi = D_x^2\Phi + \phi_x^2\Phi - \phi_x D_x^{-1}(\phi_{xx}\Phi) = D_x^2\Phi + \phi_x D_x^{-1}(\phi_x D_x\Phi). \tag{4.1}$$

The latter version results from integration by parts.

As stressed by many authors [25, 45, 55, 60], inverses of total derivatives are poorly defined. Quite often (see [39, 55]), the formal inverse D_x^{-1} is introduced by the contradictory[6] rule $D_x \circ D_x^{-1} = D_x^{-1} \circ D_x = \mathrm{Id}$. Moreover, although a formal integro-differential inverse recursion operator is available for $\mathcal{R}$, see [2, Section 3], [31, 38], it is insufficient for our purposes, mainly because we shall need to invert also $\mathcal{R} + \mathrm{Id}$, which is something very different.

This is why we adhere to Guthrie's [25] formalism, treating recursion operators as Bäcklund auto-transformations for the linearised equation [26, 40, 41, 46]. If, following this path, we identify the linearised sine-Gordon equation with the Moutard equation, we obtain Proposition 4.1 below, which, however, merely reproduces, up to notational variance, a result published by Guichard in 1890.

Proposition 4.1 ([23, equations (3) and (4)], [23, Section V]; [40, Example 4.4])**.** *Given a solution ϕ of the sine-Gordon equation, let Φ be a solution of the Moutard equation. Then the system*

$$Q_x = \Phi_x\phi_x, \qquad Q_y = \Phi\sin\phi \tag{4.2}$$

is compatible and $\Psi = \Phi_{xx} + \phi_x Q$ is another solution of the Moutard equation.

Proof. Equations (4.2) pass the cross-differentiation test

$$Q_{xy} = (\Phi_x\phi_x)_y = \Phi_{xy}\phi_x + \Phi_x\phi_{xy} = \Phi\phi_x\cos\phi + \Phi_x\sin\phi = (\Phi\sin\phi)_x = Q_{yx},$$

[6] Indeed, $D_x^{-1}D_x f(y) = f(y)$ requires that $D_x 0 = f(y)$ for every function $f(y)$.

while Ψ satisfies

$$
\begin{aligned}
\Psi_{xy} &= \Phi_{xxxy} + (\phi_x Q)_{xy} = (\Phi\cos\phi)_{xx} + \phi_{xxy}Q + \phi_{xx}Q_y + \phi_{xy}Q_x + \phi_x Q_{xy} \\
&= \Phi_{xx}\cos\phi + \phi_x Q\cos\phi = \Psi\cos\phi,
\end{aligned}
$$

which finishes the proof. ∎

To see that Proposition 4.1 actually delivers the Guthrie form of Olver's operator (4.1), it is sufficient to observe that Q can be chosen for $D_x^{-1}(\phi_x D_x \Phi)$. Thus, instead of (4.1) we can write

$$
\mathcal{R}\Phi = \Phi_{xx} + \phi_x Q, \qquad Q_x = \Phi_x\phi_x, \qquad Q_y = \Phi\sin\phi. \tag{4.3}
$$

Note that the potential Q coincides with the linearisation of the potential q_1 associated with one of the conservation laws of the sine-Gordon equation, see formula (4.9) below. This origin of Q was to be expected, cf. [40, Section 4].

Remark 4.2. To emphasise the multivalued character of the operator $\mathcal{R}$, Proposition 4.1 can be illustrated by the diagram

$$
\begin{array}{ccc}
& \boxed{\begin{aligned} Q_x &= \Phi_x\phi_x, \\ Q_y &= \Phi\sin\phi \\ \Phi_{xy} &= \Phi\cos\phi \\ \phi_{xy} &= \sin\phi \end{aligned}} & \\
\swarrow & & \searrow^{\Psi = \Phi_{xx} + \phi_x Q} \\
\boxed{\begin{aligned} \Phi_{xy} &= \Phi\cos\phi \\ \phi_{xy} &= \sin\phi \end{aligned}} & & \boxed{\begin{aligned} \Psi_{xy} &= \Psi\cos\phi \\ \phi_{xy} &= \sin\phi \end{aligned}}
\end{array}
$$

The arrow on the left forgets Q. Every solution Φ of the Moutard equation on the left has a one-parameter family of preimages (Φ, Q), which is mapped to a one-parameter family of solutions $\mathcal{R}\Phi$ of the Moutard equation on the right. The parameter is the integration constant for Q.

Remark 4.3. In the Guthrie formalism, the fact that recursion operators are multi-valued cannot be ignored. Yet one commonly writes $\Psi = \mathcal{R}\Phi$ with due reservations and caution if Ψ is among the results for a specific choice of the integration constant. When we want to stress the multi-valuedness, we write $\Psi \equiv \mathcal{R}\Phi$ or, more accurately, $\Psi \equiv \mathcal{R}\Phi \bmod \mathcal{R}0$, where $\mathcal{R}0 = \phi_x\mathbb{R}$ is the $\mathbb{R}$-ideal consisting of all results obtained from zero for all possible values of the integration constant.

Since Guthrie's recursion operators are essentially relations, the natural way to introduce their inverses is by the equivalence

$$
\Phi \equiv \mathcal{R}^{-1}\Psi \quad \overset{\text{def}}{\Leftrightarrow} \quad \Psi \equiv \mathcal{R}\Phi. \tag{4.4}
$$

Alternatively speaking, to obtain $\mathcal{R}^{-1}$ in the Guthrie form, all that is needed is to express Φ in terms of Ψ, see [25, Section 4]; for examples see also [40, Section 6], [41, Section 6.1] and Proposition 4.4 below.

When exchanging x and y in Proposition 4.1, we obviously get another recursion operator, which we shall denote by $\mathcal{R}'$, namely

$$
\mathcal{R}'\Psi = \Psi_{yy} + \phi_y Q', \qquad Q'_y = \Psi_y\phi_y, \qquad Q'_x = \Psi\sin\phi. \tag{4.5}
$$

As proved already by Guichard in 1890, $\mathcal{R}'$ is inverse to $\mathcal{R}$ up to terms induced by the inherent ambiguity; this original version can be expressed by formulas (4.7) below. The integro-differential version [39, Proposition 2] of this result was first presented only in 2021, without taking into account the ambiguities.

The ambiguity-resistant version looks as follows.

Proposition 4.4. *The inverse recursion operator $\mathcal{R}^{-1}$ defined by (4.4) coincides with the operator $\mathcal{R}'$ defined by (4.5).*

Proof. According to Proposition 4.1, the equality $\Psi = \mathcal{R}\Phi$ can be expressed by the compatible system

$$\begin{aligned}&\phi_{xy} = \sin\phi, \qquad \Phi_{xy} = \Phi\cos\phi, \qquad \Psi = \Phi_{xx} + \phi_x Q,\\ &Q_x = \Phi_x\phi_x, \qquad Q_y = \Phi\sin\phi.\end{aligned} \tag{4.6}$$

To express Φ in terms of Ψ, one can apply the standard elimination-completion algorithm, see, e.g., [50] and references therein, under a suitable elimination ordering of derivatives. The result will be an equivalent system, possessing the same solutions. To start with, we express Q from the third equation (4.6), obtaining $Q = (\Psi - \Phi_{xx})/\phi_x$. Inserting into the last two equations, we obtain

$$\Phi_{xx} = \frac{\phi_x}{\sin\phi}(\Phi_x\cos\phi - \Psi_y) + \Psi, \qquad \Phi_{xxx} = \frac{\phi_{xx}}{\sin\phi}(\Phi_x\cos\phi - \Psi_y) - \Phi_x\phi_x^2 + \Psi_x.$$

Evaluating the compatibility conditions $\Phi_{xxx} = D_x\Phi_{xx}$ and $D_x\Phi_{xy} = D_y\Phi_{xx}$, we get

$$\Psi_{xy} = \Psi\cos\phi \qquad \text{and} \qquad \Phi_x = \frac{\sin\phi}{\phi_y}(\Phi - \Psi_{yy}) + \Psi_y\cos\phi,$$

respectively. The third equation (4.6) is thereby satisfied, while the fourth yields

$$\Phi_y = \Psi_{yyy} + \frac{\phi_{yy}}{\phi_y}(\Phi - \Psi_{yy}) + \Psi_y\phi_y^2.$$

Thus, we are left with the system

$$\begin{aligned}&\phi_{xy} = \sin\phi, \qquad \Psi_{xy} = \Psi\cos\phi, \qquad \Phi_x = \frac{\sin\phi}{\phi_y}(\Phi - \Psi_{yy}) + \Psi_y\cos\phi,\\ &\Phi_y = \Psi_{yyy} + \frac{\phi_{yy}}{\phi_y}(\Phi - \Psi_{yy}) + \Psi_y\phi_y^2,\end{aligned}$$

which is compatible now and equivalent to (4.6). Substituting $\Phi = \Psi_{yy} + \phi_y Q'$, we obtain another equivalent system

$$\phi_{xy} = \sin\phi, \qquad \Psi_{xy} = \Psi\cos\phi, \qquad \Phi = \Psi_{yy} + \phi_y Q', \qquad Q'_y = \Psi_y\phi_y, \qquad Q'_x = \Psi\sin\phi.$$

These are exactly formulas (4.6) with x, y and Φ, Ψ exchanged. Therefore, $\Phi = \mathcal{R}'\Psi$, proving that $\mathcal{R}' = \mathcal{R}^{-1}$ in the sense of the equivalence (4.4). ■

Turning back to the original version published by Guichard [22, 23], it can be stated as

$$\mathcal{R}'\mathcal{R}\Phi \equiv \Phi \bmod \mathcal{R}'0, \qquad \mathcal{R}\mathcal{R}'\Psi \equiv \Psi \bmod \mathcal{R}0 \tag{4.7}$$

in the notation of Remark 4.3. In Proposition 4.6 below, we give this result yet another form, which will be useful later. First, we introduce trivial recursion operators as a tool to tackle the ambiguities.

Definition 4.5. By a *trivial* recursion operator we mean the operator of the form

$$\mathcal{Z}\Phi = W^{(1)}\Sigma^{(1)} + \cdots + W^{(n)}\Sigma^{(n)},$$

where $\Sigma^{(i)}$ are fixed symmetries and $W^{(i)}$ satisfy $D_xW^{(i)} = D_yW^{(i)} = 0$ (that is, $W^i = \mathrm{const}$).

Thus, $\mathcal{Z}$ formally satisfies Guthrie's definition ($\mathcal{Z}$ is formally a Bäcklund transformation), but $\mathcal{Z}\Phi$ does not depend of Φ.

Proposition 4.6 (cf. Guichard [22, 23])**.** *For $\mathcal{R}'$ defined by (4.5), both $\mathcal{R}'\circ\mathcal{R}-\mathrm{Id}$ and $\mathcal{R}\circ\mathcal{R}'-\mathrm{Id}$ are trivial.*

Proof. We have $\mathcal{R}'\mathcal{R}\Phi = (\mathcal{R}\Phi)_{yy} + \phi_y Q'$, where Q' satisfies the last two equations (4.5), that is, $Q'_x = (\Phi_{xx} + \phi_x Q)\sin\phi$, $Q'_y = (\Phi_x + Q\sin\phi)\phi_y$. Denoting $W = Q' + Q\cos\phi - \Phi_x\sin\phi$, we have $D_x W = 0$, $D_y W = 0$. However,

$$\begin{aligned}\mathcal{R}'\mathcal{R}\Phi &= (\Phi_{xx} + \phi_x Q)_{yy} + \phi_y Q' \\ &= \Phi_{xxyy} + \phi_{xyy}Q + 2\phi_{xy}Q_y + \phi_x Q_{yy} + (\Phi_x\sin\phi - Q\cos\phi + W)\phi_y = \Phi + W\phi_y\end{aligned}$$

by (1.1), (1.2) and (4.2). Thus, $\mathcal{R}'\circ\mathcal{R}-\mathrm{Id}$ is trivial. At the same time, we obtained the first Guichard's relation (4.7). Triviality of $\mathcal{R}\circ\mathcal{R}'-\mathrm{Id}$ as well as the second Guichard's relation follow by the $x \leftrightarrow y$ symmetry. ■

Following Olver [45], we construct an infinite hierarchy of symmetries Φ_n (and thus an infinite number of solutions to the Moutard equation) by the repeated application of the recursion operator. In Guthrie's formalism, the procedure is $\Phi_{n+1} = D_{xx}\Phi_n + \phi_x Q_n$, where Q_n satisfy $Q_{n,x} = \Phi_{n,x}\phi_x$, $Q_{n,y} = \Phi_n\sin\phi$. Depending on what we start with, we obtain two distinct hierarchies.

(1) *The pmKdV hierarchy* [45, *Example* 6]. Starting with $\Phi_1 = \phi_x$, we get

$$\begin{aligned}Q_1 &= \frac{1}{2}\phi_x^2, \\ Q_2 &= \phi_x\phi_{xxx} - \frac{1}{2}\phi_{xx}^2 + \frac{3}{8}\phi_x^4, \\ &\ \ \vdots\end{aligned}$$

whence

$$\begin{aligned}\Phi_1 &= \phi_x, \\ \Phi_2 &= \phi_{xxx} + \frac{1}{2}\phi_x^3, \\ \Phi_3 &= \phi_{xxxxx} + \frac{5}{2}\phi_x^2\phi_{xxx} + \frac{5}{2}\phi_x\phi_{xx}^2 + \frac{3}{8}\phi_x^5, \\ &\ \ \vdots\end{aligned} \tag{4.8}$$

(2) *The Khor'kova hierarchy* [29, *Example* 2]. Starting with the scaling symmetry $\bar\Phi_1 = x\phi_x - y\phi_y$, we get

$$\begin{aligned}\bar Q_1 &= xQ_1 + q_1 + y\cos\phi, \\ \bar Q_2 &= xQ_2 + q_1 Q_1 - 3q_2 + 2\phi_x\phi_{xx}, \\ &\ \ \vdots\end{aligned}$$

where the nonlocal variables q_i are to be determined from

$$\begin{aligned}q_{1,x} &= \frac{1}{2}\phi_x^2, & q_{1,y} &= -\cos\phi, \\ q_{2,x} &= \frac{1}{2}\phi_{xx}^2 - \frac{1}{8}\phi_x^4, & q_{2,y} &= \frac{1}{2}\phi_x^2\cos\phi, \\ &\ \ \vdots & &\ \ \vdots\end{aligned} \tag{4.9}$$

as potentials of the standard conservation laws [11, equation (1.5)]. Hence the nonlocal hierarchy

$$\begin{aligned}
\bar\Phi_1 &= x\phi_x - y\phi_y, \\
\bar\Phi_2 &= x\Phi_2 + q_1\Phi_1 + 2\phi_{xx}, \\
\bar\Phi_3 &= x\Phi_3 + q_1\Phi_2 - 3q_2\Phi_1 + 4\phi_{xxxx} + 7\phi_x^2\phi_{xx}, \\
&\ \ \vdots
\end{aligned} \tag{4.10}$$

The expressions $\bar\Phi_i$ represent *shadows* of nonlocal symmetries in the sense of [6, 35, 36], meaning that they satisfy equation (3.1), but depend also on the nonlocal variables q_i. All of them can be extended to full symmetries in a possibly larger covering, see op. cit.

Remark 4.7. The local symmetries Φ_n and the nonlocal symmetry shadows $\bar\Phi_n$ are referred to as members of the positive pmKdV hierarchy and the positive Khor'kova hierarchy, respectively. Using the inverse recursion operator $\mathcal{R}'$ and starting with $\Phi_{-1} = \phi_y$ and $\bar\Phi_{-1} = -\Phi_1$, we get members of the corresponding negative hierarchies, which will be denoted Φ_{-n} and $\bar\Phi_{-n}$. The explicit formulas are obtained by exchanging $x \leftrightarrow y$ and $n \leftrightarrow -n$ in (4.8), (4.9), and (4.10); cf. Proposition 4.6.

Remark 4.8. In the pmKdV and Khor'kova hierarchy above, Q_n, q_n, $\bar Q_n$ have been chosen as homogeneous[7] of weights $2n$, $2n-1$, $2n-1$, respectively, with regard to weights $[x] = -1$, $[y] = 1$, $[\phi] = 0$, which ensures that Φ_n and $\bar\Phi_n$ are homogeneous of weight $2n-1$ and $2n$, respectively. This may be viewed as setting all integration constants to zero, which is a common practice, but not always a well-defined procedure.

5 Guichard sequences of Voss surfaces

In the previous section, we recalled the infinite pmKdV hierarchy of local symmetries Φ_n and the infinite Khor'kova hierarchy of nonlocal symmetry shadows $\bar\Phi_n$, for n both positive or negative. Using them in the way described in Proposition 2.2, we obtain families of Voss surfaces, which will be referred to as *Guichard sequences*, although Guichard himself considered only Φ_n and Φ_{-n}. We say that such a sequence stops, if the resulting Voss surfaces are degenerate or differ from the previous ones by a shift or correspond to the sum of previously obtained Voss surfaces. This is so, for instance, for all travelling wave initial solutions and the whole pmKdV hierarchy Φ_n, as observed already by Guichard.

Proposition 5.1 ([22, p. 264])**.** *All symmetries Φ_n, $n \neq 0$, map every travelling wave solution of the sine-Gordon equation to a degenerate Voss surface.*

Proof. The travelling wave solutions satisfy $\phi_y = m\phi_x$, where $m = \text{const} \neq 0$. Then

$$\phi_{xx} = \frac{\sin\phi}{m}$$

in consequence of the sine-Gordon equation. Since $m\phi_x^2 + 2\cos\phi$ is a first integral, the travelling wave solutions satisfy $m\phi_x^2 + 2\cos\phi = c = \text{const}$.

Starting with $\Phi_1 = \phi_x$, we get

$$\Phi_2 = \phi_{xxx} + \frac{1}{2}\phi_x^3 = \frac{\cos\phi}{m}\phi_x + \frac{c - 2\cos\phi}{2m}\phi_x = \frac{c}{2}\phi_x = \frac{c}{2}\Phi_1.$$

Therefore, $\Phi_n = (c/2)^n\Phi_1$ by induction. However, Φ_1 yields a degenerate Voss surface by Proposition 3.2. This proves the statement for all $n > 1$. Analogously for $n < -1$. ■

[7]Here, an expression $F(x, y, \phi, \phi_x, \phi_y, \dots)$ is said to be homogeneous of weight N if F transforms to $\lambda^N F$ under $x \mapsto x/\lambda$, $y \mapsto \lambda y$, $\phi \mapsto \phi$. See, e.g., [36, Section 3.2].

Proposition 5.2. *All symmetries $\bar\Phi_n$, $n \ne 0$, map the Beltrami pseudosphere to one and the same right helicoid up to a shift.*

Proof. For $n = 1$, this is Example 3.3. For $n = 2$, we have $\bar\Phi_2 - \bar\Phi_1 = 2/\cosh(x + y)$, for which the Voss surface degenerates to the point $[0, 0, -2]$. For $n > 2$, we have $\bar\Phi_n = 2\bar\Phi_{n-1} - \bar\Phi_{n-2}$, which can be directly verified for $n = 3$ and extended to $n > 3$ by applying $\mathcal{R}$. Thus, surfaces corresponding to various $\bar\Phi_n$, $n > 1$, differ by a shift along the axis. Analogously for $n < -1$. ■

Proposition 5.3. *All symmetries $\bar\Phi_n$, $n \ne 0$, map the Dini helicoid (3.4) to one and the same Kostin helicoid (3.8) up to a shift.*

Proof. Analogously to the proof of Proposition 5.2. For $n = 1$, this is Example 3.4. For $n = 2$, we have $\bar\Phi_2 - e^{2\gamma}\bar\Phi_1 = 2e^{2\gamma}/\cosh(e^{\gamma}x + e^{-\gamma}y)$, for which the Voss surface degenerates to the point $\left[0, 0, -e^{3\gamma} - e^{\gamma}\right]$. For $n > 2$, we have $\bar\Phi_n = 2e^{2\gamma}\bar\Phi_{n-1} - e^{4\gamma}\bar\Phi_{n-2}$. Analogously for $n < -1$. ■

6 On Guichard's problem

At the end of his paper, Guichard [22] asked under what conditions the process of generating Voss surfaces stops after the n-th step. To provide an answer, we prove a lemma, which links the problem with the symmetry invariance. In what follows, by $\mathcal{R}0$ we mean the image of the zero symmetry. According to formula (4.3), $\mathcal{R}0$ coincides with the ideal $\phi_x\mathbb{R}$.

Lemma 6.1. *The Voss surface corresponding to a solution Φ of the Moutard equation by formula (2.3) is degenerate if and only if*

$$\mathcal{R}\Phi + \Phi \in \mathcal{R}0. \tag{6.1}$$

Proof. By Proposition 2.2 (ii), the surface is degenerate if and only if $X = 0$ and $Y = 0$.

The "⇒" part. The equalities $X = 0$ and $Y = 0$, together with the Moutard equation, yield the system

$$\begin{aligned} &\Phi_{xx} + \Phi - \left(\frac{\Phi_x}{\tan\phi} + \frac{\Phi_y}{\sin\phi}\right)\phi_x = 0, \qquad \Phi_{xy} - \Phi\cos\phi = 0, \\ &\Phi_{yy} + \Phi - \left(\frac{\Phi_y}{\tan\phi} + \frac{\Phi_x}{\sin\phi}\right)\phi_y = 0. \end{aligned} \tag{6.2}$$

Assuming that (6.2) holds, we easily find that

$$Q = -\frac{\Phi_x}{\tan\phi} - \frac{\Phi_y}{\sin\phi} + \text{const},$$

whence

$$\mathcal{R}\Phi = \Phi_{xx} + \phi_x Q = -\Phi + \text{const}\,\phi_x$$

by formula (4.3).

The "⇐" part. If $\Phi_{xx} + \phi_x Q = \mathcal{R}\Phi = -\Phi + \text{const}\,\phi_x$, then

$$Q = -\frac{\Phi_{xx} + \Phi}{\phi_x} + \text{const}.$$

Equating $Q_y = \Phi\sin\phi$ according to equation (4.2), we get the first line of system (6.2) by straightforward computation, which means that $X = 0$. Then also $Y = 0$ by Proposition 2.2 (iii). ■

Remark 6.2. The condition (6.1) is equivalent to

$$\mathrm{III}^{ij}\Phi_{,i}\Phi_{,j} + \Phi^2 = \mathrm{const}, \tag{6.3}$$

where

$$\mathrm{III}^{ij}\Phi_{,i}\Phi_{,j} = \frac{\Phi_x^2 + 2\Phi_x\Phi_y\cos\phi + \Phi_y^2}{\sin^2\phi}$$

is the Beltrami invariant of the Gaussian image of Φ. This follows from the fact that the expression on the left-hand side of equation (6.3) is a first integral of system (6.2).

Turning back to the Guichard problem, we reword it as follows. Given a sine-Gordon solution ϕ, consider members Φ_n of the pmKdV hierarchy, described by formula (4.8), members $\bar\Phi_n$ of the Khor'kova hierarchy, described by formula (4.10), and their negative counterparts Φ_{-n} and $\bar\Phi_{-n}$, which correspond to Φ_n and $\bar\Phi_n$ by the discrete symmetry $x \leftrightarrow y$ according to Remark 4.7. In general, Φ_n, $\bar\Phi_n$, Φ_{-n}, $\bar\Phi_{-n}$ are mutually independent except for $\bar\Phi_{-1} = -\bar\Phi_1$. The problem is to determine how many of the Voss nets obtained from these symmetries can be linearly independent, that is, what is the dimension of the $\mathbb{R}$-linear space of the Voss nets obtainable by the Guichard process.

The following examples show how to derive the answer from the symmetry properties of the initial pseudospherical surface by using Lemma 6.1.

Example 6.3. The Beltrami pseudosphere corresponds to the kink $\phi = 4\arctan(\mathrm{e}^{x+y})$, the symmetry invariance properties of which can be expressed by means of the following four easily verifiable relations

$$\Phi_2 = \Phi_1, \qquad \bar\Phi_2 = \bar\Phi_1 - 2\Phi_1, \qquad \Phi_{-1} = \Phi_1, \qquad \bar\Phi_{-2} = \bar\Phi_1 + 2\Phi_1.$$

Applying $\mathcal{R}$ to the first two and $\mathcal{R}^{-1}$ to the last two relations, it follows by induction that all Φ_n, $\bar\Phi_n$, Φ_{-n}, $\bar\Phi_{-n}$ are linear combinations of Φ_1 and $\bar\Phi_1$. Consequently, the space of generated symmetries is of dimension two. Using Lemma 6.1, we easily see that Φ_1 produces a degenerate Voss net since $\mathcal{R}\Phi_1 + \Phi_1 = 2\Phi_1 \in \mathcal{R}0$. On the other hand, $\mathcal{R}\bar\Phi_1 + \bar\Phi_1 = \bar\Phi_2 + \bar\Phi_1 = 2\bar\Phi_1 - 2\Phi_1 \equiv 2\bar\Phi_1$ mod $\mathcal{R}0$, so that $\bar\Phi_1$ produces a nontrivial Voss surface (namely the right helicoid according to Proposition 5.2). Consequently, the space of generated Voss nets is of dimension $2 - 1 = 1$.

For the Dini helicoid (3.3), the answer is 1 as well.

Example 6.4. The degenerate two-soliton solution

$$\phi = 4\arctan\left(\frac{x-y}{\cosh(x+y)}\right),$$

of the sine-Gordon equation, see equation (3.9), satisfies

$$\Phi_3 = 2\Phi_2 - \Phi_1, \qquad \bar\Phi_2 = \bar\Phi_1 - 4\Phi_1, \qquad \Phi_{-1} = -\Phi_2 + 2\Phi_1, \qquad \bar\Phi_{-2} = \bar\Phi_1 - 4\Phi_2 + 8\Phi 1.$$

Applying $\mathcal{R}$ to the first two and $\mathcal{R}^{-1}$ to the last two relations, it follows by induction that all Φ_n, $\bar\Phi_n$, Φ_{-n}, $\bar\Phi_{-n}$, $n \ge 1$, are linear combinations of Φ_1, Φ_2 and $\bar\Phi_1$. Consequently, the space of generated symmetries is of dimension three. To answer Guichard's problem with the help of Lemma 6.1, we solve the equation $(\mathcal{R} + \mathrm{Id})\big(a\Phi_1 + b\Phi_2 + c\bar\Phi_1\big) \in \mathcal{R}0$ for $a, b, c \in \mathbb{R}$. The general solution is $a = 0$, $c = -3b$, which yields a 1-parametric family of symmetries for which the resulting Voss net turns out to be degenerate. Thus, the answer is $3 - 1 = 2$ linearly independent Voss nets, for which we can choose the koru surface (see Example 3.5) and the eared screw (see Example 3.6).

Example 6.5. The degenerate three-soliton solution of the sine-Gordon equation is

$$4\arctan\left(\frac{(x+y)\cosh(x+y)-(x-y)^2\sinh(x+y)}{(x-y)^2+\cosh^2(x+y)}\right)+4\arctan(\mathrm{e}^{x+y})$$

[7, equation (2.13)] and satisfies

$$\Phi_4 = 3\Phi_3 - 3\Phi_2 + \Phi_1, \qquad \bar\Phi_2 = \bar\Phi_1 - 6\Phi_1, \qquad \Phi_{-1} = -\Phi_3 + 3\Phi_2 - 3\Phi_1,$$
$$\bar\Phi_{-2} = \bar\Phi_1 + 6\Phi_3 - 18\Phi_2 + 18\Phi_1.$$

In analogy to the previous example, $\bar\Phi_1$, Φ_3, Φ_2, Φ_1 form the basis of the four-dimensional space of generated symmetries. With the help of Lemma 6.1, we reveal one symmetry for which the resulting Voss net turns out to be degenerate, namely $\Phi_3 - 4\Phi_2 + 7\Phi_1$. Thus, the answer is that the Guichard procedure generates $4 - 1 = 3$ linearly independent Voss nets.

The overall length of Guichard's sequences is expected to be n for all n-soliton and, more generally, all n-gap solutions [64].

7 More inverse operators

Once $\mathcal{R}^{-1}$ yields practically the same results as $\mathcal{R}$, the next possibility is to invert $\mathcal{R} \pm \lambda\mathrm{Id}$, $\lambda > 0$. While the operator $\mathcal{R}$ is invariant with respect to the translations, it transforms into its multiple $\mathrm{e}^{-2\gamma}\mathcal{R}$ under the scaling $x \mapsto \mathrm{e}^{\gamma}x$, $y \mapsto \mathrm{e}^{-\gamma}y$. Indeed, the potential Q transforms to $\mathrm{e}^{-\gamma}Q$ and the rest is obvious.

Since $(\mathcal{R} \pm \lambda\,\mathrm{Id})^{-1} = \lambda^{-1}(\lambda^{-1}\mathcal{R} \pm \mathrm{Id})^{-1}$ and $\lambda^{-1}\mathcal{R}$ is related to $\mathcal{R}$ by the scaling, as we have just seen, the inversion of $\mathcal{R} \pm \lambda\,\mathrm{Id}$ can be reduced to the inversion of $\mathcal{R} \pm \mathrm{Id}$ in combination with the scaling transformation. We shall be interested in $(\mathcal{R} + \mathrm{Id})^{-1}$.

Proposition 7.1. *If Ψ satisfies the equation $\Psi_{xy} = (\cos\phi)\Psi$, then the system*

$$\Phi_{xx} = -\Phi + \frac{\phi_x}{\tan\phi}\Phi_x + \frac{\phi_x}{\sin\phi}\Phi_y + \Psi - \frac{\phi_x}{\sin\phi}\Psi_y, \qquad \Phi_{xy} = \Phi\cos\phi,$$
$$\Phi_{yy} = -\Phi + \frac{\phi_y}{\tan\phi}\Phi_y + \frac{\phi_y}{\sin\phi}\Phi_x - \frac{\phi_y}{\tan\phi}\Psi_y + \Psi_{yy} \tag{7.1}$$

is compatible and $\Phi \equiv (\mathcal{R} + \mathrm{Id})^{-1}\Psi$.

Proof. To recover the operator $(\mathcal{R} + \mathrm{Id})^{-1}$ in the Guthrie form, it is enough to express Φ from the system

$$\Psi = \Phi_{xx} + \phi_x Q + \Phi, \qquad Q_x = \Phi_x\phi_x, \qquad Q_y = \Phi\sin\phi.$$

Eliminating Q, we easily arrive at the three equations (7.1). It is also straightforward to check that the system is compatible if Ψ satisfies $\Psi_{xy} = \Psi\cos\phi$. Obviously from the second equation (7.1), Φ is a symmetry, that is, the correspondence $\Psi \mapsto \Phi$ is a recursion operator. ■

Remark 7.2. We can give an alternative proof of Lemma 6.1. Its proof uses system (6.2), which coincides with system (7.1) for $\Psi = 0$, meaning that $\Phi \in (\mathcal{R} + \mathrm{Id})^{-1}0$ by Proposition 7.1. The equivalence with equation (6.1) is obvious by the definition of inversion.

Corollary 7.3. *Denoting*

$$\boldsymbol{\Phi} = \begin{pmatrix}\Phi \\ \Phi_x \\ \Phi_y\end{pmatrix}$$

and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & \phi_x/\tan\phi & \phi_x/\sin\phi \\ \cos\phi & 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\phi & 0 & 0 \\ -1 & \phi_y/\sin\phi & \phi_y/\tan\phi \end{pmatrix},$$

the recursion operator $(\mathcal{R} + \mathrm{Id})^{-1}$ *can be written in the matrix form*

$$\mathbf{\Phi}_x = \mathbf{A}\ \mathbf{\Phi} + \begin{pmatrix} 0 \\ \Psi - \phi_x \Psi_y/\sin\phi \\ 0 \end{pmatrix}, \qquad \mathbf{\Phi}_y = \mathbf{B}\mathbf{\Phi} + \begin{pmatrix} 0 \\ 0 \\ \Psi_{yy} - \phi_y \Psi_y/\tan\phi \end{pmatrix}. \tag{7.2}$$

Proof. It is straightforward to check that systems (7.2) and (7.1) are equivalent. ■

Remark 7.4. Relative to the sine-Gordon equation, $\mathbf{A}\ \mathrm{d}x + \mathbf{B}\,\mathrm{d}y$ is a $\mathfrak{gl}_3$-valued zero-curvature representation independent of λ, which turns out to be gauge-equivalent to the adjoint representation of the well-known $\mathfrak{sl}_2$-valued zero-curvature representation [1, 64] for $\lambda = 1$. The full λ-dependent zero-curvature representation can be obtained either by applying the scaling symmetry to $\mathbf{A}\ \mathrm{d}x + \mathbf{B}\,\mathrm{d}y$ or from the inversion $(\mathcal{R} + \lambda\,\mathrm{Id})^{-1}$, see [42, p. 252] and references therein. For an interesting connection with Laplace invariants see [26].

To proceed further, we recall the Gauss–Weingarten system

$$\mathbf{r}_{xx} = \frac{\phi_x}{\tan\phi}\mathbf{r}_x - \frac{\phi_x}{\sin\phi}\mathbf{r}_y, \qquad \mathbf{r}_{xy} = \sin\phi\mathbf{n}, \qquad \mathbf{r}_{yy} = -\frac{\phi_y}{\sin\phi}\mathbf{r}_x + \frac{\phi_y}{\tan\phi}\mathbf{r}_y,$$
$$\mathbf{n}_x = \frac{1}{\tan\phi}\mathbf{r}_x - \frac{1}{\sin\phi}\mathbf{r}_y, \qquad \mathbf{n}_y = \frac{1}{\tan\phi}\mathbf{r}_y - \frac{1}{\sin\phi}\mathbf{r}_x$$

for pseudospherical surfaces in Dini's parameterisation x, y, corresponding to a solution ϕ of the sine-Gordon equation.

Proposition 7.5. *The recursion operator* $(\mathcal{R} + \mathrm{Id})^{-1}$ *can be written in the form*

$$(\mathcal{R} + \mathrm{Id})^{-1}\Psi = \mathbf{n} \cdot \mathbf{q},$$

where $\mathbf{q}$ *can be obtained by the quadratures*

$$\mathbf{q}_x = \frac{\mathbf{r}_y}{\sin\phi}\left(\frac{\phi_x}{\sin\phi}\Psi_y - \Psi\right), \qquad \mathbf{q}_y = \frac{\mathbf{r}_x}{\sin\phi}\left(\frac{\phi_y}{\tan\phi}\Psi_y - \Psi_{yy}\right). \tag{7.3}$$

Proof. Eliminating $\mathbf{r}_x$ and $\mathbf{r}_y$ from the Gauss–Weingarten equations (2.2) via

$$\mathbf{r}_x = -\frac{\mathbf{n}_x}{\tan\phi} - \frac{\mathbf{n}_y}{\sin\phi}, \qquad \mathbf{r}_y = -\frac{\mathbf{n}_x}{\sin\phi} - \frac{\mathbf{n}_y}{\tan\phi},$$

we get

$$\mathbf{n}_{xx} = -\mathbf{n} + \frac{\phi_x}{\tan\phi}\mathbf{n}_x + \frac{\phi_x}{\sin\phi}\mathbf{n}_y, \qquad \mathbf{n}_{xy} = \cos\phi\,\mathbf{n},$$
$$\mathbf{n}_{yy} = -\mathbf{n} + \frac{\phi_y}{\sin\phi}\mathbf{n}_x + \frac{\phi_y}{\tan\phi}\mathbf{n}_y.$$

In matrix form, $\mathbf{N}_x = \mathbf{A}\mathbf{N}$, $\mathbf{N}_y = \mathbf{B}\mathbf{N}$, where

$$\mathbf{N} = \begin{pmatrix} \mathbf{n} \\ \mathbf{n}_x \\ \mathbf{n}_y \end{pmatrix}$$

denotes the 3×3 matrix composed of the rows $\mathbf{n}$, $\mathbf{n}_x$, $\mathbf{n}_y$, the matrices $\mathbf{A}$, $\mathbf{B}$ being the same as above. Substituting $\mathbf{A} = \mathbf{N}_x\mathbf{N}^{-1}$, $\mathbf{B} = \mathbf{N}_y\mathbf{N}^{-1}$ into equations (7.2), the recursion operator becomes

$$\mathbf{\Phi}_x = \mathbf{N}_x\mathbf{N}^{-1}\mathbf{\Phi} + \begin{pmatrix} 0 \\ \Psi - \phi_x\Psi_y/\sin\phi \\ 0 \end{pmatrix}, \qquad \mathbf{\Phi}_y = \mathbf{N}_x\mathbf{N}^{-1}\mathbf{\Phi} + \begin{pmatrix} 0 \\ 0 \\ \Psi_{yy} - \phi_y\Psi_y/\tan\phi \end{pmatrix}.$$

Equivalently,

$$\left(\mathbf{N}^{-1}\mathbf{\Phi}\right)_x = \mathbf{N}^{-1}\begin{pmatrix} 0 \\ \Psi - \phi_x\Psi_y/\sin\phi \\ 0 \end{pmatrix}, \qquad \left(\mathbf{N}^{-1}\mathbf{\Phi}\right)_y = \mathbf{N}^{-1}\begin{pmatrix} 0 \\ 0 \\ \Psi_{yy} - \phi_y\Psi_y/\tan\phi \end{pmatrix}. \tag{7.4}$$

Therefore, $\mathbf{N}^{-1}\mathbf{\Phi}$ can be identified with the vector $\mathbf{q}$ introduced by system (7.3). Now, recall that the 3×3 matrix $\mathbf{N}$ has vectors $\mathbf{n}, \mathbf{n}_x, \mathbf{n}_y$ as rows. Then $\mathbf{N}^{-1}$ is the 3×3 matrix

$$\mathbf{N}^{-1} = \frac{\operatorname{adj}\mathbf{N}}{\det\mathbf{N}} = \begin{pmatrix} \dfrac{\mathbf{n}_x \times \mathbf{n}_y}{[\mathbf{n}, \mathbf{n}_x, \mathbf{n}_y]} & \dfrac{\mathbf{n}_y \times \mathbf{n}}{[\mathbf{n}, \mathbf{n}_x, \mathbf{n}_y]} & \dfrac{\mathbf{n} \times \mathbf{n}_x}{[\mathbf{n}, \mathbf{n}_x, \mathbf{n}_y]} \end{pmatrix}^\top = \begin{pmatrix} \mathbf{n} & -\dfrac{\mathbf{r}_y}{\sin\phi} & -\dfrac{\mathbf{r}_x}{\sin\phi} \end{pmatrix}^\top$$

column-wise. Substituting this expression into equation (7.4), we get the required formulas (7.3) in the first row. ■

The following proposition immediately leads to the result of Example 2.5 above.

Proposition 7.6. *The ideal* $(\mathcal{R} + \mathrm{Id})^{-1}0$ *consists of the projections of the normal vector* $\mathbf{n}$ *to arbitrary directions, that is, of the dot products* $\mathbf{n} \cdot \mathbf{c}$*, where* $\mathbf{c}$ *is an arbitrary constant vector.*

Proof. Substituting $\Psi = 0$ into equation (7.3), we obtain $\mathbf{q}_x = 0$ and $\mathbf{q}_y = 0$. Hence the statement. ■

Denoting $\tilde{\Phi}_0 = \mathbf{n} \cdot \mathbf{c}$, the shadow $\tilde{\Phi}_0$ can be interpreted as the starting member of what we shall call the *negative* $\mathcal{R} + \mathrm{Id}$ *hierarchy.*

Proposition 7.7. *The first member* $(\mathcal{R}+\mathrm{Id})^{-1}(\mathbf{n}\cdot\mathbf{c})$ *of the negative* $\mathcal{R}+\mathrm{Id}$ *hierarchy coincides with* $\tilde{\Phi}_1 = \mathbf{n} \cdot \mathbf{q}_1$*, where* $\mathbf{q}_1$ *is determined by*

$$\mathbf{q}_{1,x} = -\frac{\mathbf{r}_y}{\sin\phi}\left(\mathbf{n} - \frac{\phi_x}{\sin\phi}\mathbf{n}_y\right) \cdot \mathbf{c}, \qquad \mathbf{q}_{1,y} = \frac{\mathbf{r}_x}{\sin\phi}\left(\mathbf{n} - \frac{\phi_y}{\sin\phi}\mathbf{n}_x\right) \cdot \mathbf{c}. \tag{7.5}$$

Proof. Substituting $\Psi = \mathbf{n} \cdot \mathbf{c}$ into system (7.3), we obtain exactly system (7.5). ■

8 Beyond the Guichard sequence

It turns out that the vector $\mathbf{q}$ determined by formulas (7.3) determines a Voss surface away from the Guichard sequence and independent of it.

Proposition 8.1. *Let* $\mathbf{r}(x, y)$ *be the position vector of a pseudospherical surface in the Dini parameterisation. Let* Ψ *be any solution of the Moutard equation* $\Psi_{xy} = \Psi\cos\phi$*. Let*

$$\tilde{X} = \frac{\phi_x}{\sin\phi}\Psi_y - \Psi, \qquad \tilde{Y} = \frac{\phi_y}{\tan\phi}\Psi_y - \Psi_{yy}.$$

(*i*) *Introduce a vector* $\tilde{\mathbf{r}}$ *by*

$$\tilde{\mathbf{r}}_x = \frac{\tilde{X}}{\sin\phi}\mathbf{r}_y, \qquad \tilde{\mathbf{r}}_y = \frac{\tilde{Y}}{\sin\phi}\mathbf{r}_x. \tag{8.1}$$

Then the net $\tilde{\mathbf{r}}(x, y)$*, if nonsingular, is a Voss net. The corresponding fundamental forms are*

$$\tilde{\mathrm{I}} = \frac{\tilde{X}^2\,\mathrm{d}x^2 + 2\tilde{X}\tilde{Y}\cos\phi\,\mathrm{d}x\,\mathrm{d}y + \tilde{Y}^2\,\mathrm{d}y^2}{\sin^2\phi}, \qquad \tilde{\mathrm{II}} = \tilde{X}\,\mathrm{d}x^2 + \tilde{Y}\,\mathrm{d}y^2.$$

(*ii*) *The singular points* $\det\tilde{\mathrm{I}} = 0$ *of* $\tilde{\mathbf{r}}$ *are determined by* $\tilde{X}\tilde{Y} = 0$.

(*iii*) *The functions* $\tilde{X}$, $\tilde{Y}$ *obtained for* $\Psi = \mathcal{R}\Phi + \Phi$ *coincide with the functions* X, Y *obtained for* Φ *via Proposition* 2.2.

Proof. (i) Obviously, $\tilde{\mathbf{r}}_x \cdot \mathbf{n} = \tilde{\mathbf{r}}_y \cdot \mathbf{n} = 0$. Viewing $\tilde{\mathbf{r}}$ as the position vector of a surface, we see that $\tilde{\mathbf{r}}$ and $\mathbf{r}$ are related by the parallelism of normals.

Obviously from $\tilde{\mathrm{II}}_{12} = 0$, the parameters x, y are conjugate. Next, computing $\tilde{\mathbf{r}}_x \times \tilde{\mathbf{r}}_{xx}$ and $\tilde{\mathbf{r}}_y \times \tilde{\mathbf{r}}_{yy}$, one easily sees that they are parallel to $\mathbf{n}_y$ and $\mathbf{n}_x$, respectively. Consequently, both are orthogonal to $\mathbf{n}$ and the geodesic curvatures

$$\tilde{\mathrm{gc}}_1 = \frac{[\tilde{\mathbf{r}}_x, \tilde{\mathbf{r}}_{xx}, \mathbf{n}]}{(\tilde{\mathbf{r}}_x \cdot \tilde{\mathbf{r}}_x)^{3/2}}, \qquad \tilde{\mathrm{gc}}_2 = \frac{[\tilde{\mathbf{r}}_y, \tilde{\mathbf{r}}_{yy}, \mathbf{n}]}{(\tilde{\mathbf{r}}_y \cdot \tilde{\mathbf{r}}_y)^{3/2}}$$

along the coordinate lines are zero.

(ii) It is easy to check that $\det\tilde{I} = \tilde{X}\tilde{Y}/\sin^2\phi$.

(iii) This can be seen by comparing formulas (2.8) and (8.1). ■

When applying Proposition 8.1 to $\Psi = \mathbf{n}\cdot\mathbf{c}$, we obtain $\mathbf{q}_1$ of Proposition 7.7. The resulting Voss net can be nontrivial and linearly independent of the nets of the Guichard sequence, i.e., sequence obtained from the pmKdV hierarchy or the Khor'kova hierarchy via Proposition 2.2.

Example 8.2 (the *arch*)**.** Let us apply Proposition 7.7 (that is, Proposition 8.1 with $\Psi = \mathbf{n}\cdot\mathbf{c}$) to the Beltrami pseudosphere, that is, to $\mathbf{r}$ given by equation (3.6) and ϕ given by equation (3.5). Let $\mathbf{c} \in \mathbb{S}^2$ be an arbitrary unit vector. Choosing $\mathbf{c} = [0, 0, 1]$, we get the constant $\tilde{\mathbf{r}} = \mathbf{0}$. Choosing $\mathbf{c} = [\sin\alpha, \cos\alpha, 0]$ orthogonal to $[0, 0, 1]$, we obtain a family of Voss nets rotated around the axis $[0, 0, 1]$ by α. Thus, we are left with a single Voss net

$$\tilde{\mathbf{r}} = \left[\frac{1 - \cos(2x - 2y)}{4\tanh(x + y)} - \frac{x + y}{2}, -\frac{\sin(2x - 2y)}{4\tanh(x + y)}, \frac{1}{2}\cos(x - y)\cosh(x + y)\right]$$

up to the rigid motions. Computing the loci of singular points according to part (ii) of Proposition 8.1, we get the curves $\tanh(x+y) = \pm\tan(x-y)$, known as the pseudo-elliptic radioids [59]. The part of the Voss surface delimited by two pseudo-elliptic radioids, $\tanh(x+y) < |\tan(x-y)|$, is shown in Figure 9. The "spinal curve" $y = x$ is the catenary. Therefore, this part of the surface can be viewed as a bent lath version of the common catenary arch. It would be interesting to investigate to what extent it can support its own weight without collapsing.

We finish this paper with a result analogous to Lemma 6.1 in the context of Proposition 8.1.

Lemma 8.3. *Formula* (8.1) *turns a solution* Ψ *of the Moutard equation into a degenerate Voss surface if and only if* $\Psi \in \mathcal{R}0$.

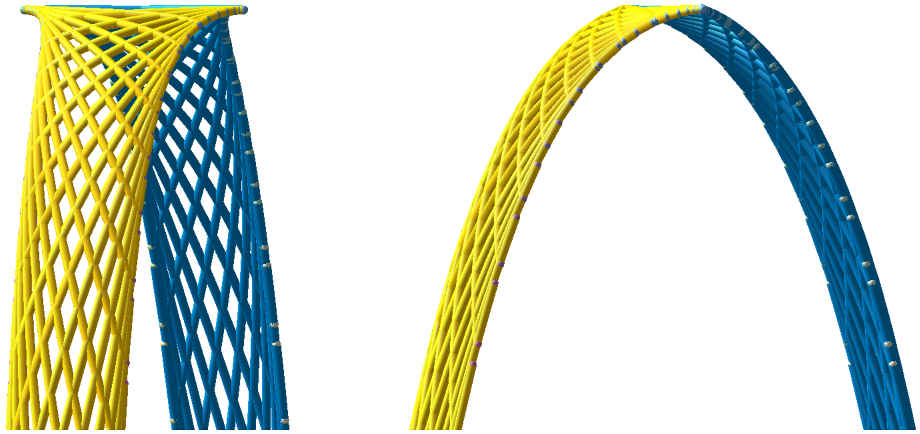

Figure 9. The Voss arch from two different viewpoints.

Proof. The lemma is a consequence of Lemma 6.1 and Proposition 8.1 (iii). A short direct proof is as follows. By Proposition 8.1 (ii), the surface (8.1) is degenerate if and only if $\tilde{X} = 0$ and $\tilde{Y} = 0$. The overdetermined system formed by $\tilde{X} = 0$, $\tilde{Y} = 0$ and the Moutard equation for Ψ reduces to the passive form

$$\Psi_x = \frac{\phi_{xx}}{\phi_x}\Psi, \qquad \Psi_y = \frac{\phi_{xy}}{\phi_x}\Psi,$$

with the general solution $\Psi = \mathrm{const}\,\phi_x$ or, equivalently, $\Psi \in \mathcal{R}0$. ■

9 The division algebra of recursion operators

Recursion operators form an associative $\mathbb{R}$-algebra with respect to operations defined by

$$(\mathcal{U} \circ \mathcal{V})\Phi = \mathcal{U}\mathcal{V}\Phi, \qquad (\mathcal{U} + \mathcal{V})\Phi = \mathcal{U}\Phi + \mathcal{V}\Phi, \qquad (r\mathcal{U})\Phi = r\mathcal{U}\Phi, \qquad r \in \mathbb{R}.$$

Previously, this algebra was studied for the heat equation [33, 35, 36] (is non-commutative), the Krichever–Novikov and the Landau–Lifshitz equation [8]. For evolution equations like these, ambiguities do not pose a serious problem. To cope with ambiguities discussed in Section 4, we factor out the ideal formed by the trivial operators in the sense of Definition 4.5.

Proposition 9.1. *Trivial recursion operators form a two-sided ideal in the associative* $\mathbb{R}$*-algebra of all recursion operators.*

Proof. Indeed, if $\mathcal{Z}$ is trivial, then $\mathcal{Z}\mathcal{U}\Phi = \mathcal{Z}\Phi$ by definition. Moreover, if $\mathcal{Z}\Phi = W^{(1)}\Sigma^{(1)} + \cdots + W^{(n)}\Sigma^{(n)}$, then $\mathcal{U}\mathcal{Z}\Phi = W^{(1)}\mathcal{U}\Sigma^{(1)} + \cdots + W^{(n)}\mathcal{U}\Sigma^{(n)}$ by linearity. ■

The two-sided ideal induces a congruence relation on the $\mathbb{R}$-algebra of all recursion operators.

Definition 9.2. Two recursion operators $\mathcal{U}$ a $\mathcal{V}$ are said to be *congruent*, if $\mathcal{U} - \mathcal{V}$ is trivial. We write $\mathcal{U} \equiv \mathcal{V}$.

Now, consider the associative algebra $\langle\mathcal{R}\rangle$ generated by the operator (4.3). Let $\mathfrak{R}$ denote the quotient algebra $\langle\mathcal{R}\rangle/{\equiv}$. Elements of $\mathfrak{R}$ are recursion operators modulo the congruence $\equiv$.

In consequence of Proposition 4.6, we get the following corollary.

Corollary 9.3. *As elements of* $\mathfrak{R}$*, the operators* $\mathcal{R}$*,* $\mathcal{R}'$ *are mutually inverse.*

Proposition 9.4. *The associative algebra $\mathfrak{R}$ is commutative and isomorphic to the algebra $\mathbb{R}[\mathcal{R}]$ of polynomials in $\mathcal{R}$.*

Proof. Obviously, the associative algebra $(\mathfrak{R}, +, 0, -, \circ, \mathbb{R})$ generated by the single element $\mathcal{R}$ is commutative and consists of polynomials $\mathcal{P} = \mathcal{R}^n + \nu_{n-1}\mathcal{R}^{n-1} + \cdots + \nu_1\mathcal{R} + \nu_0 \operatorname{Id}$. To show that $\mathfrak{R} \cong \mathbb{R}[\mathcal{R}]$, we have to verify that no non-constant polynomial $\mathcal{P}$ is congruent to zero. To this end, consider $\mathcal{P}\phi_x = \mathcal{R}^n\phi_x + \nu_{n-1}\mathcal{R}^{n-1}\phi_x + \cdots + \nu_1\mathcal{R}\phi_x$. According to Remark 4.7, symmetries $\mathcal{R}^i\phi_x$ are members of the pmKdV hierarchy of ever increasing order. Consequently, $\mathcal{P}\phi_x$ is nonzero in $\mathfrak{R}$. Hence the statement. ∎

Finally, consider the division algebra $\mathfrak{Q}$ of $\mathfrak{R}$. By the previous proposition, $\mathfrak{Q}$ is isomorphic to $\mathbb{R}(\mathcal{R})$, the division algebra of $\mathbb{R}[\mathcal{R}]$. The elements of $\mathbb{R}(\mathcal{R})$ can be viewed as formal fractions $\mathcal{P}/\mathcal{Q}$, where $\mathcal{P}, \mathcal{Q} \in \mathbb{R}[\mathcal{R}]$, $\mathcal{Q} \not\equiv 0$. Since not all recursion operators are invertible (for instance, the trivial ones), some fractions need not correspond to true recursion operators, in principle. The next proposition proves that they actually do.

Proposition 9.5. *All operators $\mathcal{P} \circ \mathcal{Q}^{-1}$, corresponding to formal fractions $\mathcal{P}/\mathcal{Q}$, where $\mathcal{Q} \not\equiv 0$, are Bäcklund auto-transformations of the Moutard equation.*

Proof. It suffices to show that all inverses $(\mathcal{R} + \lambda \operatorname{Id})^{-1}$, corresponding to linear factors $\mathcal{R} + \lambda \operatorname{Id}$ of $\mathcal{Q}$, are true Bäcklund transformations. For $\lambda = 0$, this follows from Proposition 4.6. For $\lambda = 1$, see Proposition 7.1, which easily extends to arbitrary $\lambda \neq 0$. ∎

The entire division algebra will be employed in a future work.

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