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Relativistic Toda Lattice and Equivariant K-Homology of Affine Grassmannian

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Abstract. We investigate the phenomenon known as "quantum equals affine" in the setting of T-equivariant quantum K-theory of the flag variety G/B, as established by Kato for any semisimple algebraic group G. In particular, we focus on the K-Peterson isomorphism between the T-equivariant quantum K-ring $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$ and the T-equivariant K-homology ring $K_*^T(\mathrm{Gr}_{\mathrm{SL}_n})$ of the affine Grassmannian, after suitable localizations on both sides. Building on an earlier work by Ikeda, Iwao, and Maeno, we present an explicit algebraic realization of the K-Peterson map via a rational substitution that sends the generators of the quantum K-theory ring to explicit rational expressions in the fundamental generators of $K_*^T(\mathrm{Gr}_{\mathrm{SL}_n})$, thereby matching the Schubert bases on both sides. Our approach builds on recent developments in the theory of $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$ by Maeno, Naito, and Sagaki, as well as the theory of K-theoretic double k-Schur functions introduced by Ikeda, Shimozono, and Yamaguchi. This concrete formulation provides new insight into the combinatorial structure of the K-Peterson isomorphism in the equivariant setting. As an application, we establish a factorization formula for the K-theoretic double k-Schur function associated with the maximal k-irreducible k-bounded partition.

Key words: equivariant quantum K-theory; affine Grassmannian; Peterson isomorphism; relativistic Toda lattice; k-Schur functions

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1 Introduction

Let G be a simple, simply-connected algebraic group over $\mathbb{C}$. Fix a Borel subgroup B of G, and a maximal torus T contained in B. We study a remarkable relation between the T-equivariant quantum K-ring $QK_T(G/B)$ and the T-equivariant K-homology ring $K_*^T(\mathrm{Gr}_G)$ of the affine Grassmannian Gr_G of G. This viewpoint, commonly referred to as the "quantum equals affine" phenomenon, was originally introduced by Peterson [29] in the (co)homology context (see also [20]), and its K-theory analogue has since been investigated by several authors (see the next paragraph for references).

There exists a map, known as the *K-Peterson map*, which connects the rings $QK_T(G/B)$ and $K_*^T(\mathrm{Gr}_G)$, after appropriate localization. The purpose of this paper is to study the K-Peterson map for $G = \mathrm{SL}_n$ by realizing it through an explicit rational substitution, which establishes a correspondence between the Schubert bases at the combinatorial level.

A heuristic version of "quantum equals affine" phenomena in K-theory was explored for $G = \mathrm{SL}_n$ in non-equivariant setting [8]. This approach utilized an integrable system, called the *relativictic Toda lattice*, due to Ruijsenaars [30], which can be seen as the group version of the ordinary Lie algebra version of the Toda lattice. The construction in [8] was somewhat *ad hoc* as it relied on an unsolved conjecture by Kirillov and Maeno concerning the ring presentation of $QK(\mathrm{SL}_n(\mathbb{C})/B)$ at that time. However, this conjecture has since been resolved, with a minor correction, by Maeno, Naito, and Sagaki [26]; we also refer the reader to [1, 7, 15, 16] (see Remark 2.3 below). This advancement has improved the situation significantly. The K-theoretic Peterson isomorphism for general G was conjectured by Lam, Li, Mihalcea, and Shimozono [18], and later proved by Kato [13] by using semi-infinite flag manifolds. An alternative proof was also provided by Chow and Leung [3]. It is shown in [9] that the map studied in [8] coincides with the map by Kato, up to a natural ring automorphism σ (see Section 4.1).

The aim of this article is to extend the study in [8] to the equivariant case, building on more recent developments. Specifically, Maeno, Naito, and Sagaki [26, 27] established that the quantum double Grothendieck polynomials $\mathfrak{G}_w^Q$ of Lenart–Maeno represent the Schubert classes $\mathcal{O}^w \in QK_T(\mathrm{SL}_n(\mathbb{C})/B)$. On the affine side, Ikeda, Shimozono, and Yamaguchi [10] provided a realization of $K_*^T(\mathrm{Gr}_{\mathrm{SL}_n})$ in terms of equivariantly deformed symmetric functions. They introduced a family of special functions, $\tilde{g}_x^{(k)}(y|b)$, called the *K-theoretic double k-Schur functions*, which are identified with the Schubert classes $\mathcal{O}_x$. The primary goal of this paper is to realize the K-Peterson map explicitly through an algebraic substitution. This provides a concrete connection between these Schubert representatives in both the quantum and affine settings.

1.1 K-theoretic Peterson map: Abstract form

To describe the K-Peterson map more precisely, we fix some notation of the affine Weyl group. Let W_G be the Weyl group of (G, T), and $\hat{W}_G = W \ltimes Q^\vee$ the affine Weyl group, where $Q^\vee$ is the coroot lattice. Let $\hat{W}_G^0$ be the set of minimal-length coset representatives for $\hat{W}_G/W_G$. For $x \in \hat{W}_G^0$, there is an associated Schubert structure sheaf $\mathcal{O}_x$ in $K_*^T(\mathrm{Gr}_G)$. These sheaves form an $R(T)$-basis of $K_*^T(\mathrm{Gr}_G)$, where $R(T)$ denotes the representation ring of T. On the quantum side, we have the Schubert class $\mathcal{O}^w$ for each $w \in W_G$. For $x \in \hat{W}_G^0$, write it as $x = wt_\xi$ with $w \in W$, $\xi \in Q^\vee$, where t_ξ is the translation element corresponding to $\xi \in Q^\vee$. Let $QK_T(G/B)_Q$ denote the localization of $QK_T(G/B)$ by the Novikov variables $Q_1, \ldots, Q_r$, where r is the rank of G. The K-Peterson map, at the abstract level of Schubert bases, maps $Q^\xi \mathcal{O}^w \in QK_T(G/B)_Q$ to $\mathcal{O}_x \in K_*^T(\mathrm{Gr}_G)$, where Q^ξ denote the product of the Novikov variables corresponding to ξ. This map establishes a correspondence between the quantum and affine Schubert calculus.

1.2 K-Peterson map from the Relativistic Toda lattice

Let us explain the key idea of our construction before going into the details. By solving the relativistic Toda lattice, we obtain a birational map between the phase space $\mathcal{Y}$ of the relativistic Toda lattice and a certain centralizer family $\mathcal{Z}$ associated with $\mathrm{PGL}_n(\mathbb{C})$. Both varieties are defined over T, the maximal torus of $\mathrm{SL}_n(\mathbb{C})$, and their coordinate rings have the following geometric interpretations:

$$\mathcal{O}(\mathcal{Y}) \cong QK_T^{\mathrm{pol}}(\mathrm{SL}_n(\mathbb{C})/B), \qquad \mathcal{O}(\mathcal{Z}) \cong K_*^T(\mathrm{Gr}_{\mathrm{SL}_n}). \tag{1.1}$$

Here $QK_T^{\mathrm{pol}}(\mathrm{SL}_n(\mathbb{C})/B)$ is the polynomial version of the quantum K-ring of the flag manifold (see Section 2.2). The second isomorphism of (1.1) was proved in [10], which is a K-theory analogue of a result for $H_*^T(\mathrm{Gr}_G)$ due to Peterson [29] and Ginzburg [4] independently. We obtain the following commutative diagram (see Theorem 1.1)

$$\begin{tikzcd}
QK_T^{\mathrm{pol}}(\mathrm{SL}_n(\mathbb{C})/B)_Q \arrow[r] \arrow[d] & K_*^T(\mathrm{Gr}_{\mathrm{SL}_n})\left[\sigma_i^{-1}, \tau_i^{-1} \mid 1 \le i \le n\right] \arrow[d] \\
\mathcal{O}(\mathcal{Y}^\circ) \arrow[r, "\widetilde{\Phi}_n"'] & \mathcal{O}(\mathcal{Z}^\circ),
\end{tikzcd} \tag{1.2}$$

where the top arrow is Kato's map, and the bottom arrow is defined by (1.4) below. In this diagram, all maps are isomorphisms, and $\mathcal{Y}^\circ$ and $\mathcal{Z}^\circ$ are certain open dense subsets of $\mathcal{Y}$ and $\mathcal{Z}$ respectively, and σ_i, τ_i are discussed in detail in Section 2.4. This perspective extends the work [21, 22] by Lam and Shimozono for (co)homology and the classical Toda lattice.

1.3 K-theoretic double k-Schur function

Another ingredient of our work is the *K-theoretic double k-Schur function* introduced in [10], which is an equivariant deformation of the K-theoretic k-Schur function introduced by Lam, Schilling, and Shimozono [19]. The representation ring $R(T)$ is given as

$$\mathbb{Z}\left[\mathrm{e}^{\pm a_i}(1 \le i \le n)\right] / \left(\mathrm{e}^{a_1+\cdots+a_n} - 1\right).$$

For $x \in \hat{W}_G^0$, the K-theoretic double k-Schur function $g_x^{(k)}(y|b)$ is a symmetric formal power series in the infinitely many variables $y = (y_1, y_2, \dots)$ with coefficients in $R(T)$, where we set $k = n-1$. It depends on the sequence $b = (b_1, \dots, b_n)$ of equivariant parameters, where each b_i is identified with $1 - \mathrm{e}^{-a_i} \in R(T)$. Let $\hat{\Lambda}_{(n)}^{R(T)}$ be the $R(T)$-span of $g_x^{(k)}(y|b)$, $x \in \hat{W}_G^0$. Then we have an isomorphism

$$K_*^T(\mathrm{Gr}_{\mathrm{SL}_n}) \cong \hat{\Lambda}_{(n)}^{R(T)}$$

of $R(T)$-algebras such that the structure sheaf $\mathcal{O}_x$ corresponds to the *closed* K-theoretic double k-Schur function $\tilde{g}_x^{(k)}(y|b) := \sum_{z \le x} g_z^{(k)}(y|b)$, where $\le$ denotes the Bruhat order on $\hat{W}_G^0$.

There is a bijection $\hat{W}_G^0 \cong \mathcal{P}^{(k)}$, where $\mathcal{P}^{(k)}$ denotes the set of k-bounded partitions, i.e., the partition $\lambda = (\lambda_1, \dots, \lambda_i)$ such that $\lambda_1 \le k$. If $x \in \hat{W}_G^0$ corresponds to $\lambda \in \mathcal{P}^{(k)}$, then we write $g_x^{(k)}(y|b)$ $\big($resp. $\tilde{g}_x^{(k)}(y|b)\big)$ as $g_\lambda^{(k)}(y|b)$ $\big($resp. $\tilde{g}_\lambda^{(k)}(y|b)\big)$.

We have derived determinantal formulas for $g_\lambda^{(k)}(y|b)$ and $\tilde{g}_\lambda^{(k)}(y|b)$ (see Theorems 6.7 and 6.16) for a *k-small* k-bounded partition λ; a partition $\lambda \in \mathcal{P}^{(k)}$ is said to be k-small if $\lambda_1 + \ell(\lambda) \le n$, where $\ell(\lambda)$ is the number of nonzero parts of λ.

1.4 k-rectangles and the τ-functions

The so-called *τ-functions* in the theory of integrable systems also play an important role in our geometric context. For $1 \le i \le n$, we define $\tau_i, \sigma_i \in \mathcal{O}(\mathcal{Z})$ as the i-th principal minor determinants of certain matrices related to the centralizer family (see Section 2.4).

For $1 \le i \le k$, let R_i denote the partition $(i)^{n-i} \in \mathcal{P}^{(k)}$ of rectangular shape, and $R_n = \varnothing$. Note that each R_i $(1 \le i \le n-1)$ is a maximal k-small k-bounded partition. Under the isomorphism $\mathcal{O}(\mathcal{Z}) \cong \hat{\Lambda}_{(n)}^{R(T)}$, τ_i and σ_i correspond to $g_{R_{n-i}}^{(k)}(y|b)$ and $\tilde{g}_{R_{n-i}}^{(k)}(y|b)$ up to some simple factors respectively (see Corollary 6.15).

These functions are fundamental because $\tilde{g}_\lambda^{(k)}(y|b)$ satisfies the k-rectangle factorization property (see Theorem 7.2)

$$\tilde{g}_{R_i \cup \lambda}^{(k)}(y|b) = \tilde{g}_{R_i}^{(k)}(y|b)\tilde{g}_\lambda^{(k)}(y|\omega^i b),$$

where ω is the permutation sending b_i to b_{i+1} with $b_{n+1} = b_1$. Thanks to this result, we only need to study the functions $\tilde{g}^{(k)}_\lambda(y|b)$ associated with the *k-irreducible* k-bounded partition λ; i.e., an element of $\mathcal{P}^{(k)}$ which is not expressed as $R_i \cup \mu$ for $1 \le i \le n-1$ and $\mu \in \mathcal{P}^{(k)}$ with $\mu \neq \varnothing$.

1.5 Quantum double Grothendieck polynomials

For $w \in S_n$, let $\mathfrak{G}^Q_w(z|\eta)$ be the *quantum double Grothendieck polynomial* due to Lenart and Maeno [23]. This is a polynomial in two sets of variables $z_1, \dots, z_n$ and $\eta_1, \dots, \eta_n$ with coefficients in $\mathbb{Z}[Q_1, \dots, Q_{n-1}]$. We basically follow the notation in [27], however, there are some differences in the identification of the equivariant parameters. In particular, η_i is identified with $1 - \mathrm{e}^{a_{n-i+1}} \in R(T)$ (see (5.6) for more details). We denote $R(T)[z_1, \dots, z_n, Q_1, \dots, Q_{n-1}]$ by $R(T)[z, Q]$.

In the context of the relativistic Toda lattice, z_i, Q_i are interpreted as dynamical variables (see Section 2.1). There are conserved quantities $F_i(z, Q) \in R(T)[z, Q]$ $(1 \le i \le n)$ of the relativistic Toda lattice. Let $\mathcal{I}^{Q,\mathrm{pol}}_n$ be the ideal generated by $F_i(z, Q) - e_i(\mathrm{e}^{-a_1}, \dots, \mathrm{e}^{-a_n})$ $(1 \le i \le n)$, where e_i denotes the i-th elementary symmetric polynomial. The ring $R(T)[z, Q]/\mathcal{I}^{Q,\mathrm{pol}}_n$ is by our definition the ring of regular functions $\mathcal{O}(\mathcal{Y})$ on the phase space $\mathcal{Y}$.

Let $\mathcal{Y}^\circ$ be the open set of $\mathcal{Y}$ defined as the complement of the divisor given by the equation $Q_1 \cdots Q_{n-1} = 0$. Due to [27], the ring $\mathcal{O}(\mathcal{Y}^\circ) = \mathcal{O}(\mathcal{Y})\big[Q_i^{-1} \mid 1 \le i \le n-1\big]$ is identified with $QK^{\mathrm{pol}}_T(\mathrm{SL}_n(\mathbb{C})/B)_Q$ and $\mathfrak{G}^Q_w(z|\eta)$ represents the Schubert structure sheaf $\mathcal{O}^w$ (see Remark 2.4 for more details).

1.6 Correspondence of Schubert bases

Following an analogous approach to that used in Kostant's construction of solutions to the Toda lattice, we obtain a map

$$\Phi_n\colon\ \mathcal{O}(\mathcal{Y}^\circ) \longrightarrow \hat{\Lambda}^{R(T)}_{(n)}\big[\tau_i^{-1}, \sigma_i^{-1} \mid 1 \le i \le n\big] \cong \mathcal{O}(\mathcal{Z}^\circ)$$

of $R(T)$-algebras as

$$z_i \mapsto \frac{\tau_i \sigma_{i-1}}{\sigma_i \tau_{i-1}}, \qquad Q_i \mapsto \frac{\tau_{i-1}\tau_{i+1}}{\tau_i^2}. \tag{1.3}$$

Let ω_k, with $k = n-1$, be an involution on $\hat{W}^0_G$ called the *k-conjugation* defined by replacing s_i with s_{n-i} for $i \in I := \{1, \dots, n-1\}$ in any reduced expression of $x \in \hat{W}^0_G$. There is an automorphism σ of $\hat{\Lambda}^{R(T)}_{(n)}$ as an $R(T)$-algebra sending $h_i(y)$ $(i \in \mathbb{N})$ to $1 + h_1(y) + \cdots + h_i(y)$, where $h_i(y)$ is the i-th complete symmetric function. In particular, we have $\sigma(\tau_i) = \sigma_i$. Define the map $\tilde{\Phi}_n$ as follows

$$\tilde{\Phi}_n = \sigma \circ \Phi_n. \tag{1.4}$$

The main result of this paper is the following.

Theorem 1.1. *We have the commutative diagram* (1.2)*. More precisely, for $x \in \hat{W}^0_G$, write $x = wt_\xi$ with $w \in W$, $\xi \in Q^\vee$. Then*

$$\tilde{\Phi}_n\big(Q^\xi \mathfrak{G}^Q_w(z|\eta)\big) = \tilde{g}^{(k)}_{x^{\omega_k}}(y|b).$$

Thus, the substitution map Φ_n agrees with Kato's map up to a twist, after identifying $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$ with $\mathcal{O}(\mathcal{Y})$ $\big($resp. $K^T_*(\mathrm{Gr}_{\mathrm{SL}_n})$ with $\mathcal{O}(\mathcal{Z})\big)$.

Let us outline the proof. We begin by proving the theorem for $x = s_0$ in a purely combinatorial manner (see Proposition 5.10). In this proof, we utilize basic properties of the quantum double Grothendieck polynomials, the realization of $K^T_*(\mathrm{Gr}_{SL_n})$ as $\mathcal{O}(\mathcal{Z})$, and the explicit description of Φ_n. This step forms the technical core of the paper. An essential idea to prove the general case is the use of the action of Demazure operators D_x^Q, $x \in \hat{W}_G^0$, on $QK_T(\mathrm{SL}_n(\mathbb{C})/B)_Q$.

As a straightforward consequence of the theorem for the specific case $x = s_0$, we demonstrate that the map $\tilde{\Phi}_n$ intertwines the actions of the Demazure operators on both the quantum and affine sides (see Corollary 5.15). The final result needed to complete the proof is the precise formula for the action of D_i^Q on the Schubert structure sheaves $\mathcal{O}^w$ in $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$ (see Proposition C.2).

1.7 Factorization formula for the maximal k-irreducible partition ν_n

As an application of Theorem 1.1, we derive a factorization formula of $\tilde{g}_{\nu_n}^{(k)}(y|b)$, where

$$\nu_n = \bigcup_{i=1}^{n-2} (n-i-1)^i.$$

This partition ν_n is important because it is the unique maximal k-irreducible k-bounded partition. The non-equivariant version of the following result was conjectured in [8], and it was proved by Blasiak, Morse, and Seelinger [2].

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to $x \in \mathbb{R}$.

Theorem 1.2. *Let n be even and write $n = 2m$. Then*

$$\tilde{g}_{\nu_n}^{(k)}(y|b) = \prod_{i=0}^{\lfloor (m-1)/2 \rfloor} \frac{\Omega(b_{m-1-2i}|y)}{\Omega(b_{m+2-2i}|y)} \prod_{i=1}^{n-2} \tilde{g}_{(n-i-1)^i}^{(k)}\big(y|\omega^{m+2i+1}b\big), \tag{1.5}$$

where $\Omega(b_i|y)$ is defined in (3.3). Let n be odd. Then

$$\tilde{g}_{\nu_n}^{(k)}(y|b) = \prod_{i=1}^{n-2} \tilde{g}_{(n-i-1)^i}^{(k)}\big(y|\omega^{2i+1}b\big). \tag{1.6}$$

Note that all the factors $\tilde{g}_{(n-i-1)^i}^{(k)}\big(y|\omega^{2i+1}b\big)$ are expressed as determinants because $(n-i-1)^i$ is k-small.

1.8 Organization

The paper is organized as follows. In Section 2, we first define the relativistic Toda lattice and explain the presentation for $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$ due to Maeno, Naito, and Sagaki [26, 27]. We also review the centralizer family $\tilde{\mathcal{Z}}$ given in [10]. We introduce the τ-functions as elements of $\mathcal{O}\big(\tilde{\mathcal{Z}}\big)$. At the last part of this section we give a detailed review of the construction of Φ_n given in [8]. In Section 4, we collect results on some automorphisms used in the rest of the paper. In Section 3, we review the definition and basic results on the K-theoretic double k-Schur functions. In Section 5, we introduce the quantum double Grothendieck polynomials and prove Theorem 1.1. As an application we prove a formula giving a relation between $g_\lambda^{(k)}(y|b)$ and $\tilde{g}_\lambda^{(k)}(y|b)$. In Section 6, we prove determinantal formulas for the K-theoretic double k-Schur functions associated with k-small partitions. In Section 7, we prove the k-rectangle factorization property. In Section 8, we prove Theorem 1.2. In Appendix A, we discuss the meaning of the automorphism σ in the context of discrete integrable systems. In Appendix B, we prove a formula for the conserved quantities of the relativistic Toda lattice. In Appendix C, we discuss the affine K-nil-Hecke action on the quantum K-theory ring and give a proof of Proposition 5.8.

List of symbols

- I, $\tilde{I} = I \cup \{0\}$: (affine) Dynkin index set, Section 3.1,
- Q_i: Novikov variables, Section 2.2,
- e^{a_i}, b_i: the equivariant parameters, Section 1.3, (3.2),
- σ_i, τ_i: the tau-functions, (2.9), (2.10),
- T_i, D_i: the Demazure operators, (3.1),
- $\Omega(b_i|y)$: (3.3),
- $\varrho_i(y)$: (7.1),
- $\xi_\lambda(y)$: (6.9),
- Φ_n, $\tilde{\Phi}_n$: the K-Peterson maps, (1.3), (1.4),
- ι: an involution, Section 4.4,
- T_i^Q, D_i^Q: Demazure operators on the quantum K-ring, (5.1), (5.13),
- $F_i(z, Q)$: the conserved quantities of the relativistic Toda lattice, (2.3),
- $F_j^{(i)}$: (4.15),
- $g_x^{(k)}(y|b)$, $\tilde{g}_x^{(k)}(y|b)$: K-theoretic double k-Schur functions, (3.4),
- $\mathcal{Z}$, $\tilde{\mathcal{Z}}$: centralizer families, Section 2.3,
- $\mathcal{Z}^\circ$, $\tilde{\mathcal{Z}}^\circ$: open parts of centralizer families, Section 2.4,
- $Z = (z_{ij})$: matrix of coordinate functions of $\tilde{\mathcal{Z}}$, (2.5),
- A: (2.5),
- C_A: companion matrix of A, (2.7),
- P: transition matrix, from A to C_A, (2.8),
- L (and M, N): Lax matrix, Section 2.1,
- $\mathcal{Y}$, $\mathcal{Y}^\circ$: family of isolevel sets and its open part, Section 2.2, Remark 2.4,
- c_i, $c_i^{(j)}$: Section 4.3,
- $\mathfrak{G}_w^Q(z|\eta)$: quantum double Grothendieck polynomials, Section 5.1,
- ψ_i: (5.3),
- z_i: variables of $\mathfrak{G}_w^Q(z|\eta)$, Section 5.1,
- η_i: equivariant parameters of $\mathfrak{G}_w^Q(z|\eta)$, (5.6),
- σ: automorphism, Section 4.1,
- ω: a cyclic permutation, Section 3.1,
- ω_k: involution, (3.7),
- R_i: k-rectangle, the partition $(i)^{n-i}$,
- ν_n: the maximal k-irreducible k-bounded partition, Section 1.7,
- M_λ: (6.11),
- $\mathcal{P}^{(k)}$: set of k-bounded partitions, Section 1.3,
- W_G: Weyl group,
- $\hat{W}_G$: affine Weyl group, Section 3.1,
- $\hat{W}_G^0$: mimimal-length coset representatives for $\hat{W}_G/W_G$.

2 Relativistic Toda lattice and the centralizer family

The aim of this section is to explain how the map Φ_n is obtained by solving the relativistic Toda lattice. Taking into account recent developments, we will review the construction in [8].

2.1 Relativistic Toda lattice

The *relativistic Toda lattice equation* was introduced by Ruijsenaars [30]. In this paper, we start from the Lax equation due to Suris [31]. Let

$$M = \begin{pmatrix} z_1 & -1 & 0 & \cdots & 0 \\ & z_2 & -1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & -1 \\ & & & & z_n \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -Q_1 z_1 & 1 & 0 & \ddots & \vdots \\ & -Q_2 z_2 & \ddots & \ddots & 0 \\ & & \ddots & 1 & 0 \\ & & & -Q_{n-1} z_{n-1} & 1 \end{pmatrix} \tag{2.1}$$

and $L := MN^{-1}$, which we call the Lax matrix. We consider the system of partial differential equations

$$\partial L / \partial t_i = \left[L, \left(L^i\right)_{<}\right] \qquad \text{for} \quad 1 \le i \le n-1, \tag{2.2}$$

where $\left(L^i\right)_{<}$ is the strictly lower triangular part of L^i. The equation is a group version of the finite open Toda lattice.

2.2 Quantum K-ring of the flag variety

The integrable system is related to the quantum K-ring of the flag variety $\mathrm{SL}_n(\mathbb{C})/B$, analogously to a result of Givental and Kim [5] for the quantum cohomology ring.

The conserved quantities $F_i(z, Q)$ of the system (2.2) are given by

$$\det(\zeta E - L) = \sum_{i=0}^{n} (-1)^i F_i(z, Q) \zeta^{n-i}.$$

Explicitly, we have (see Appendix B)

$$F_i(z, Q) = \sum_{\substack{J \subset \{1, \dots, n\} \\ |J| = i}} \prod_{j \in J, j+1 \notin J} (1 - Q_j) \prod_{j \in J} z_j. \tag{2.3}$$

Let $R(T)[\![Q]\!]$ denote the ring of formal power series in the variables $Q_1, \dots, Q_{n-1}$ with coefficients in $R(T)$. The quantum K-ring $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$ is a commutative $R(T)[\![Q]\!]$-algebra.

Theorem 2.1 (Maeno, Naito, Sagaki [26])**.** *There exists an isomorphism of $R(T)[\![Q]\!]$-algebras*

$$QK_T(\mathrm{SL}_n(\mathbb{C})/B) \cong R(T)[\![Q]\!][z_1, \dots, z_n]/\mathcal{I}_n^Q,$$

where the ideal $\mathcal{I}_n^Q$ is generated by the elements

$$F_i(z, Q) - e_i(\mathrm{e}^{-a_1}, \dots, \mathrm{e}^{-a_n}) \qquad \textit{for} \quad 1 \le i \le n. \tag{2.4}$$

Remark 2.2. Our a_i is $-\epsilon_i$ in [26].

In this geometric context, z_i is related to the class of the universal line bundle (see [26] for details), and Q_i $(1 \le i \le n-1)$ are the Novikov variables.

Remark 2.3. It follows from [6, Corollary 2] that the elements in (2.4) are contained in the defining ideal of $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$ (see also [1, 26] for details). We also note that Koroteev–Puskar–Smirnov–Zeitlin [16] proved that the elements in (2.4) generate the defining ideal of $QK_T^{\mathrm{QM}}\big(\mathrm{SL}_n(\mathbb{C})/B\big)$, which is the limit of quasimap quantum K-ring of $T^*(\mathrm{SL}_n(\mathbb{C})/B)$ at $\hbar=\infty$. Moreover, Huq–Kuruvilla [7] established that this limit is isomorphic to $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$ (more generally, for partial flag varieties).

Let $\mathcal{Y}$ be the affine subscheme of $\mathbb{A}^{2n-1}$ with coordinates $Q_1,\dots,Q_{n-1}$, $z_1,\dots,z_n$ whose defining ideal is generated by the polynomials (2.4). $\mathcal{Y}$ is a scheme over T whose fibers are the isolevel sets. The coordinate ring, i.e., the ring of regular functions $\mathcal{O}(\mathcal{Y})$ is considered to be a polynomial version $QK_T^{\mathrm{pol}}(\mathrm{SL}_n(\mathbb{C})/B)$ of $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$.

Remark 2.4. Let us denote the ideal of $R(T)[z,Q]$ generated by the polynomials (2.4) by $\mathcal{I}_n^{Q,\mathrm{pol}}$. The quotient ring $\mathcal{O}(\mathcal{Y})=R(T)[z,Q]/\mathcal{I}_n^{Q,\mathrm{pol}}$ is $QK_T^{\mathrm{pol}}(\mathrm{SL}_n(\mathbb{C})/B)$ in the notation used in the introduction. Although there are subtleties in relating $QK_T^{\mathrm{pol}}(\mathrm{SL}_n(\mathbb{C})/B)$ to $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$, $QK_T^{\mathrm{pol}}(\mathrm{SL}_n(\mathbb{C})/B)_Q=\mathcal{O}(\mathcal{Y}^\circ)$ is actually a subring of $QK_T(\mathrm{SL}_n(\mathbb{C})/B)_Q$. Furthermore, the residue class of $\mathfrak{G}_w^Q$ in $QK_T^{\mathrm{pol}}(\mathrm{SL}_n(\mathbb{C})/B)_Q$ corresponds to $\mathcal{O}^w$ in $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$. For these facts, see [9, Sections 4.1 and 4.2]; it is straightforward to have the equivariant version based on [26, Remark 6.3].

If we consider the special fiber $\mathcal{Y}_{\mathrm{uni}}$ corresponding to the case when the Lax matrix is *unipotent*, and the corresponding centralizer $\mathcal{Z}_{\mathrm{uni}}$, we obtain the non-equivariant K-Peterson isomorphism $QK(\mathrm{SL}_n(\mathbb{C})/B)^{\mathrm{pol}}_{\mathrm{loc}}\cong K_*(\mathrm{Gr}_{\mathrm{SL}_n})_{\mathrm{loc}}$ studied in [8]. In fact, the element of $\mathcal{Z}_{\mathrm{uni}}$ is of the form

$$\begin{pmatrix} 1 & h_1 & h_2 & \cdots & h_{n-1} \\ & 1 & h_1 & \ddots & \vdots \\ & & \ddots & \ddots & h_2 \\ & & & \ddots & h_1 \\ & & & & 1 \end{pmatrix},$$

so $\mathcal{O}(\mathcal{Z}_{\mathrm{uni}})$ is a polynomial ring of $(n-1)$ variables, which can be identified with the non-equivariant K-homology ring $K_*(\mathrm{Gr}_{\mathrm{SL}_n})$ studied by Lam, Schilling, and Shimozono. If we identify the generators of the polynomial ring with the complete symmetric functions $h_1,\dots,h_{n-1}$, the Schubert structure sheaves are given by the K-theoretic closed k-Schur functions. See [9] for a more precise correspondence.

2.3 Centralizer family $\tilde{\mathcal{Z}}$

We construct the solutions of (2.2) from a centralizer family $\mathcal{Z}$ defined below. Consider the matrix equation

$$[A,Z]=0,\qquad A=\begin{pmatrix} \mathrm{e}^{-a_1} & -1 & 0 & \cdots & 0 \\ & \mathrm{e}^{-a_2} & -1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & -1 \\ & & & & \mathrm{e}^{-a_n} \end{pmatrix}, \tag{2.5}$$

where Z is an upper triangular matrix with the entries z_{ij} for $1\le i\le j\le n$. We assume $z_{11}\cdots z_{nn}\neq 0$. $[A,Z]=0$ is equivalent to the equations

$$(b_i-b_j)z_{ij}=z_{i,j-1}-z_{i+1,j}\qquad\text{for}\quad 1\le i\le j\le n. \tag{2.6}$$

Let us denote the affine variety over T defined by these equations by $\tilde{\mathcal{Z}}$. So we have

$$\begin{aligned}&\mathcal{O}\big(\tilde{\mathcal{Z}}\big) = R(T)\big[z_{ii}^{\pm 1}\ (1 \le i \le n),\ z_{ij}\ (1 \le i < j \le n)\big]/I,\\ &I = \langle (b_i - b_j) z_{ij} - z_{i,j-1} + z_{i+1,j} \mid 1 \le i < j \le n \rangle.\end{aligned}$$

There is a natural $\mathbb{C}^\times$-action on $\tilde{\mathcal{Z}}$ by scalar multiplication $z_{ij} \mapsto c z_{ij}$ ($c \in \mathbb{C}^\times$). The variety $\mathcal{Z}$ is defined to be $\tilde{\mathcal{Z}}/\mathbb{C}^\times$. Thus the coordinate ring $\mathcal{O}(\mathcal{Z})$ is the $R(T)$-subalgebra of $\mathcal{O}\big(\tilde{\mathcal{Z}}\big)$ generated by z_{ij}/z_{11} ($1 \le i \le j \le n$). $\mathcal{Z}$ is a closed subscheme of $T \times B^\vee$, where $B^\vee$ is the Borel subgroup of $\mathrm{PGL}_n(\mathbb{C})$.

Remark 2.5. In [10], $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ are denoted by $\mathcal{Z}_{\mathrm{PGL}_n}$ and $\mathcal{Z}_{\mathrm{GL}_n}$ respectively.

Let $R(T)^\Delta$ be the localization of $R(T)$ by the multiplicative set generated by $1 - \mathrm{e}^{a_i - a_j}$ ($i \ne j$). Set $\mathcal{O}\big(\tilde{\mathcal{Z}}\big)^\Delta := R(T)^\Delta \otimes_{R(T)} \mathcal{O}\big(\tilde{\mathcal{Z}}\big)$. We often use the following.

Proposition 2.6. *$\mathcal{O}\big(\tilde{\mathcal{Z}}\big)^\Delta$ is generated by z_{ii} ($1 \le i \le n$) as an $R(T)^\Delta$-algebra.*

2.4 τ-functions

We will explain that a Zariski open subset $\mathcal{Z}^\circ$ of $\mathcal{Z}$ is isomorphic to $\mathcal{Y}^\circ$, the compliment of the divisors defined by Q_i ($1 \le i \le n-1$), whereas $\mathcal{Z}^\circ$ is the complement of the divisor defined by the so-called *τ-functions*.

Let $\mathrm{e}_i^{(m)} := e_i(\mathrm{e}^{-a_1}, \mathrm{e}^{-a_2}, \dots, \mathrm{e}^{-a_m})$ denote the i-th elementary symmetric polynomial in m variables $\mathrm{e}^{-a_1}, \mathrm{e}^{-a_2}, \dots, \mathrm{e}^{-a_m}$. The companion matrix C_A of A is described as

$$C_A := P^{-1} A P = \begin{pmatrix} 0 & -1 & & & \\ & 0 & -1 & & \\ & & \ddots & \ddots & \\ & & & 0 & -1 \\ \mathrm{e}_n^{(n)} & \mathrm{e}_{n-1}^{(n)} & \cdots & \mathrm{e}_2^{(n)} & \mathrm{e}_1^{(n)} \end{pmatrix}, \tag{2.7}$$

where

$$P = \begin{pmatrix} 1 & & & & & \\ \mathrm{e}_1^{(1)} & 1 & & & & \\ \mathrm{e}_2^{(2)} & \mathrm{e}_1^{(2)} & 1 & & & \\ \mathrm{e}_3^{(3)} & \mathrm{e}_2^{(3)} & \mathrm{e}_1^{(3)} & 1 & & \\ \vdots & \vdots & \vdots & & \ddots & \\ \mathrm{e}_{n-1}^{(n-1)} & \mathrm{e}_{n-2}^{(n-1)} & \cdots & \cdots & \mathrm{e}_1^{(n-1)} & 1 \end{pmatrix}. \tag{2.8}$$

For later use, we record the following formula:

$$\big(P^{-1}\big)_{ij} = (-1)^{i-j} h_{i-j}(\mathrm{e}^{-a_1}, \dots, \mathrm{e}^{-a_j}).$$

Let I, J be subsets of $[1,n] := \{1, \dots, n\}$, with $I = \{i_1, \dots, i_p\}$ and $J = \{j_1, \dots, j_q\}$. Let X_I^J denote the submatrix of X consisting of its $i_1, \dots, i_p$-th rows and $j_1, \dots, j_q$-th columns.

Definition 2.7. For $1 \le i \le n$, define $\tau_i, \sigma_i \in \mathcal{O}\big(\tilde{\mathcal{Z}}\big)$ by

$$\tau_i = \det(ZAP)_{[1,i]}^{[1,i]}, \tag{2.9}$$

$$\sigma_i = \det(ZP)_{[1,i]}^{[1,i]}, \tag{2.10}$$

and $\tau_0 = \sigma_0 = 1$.

In particular, we have $\sigma_n = \det(Z) = z_{11} \cdots z_{nn}$, $\tau_n = \mathrm{e}^{a_1+\cdots+a_n}\sigma_n$. The open set $\tilde{\mathcal{Z}}^\circ$ of $\tilde{\mathcal{Z}}$ is the complement of the closed set defined by $\tau_i = 0$, $\sigma_i = 0$ $(1 \le i \le n-1)$.

We often use the following results on the determinants.

Proposition 2.8. *Let A, B be square matrices of size n, and $1 \le i \le n$. If A is lower triangular or B is upper triangular, then*

$$\det(AB)^{[1,i]}_{[1,i]} = \det A^{[1,i]}_{[1,i]} \det B^{[1,i]}_{[1,i]}.$$

Proposition 2.9. *Let A, B be square matrices of size n such that B is invertible, and $1 \le i \le n-1$.*

$$\det(AB)^{[i+1,n]}_{[i+1,n]} = \det \begin{pmatrix} (B^{-1})^{[1,n]}_{[1,i]} \\ \hline A^{[1,n]}_{[i+1,n]} \end{pmatrix} \cdot \det B.$$

2.5 Construction of Φ_n

Theorem 2.10 ([8, Section 3])**.** *By the map defined by (1.3), we have an isomorphism $\mathcal{Y}^\circ \cong \mathcal{Z}^\circ$ of varieties over T.*

An element of $\tilde{\mathcal{Z}}$ is an algebraic family $\{Z_t\}$ of invertible upper triangular matrices parametrized by $t \in T$ such that Z_t commute with A_t. We denote the family $\{Z_t\}$ simply by Z. If we assume $Z \in \tilde{\mathcal{Z}}^\circ$, there exists an upper triangular matrix R and a unipotent lower triangular matrix U (see [8, Proposition 3.2]), both defined as families over T, such that

$$P^{-1}ZAP = U^{-1}R. \tag{2.11}$$

Because the left-hand side of (2.11) commutes with C_A, we have

$$UC_AU^{-1} = RC_AR^{-1}. \tag{2.12}$$

Let L denote the matrix (2.12). If we replace Z by cZ with $c \in \mathbb{C}^\times$, then R becomes cR and we obtain the same matrix L. We apply the Gauss decomposition to L as $L = MN^{-1}$. Then M, N are matrices of the forms given in (2.1) for unique $z_1, \dots, z_n, Q_1, \dots, Q_{n-1}$ (see [8, Section 3.2]). Thus we obtain functions z_i, Q_i on $\mathcal{Z}^\circ = \tilde{\mathcal{Z}}^\circ/\mathbb{C}^\times$. In view of (2.12), we have $\det(\zeta E - L) = \det(\zeta E - C_A)$, which means that the regular functions $z_1, \dots, z_n$, $Q_1, \dots$, Q_{n-1} satisfy the definiting equation for $\mathcal{Y}$. We can also check $Q_i \neq 0$ for $1 \le i \le n-1$ (see [8, Section 3.5]). In this way, we have a ring homomorphism $\Phi_n \colon \mathcal{O}(\mathcal{Z}^\circ) \to \mathcal{O}(\mathcal{Y}^\circ)$, which is naturally a homomorphism of $R(T)$-algebras.

The explicit form of Φ_n is determined as follows. By abuse of notation we simply denote $\Phi_n(z_i)$, $\Phi_n(Q_i)$ by z_i, Q_i. Let r_{ij} be the (i,j)-th entry of R. Then, from (2.11) and the Cauchy–Binet formula, we obtain

$$r_{11}r_{22}\cdots r_{ii} = \det R^{[1,i]}_{[1,i]} = \det\bigl(UP^{-1}ZAP\bigr)^{[1,i]}_{[1,i]} = \det(ZAP)^{[1,i]}_{[1,i]} = \tau_i, \tag{2.13}$$

which implies $r_{ii} = \tau_i/\tau_{i-1}$. Comparing the $(i+1,i)$-th entries on both sides of $NM^{-1} = RC_A^{-1}R^{-1}$, which is derived from (2.12), we obtain

$$Q_i = \frac{r_{i+1,i+1}}{r_{ii}} = \frac{\tau_{i+1}\tau_{i-1}}{\tau_i^2}. \tag{2.14}$$

On the other hand, from (2.1) and (2.12), z_i is expressed as

$$z_i = \frac{\det\bigl(L^{-1}\bigr)^{[1,i-1]}_{[1,i-1]}}{\det\bigl(L^{-1}\bigr)^{[1,i]}_{[1,i]}} = \frac{\det\bigl(UC_A^{-1}U^{-1}\bigr)^{[1,i-1]}_{[1,i-1]}}{\det\bigl(UC_A^{-1}U^{-1}\bigr)^{[1,i]}_{[1,i]}} = \frac{\det\bigl(C_A^{-1}U^{-1}\bigr)^{[1,i-1]}_{[1,i-1]}}{\det\bigl(C_A^{-1}U^{-1}\bigr)^{[1,i]}_{[1,i]}}.$$

As $C_A^{-1}U^{-1} = P^{-1}ZPR^{-1}$, we have

$$\det\bigl(C_A^{-1}U^{-1}\bigr)^{[1,i]}_{[1,i]} = \det\bigl(P^{-1}ZPR^{-1}\bigr)^{[1,i]}_{[1,i]} = \frac{\det\bigl(P^{-1}ZP\bigr)^{[1,i]}_{[1,i]}}{r_{11}\cdots r_{ii}} = \frac{\det(ZP)^{[1,i]}_{[1,i]}}{r_{11}\cdots r_{ii}} = \frac{\sigma_i}{\tau_i},$$

which gives

$$z_i = \frac{\tau_i\sigma_{i-1}}{\tau_{i-1}\sigma_i}. \tag{2.15}$$

We can construct the inverse of Φ_n (see [8, Section 3.4]). Here we provide an expression for Φ_n^{-1}. From (2.11), we have $P^{-1}ZP = U^{-1}RC_A^{-1}$. The entries of U^{-1} can be expressed as polynomials in z_i, Q_i, while the entries of R can be written as Laurent polynomials in z_i, Q_i. This gives rise to an expression for the entries of Z as Laurent polynomials in z_i, Q_i, which are considered as elements in $\mathcal{O}(\mathcal{Y}^\circ)$.

Example 2.11. For $n = 2$, we have

$$Z = \frac{1}{z_1z_2Q_1}\begin{pmatrix} z_2 - \mathrm{e}^{-a_1} & 1 \\ 0 & z_2 - \mathrm{e}^{-a_2}\end{pmatrix}.$$

For $n = 3$, we have

$$\begin{aligned} z_{11} &= (z_1z_2z_3Q_1Q_2)^{-1}\bigl(z_2z_3 - \mathrm{e}^{-a_1}(z_2(1-Q_2)+z_3) + \mathrm{e}^{-2a_1}\bigr),\\ z_{12} &= (z_1z_2z_3Q_1Q_2)^{-1}\bigl(z_2(1-Q_2) + z_3 - \mathrm{e}^{-a_1} - \mathrm{e}^{-a_2}\bigr),\\ z_{13} &= (z_1z_2z_3Q_1Q_2)^{-1}. \end{aligned}$$

The other entries are determined by $z_{i+1,j+1} = \omega(z_{ij})$.

Remark 2.12. As a complex manifold, $\mathcal{Z}^\circ$ is parameterized as

$$Z = \exp\bigl(At_1 + A^2t_2 + \cdots + A^{n-1}t_{n-1}\bigr) \in \mathcal{Z}^\circ \tag{2.16}$$

by the complex parameters $t_1, t_2, \dots, t_{n-1}$. From (2.16), we deduce the differential equation of motion $\frac{\partial}{\partial t_i}Z = A^iZ$. Then, computing the differential $\frac{\partial}{\partial t_i}\bigl(P^{-1}ZAP\bigr)$, we obtain

$$\frac{\partial}{\partial t_i}\bigl(P^{-1}ZAP\bigr) = P^{-1}\left(\frac{\partial}{\partial t_i}Z\right)AP = P^{-1}A^iZAP = C_A^iU^{-1}R,$$

where, for the last equality, we used (2.11). On the other hand, by using (2.11) again, we also obtain

$$\frac{\partial}{\partial t_i}\bigl(P^{-1}ZAP\bigr) = \frac{\partial}{\partial t_i}\bigl(U^{-1}R\bigr) = U^{-1}\left(\frac{\partial}{\partial t_i}R\right) - U^{-1}\left(\frac{\partial}{\partial t_i}U\right)U^{-1}R.$$

Comparing these equations, we have

$$L^i = UC_A^iU^{-1} = \left(\frac{\partial}{\partial t_i}R\right)R^{-1} - \left(\frac{\partial}{\partial t_i}U\right)U^{-1}.$$

Since $\bigl(\frac{\partial}{\partial t_i}R\bigr)R^{-1}$ is upper triangular, we obtain $\bigl(L^i\bigr)_< = -\bigl(\frac{\partial}{\partial t_i}U_{t_i}\bigr)U^{-1}$ by (2.12). Then, we have the relativistic Toda lattice (2.2) as follows:

$$\begin{aligned} \frac{\partial}{\partial t_i}L = \frac{\partial}{\partial t_i}\bigl(UC_AU^{-1}\bigr) &= \left(\frac{\partial}{\partial t_i}U\right)C_AU^{-1} - UC_AU^{-1}\left(\frac{\partial}{\partial t_i}U\right)U^{-1}\\ &= \left(\frac{\partial}{\partial t_i}U\right)U^{-1}L - L\left(\frac{\partial}{\partial t_i}U\right)U^{-1} = \bigl[L, \bigl(L^i\bigr)_<\bigr]. \end{aligned}$$

3 *K*-theoretic double *k*-Schur functions

In order to study the map Φ_n in more detail, we use a recent work [10] on a symmetric function realization of $K_*^T(\mathrm{Gr}_{\mathrm{SL}_n})$.

3.1 Definition of *K*-theoretic double *k*-Schur functions

The simple roots of $G = SL_n(\mathbb{C})$ are given by $\alpha_i = a_i - a_{i+1}$ for $i \in I$, where $I = \{1, \ldots, n-1\}$. Let $\theta = a_1 - a_n$ denote the highest root, and let $\theta^\vee$ be its corresponding coroot. We define an action of $\tilde{W}_G$ on $R(T)$, referred to as the *level zero* action. The coroot lattice $Q^\vee$ acts by the identity, while W_G acts naturally. In particular, s_0 acts as s_θ where θ is the highest root.

We will work in the *level zero* affine setting so we set $\alpha_0 := -\theta$. Let $\tilde{W}_G$ denote the corresponding affine Weyl group, with the standard generators $s_0, s_1, \ldots, s_{n-1}$. Let $R(T)^\Delta$ denote the localization of $R(T)$ by the multiplicative set generated by $1 - \mathrm{e}^\alpha$ where α are any roots. The *twisted group algebra* $R(T)^\Delta[\tilde{W}_G]$ is $\bigoplus_{w\in\tilde{W}_G} R(T)^\Delta w$ with product defined by

$$(f_1w_1)(f_2w_2) = (f_1w_1(f_2))(w_1w_2) \qquad \text{for} \quad f_1, f_2 \in R(T)^\Delta, \quad w_1, w_2 \in \tilde{W}_G.$$

with the level zero action of $\tilde{W}_G$ on $R(T)^\Delta$.

For $i \in \tilde{I} = I \cup \{0\}$, define the *Demazure operators*

$$T_i = (1 - \mathrm{e}^{\alpha_i})^{-1}(s_i - 1), \qquad D_i = T_i + 1, \tag{3.1}$$

which are considered as elements of $R(T)^\Delta[\tilde{W}_G]$.

T_i satisfies $T_i^2 = -T_i$ and the braid relations of type $A_{n-1}^{(1)}$. Similarly, D_i satisfies $D_i^2 = D_i$ and the braid relation of type $A_{n-1}^{(1)}$. Then, for any $w \in \hat{W}_G$ written as $w = s_{i_1} \cdots s_{i_l}$, the product $T_w = T_{i_1} \cdots T_{i_l}$ and the product $D_w = D_{i_1} \cdots D_{i_l}$ depend only on w (see [19], see also [10]).

The K-theoretic nil-Hecke algebra $\mathbb{K}_G$ is defined to be the left $R(T)$-module generated by D_w for $w \in \hat{W}_G$ (see [19], see also [10]).

We use notation

$$b_i := 1 - \mathrm{e}^{-a_i}, \qquad 1 \leq i \leq n. \tag{3.2}$$

We consider the ring $\hat{\Lambda}^{R(T)}$ of symmetric formal power series in the infinitely many variables $y = (y_1, y_2, \ldots)$ with coefficients in $R(T)$. We can define an action of Demazure operators D_i $(0 \leq i \leq n-1)$ on $\hat{\Lambda}^{R(T)}$ (see [10]). For $1 \leq i \leq n-1$, we let s_i act by exchanging a_i and a_{i+1} in the coefficients, and s_0 by the formula

$$s_0(f) = \frac{\Omega(b_1|y)}{\Omega(b_n|y)} s_\theta(f), \qquad \Omega(b_i|y) := \frac{1}{\prod_{j=1}^\infty (1 - b_i y_j)}, \tag{3.3}$$

where s_θ is the reflection with respect to θ, exchanging a_1 and a_n.

The finite Weyl group $W = \langle s_1, \ldots, s_{n-1} \rangle$ is the symmetric group S_n. Let $\hat{W}_G^0$ denote the set of minimal-length coset representatives of $\hat{W}_G/W_G$. The set $\hat{W}_G^0$ naturally indexes the set of the Schubert classes of $K_*^T(\mathrm{Gr}_G)$, and we refer to an element of $\hat{W}_G^0$ as an affine Grassmann element. For each $x \in \hat{W}_G^0$ [10], the *double K-theoretic k-Schur functions* are defined as

$$\tilde{g}_x^{(k)}(y|b) := D_x(1), \qquad g_x^{(k)}(y|b) := T_x(1). \tag{3.4}$$

The $R(T)$-span

$$\hat{\Lambda}_{(n)}^{R(T)} := \bigoplus_{x\in\hat{W}_G^0} R(T)\tilde{g}_x^{(k)}(y|b)$$

is an $R(T)$-subalgebra of $\hat{\Lambda}_{R(T)}$ which is isomorphic to $K_*^T(\mathrm{Gr}_{\mathrm{SL}_n})$ and the Schubert structure sheaf $\mathcal{O}_x$ is identified with $\tilde{g}_x^{(k)}(x|b)$. Let $\omega \in S_n$ be the cyclic permutation sending i to $i+1$ with $n+1 = 1$ by convention.

Theorem 3.1 ([10])**.** *There are isomorphisms β, κ of $R(T)$-algebras*

$$\mathcal{O}(\mathcal{Z}) \xrightarrow{\beta} \hat{\Lambda}^{R(T)}_{(n)} \xrightarrow{\kappa} K^T_*(\mathrm{Gr}_{\mathrm{SL}_n}),$$

such that

$$\begin{aligned}&\beta(z_{ij}/z_{11}) = \mathrm{e}^{a_i+\cdots+a_{j-1}}\Omega(b_i|y)\Omega(b_1|y)^{-1}g_{\rho_{j-i}}\big(y|\omega^i(b)\big),\\ &\kappa\big(\tilde{g}^{(k)}_x(y|b)\big) = \mathcal{O}_x, \qquad \kappa\big(g^{(k)}_x(y|b)\big) = \mathcal{I}_x,\end{aligned} \tag{3.5}$$

where $\rho_l = s_{l-1}\cdots s_1 s_0$ for $1 \le l \le n-1$.

Remark 3.2. In [10], it is proven that each of the three $R(T)$-algebras has a natural $R(T)$-Hopf-algebra structure.

Remark 3.3. It is natural to define a map $\beta\colon \mathcal{O}\big(\tilde{\mathcal{Z}}\big) \to \hat{\Lambda}^{R(T)}$. We have

$$\beta(z_{ij}) = \mathrm{e}^{a_i+\cdots+a_{j-1}}\Omega(b_i|y)g_{\rho_{j-i}}\big(y|\omega^i(b)\big), \tag{3.6}$$

where T is the maximal torus of $\mathrm{GL}_n(\mathbb{C})$. See [10, Theorem 1.2] for more details.

3.2 Preliminaries on Demazure operators

We collect some properties of T_i and D_i which are used in Section 6.2. Let ι be the involution on $\hat{\Lambda}^{R(T)}$ such that

$$\iota(h_i(y)) = \sum_{r=0}^{i-1}\binom{i-1}{r}e_{r+1}(y), \qquad \iota(\mathrm{e}^{a_i}) = \mathrm{e}^{-a_{n-i+1}}.$$

Proposition 3.4. *For $i \in I$, $\iota\circ T_i = T_{n-i}\circ\iota$, $\iota\circ D_i = D_{n-i}\circ\iota$.*

Proof. Straightforward. ■

Let ω_k be the group automorphism of $\hat{W}_G$ such that

$$\omega_k(s_i) = s_{-i}, \tag{3.7}$$

with indices taken mod $n\mathbb{Z}$.

Proposition 3.5 ([10, Proposition 3.14])**.** *We have $\iota\big(\tilde{g}^{(k)}_x\big(y|\omega^i b\big)\big) = \tilde{g}^{(k)}_{x^{\omega_k}}\big(y|\omega^{-i}b\big)$.*

We define elements T_θ and D_θ of the twisted group algebra $R(T)^\Delta\big[\tilde{W}_G\big]$:

$$T_\theta := \omega^{-i}\circ T_i\circ\omega^i = \frac{s_\theta - 1}{1-\mathrm{e}^{-\theta}}, \qquad D_\theta := \omega^{-i}\circ D_i\circ\omega^i = T_\theta + 1. \tag{3.8}$$

Remark 3.6. For any $\alpha \in Q$, an element T_α of $R(T)^\Delta\big[\tilde{W}_G\big]$ is defined. See [10, Section 2.2.4] for more details.

Lemma 3.7. *For $i \in I$, $T_i\circ\mathrm{e}^{-a_{i+1}} = \mathrm{e}^{-a_i}\circ D_i$, and $T_\theta\circ\mathrm{e}^{-a_1} = \mathrm{e}^{-a_n}\circ D_\theta$.*

Proof. The lemma is shown by direct calculations: $T_j(\mathrm{e}^{-a_{j+1}}f) = T_j(\mathrm{e}^{-a_{j+1}})f + s_j(\mathrm{e}^{-a_{j+1}})T_j(f)$ $= \mathrm{e}^{-a_j}f + \mathrm{e}^{-a_j}T_j(f) = \mathrm{e}^{-a_j}D_j(f)$. ■

Proposition 3.8. *We have*

$$T_0 = \Omega(b_1|y)\circ T_\theta\circ\Omega(b_1|y)^{-1} = \Omega(b_n|y)^{-1}\circ T_\theta\circ\Omega(b_n|y).$$

Proof. The first equality is shown as follows:

$$\begin{aligned}
&\Omega(b_1|y)\circ T_\theta\circ\Omega(b_1|y)^{-1}(f)\\
&\quad=\Omega(b_1|y)T_\theta\left(\frac{f}{\Omega(b_1|y)}\right)=\frac{1-b_n}{b_n-b_1}\Omega(b_1|y)\left(\frac{f}{\Omega(b_1|y)}-\frac{s_\theta(f)}{\Omega(b_n|y)}\right)\\
&\quad=\frac{1-b_n}{b_n-b_1}\left(f-\frac{\Omega(b_1|y)}{\Omega(b_n|y)}s_\theta(f)\right)=T_0(f).
\end{aligned}$$

The second equality follows from the fact that $\Omega(b_1|y)\Omega(b_n|y)$ commutes with T_θ. ■

Lemma 3.9. *For $1\le i\le n-1$ and $j-i\le n-1$, and $m\ge 0$,*

$$D_ih_m(\mathrm{e}^{-a_{i+1}},\mathrm{e}^{-a_{i+2}},\dots,\mathrm{e}^{-a_j})=h_m(\mathrm{e}^{-a_i},\mathrm{e}^{-a_{i+1}},\mathrm{e}^{-a_{i+2}},\dots,\mathrm{e}^{-a_j}). \tag{3.9}$$

For $1\le j\le n-1$, and $m\ge 0$,

$$D_\theta h_m(\mathrm{e}^{-a_1},\mathrm{e}^{-a_2},\dots,\mathrm{e}^{-a_j})=h_m(\mathrm{e}^{-a_n},\mathrm{e}^{-a_1},\mathrm{e}^{-a_2},\dots,\mathrm{e}^{-a_j}). \tag{3.10}$$

Proof. Note

$$D_i(1-\mathrm{e}^{-a_{i+1}}u)^{-1}=(1-\mathrm{e}^{-a_i}u)^{-1}(1-\mathrm{e}^{-a_{i+1}}u)^{-1}.$$

As $\prod_{s=i+2}^{j}(1-\mathrm{e}^{-a_s}u)^{-1}$ is invariant under s_i, we obtain $D_i\prod_{s=i+1}^{j}(1-\mathrm{e}^{-a_s}u)^{-1}=\prod_{s=i}^{j}(1-\mathrm{e}^{-a_s}u)^{-1}$, which implies (3.9). The proof of (3.10) is given similarly. ■

4 Some automorphisms

For later use, we introduce some automorphisms of the algebras $\mathcal{O}(\tilde{\mathcal{Z}}),\hat{\Lambda}^{R(T)}$ and study their properties.

4.1 Automorphism σ

We define an automorphism σ of $\mathcal{O}(\tilde{\mathcal{Z}})$ as an $R(T)$-algebra by

$$\sigma(Z)=ZA^{-1}. \tag{4.1}$$

Explicitly, for $1\le i\le j\le n$, we have

$$\sigma(z_{ij})=\mathrm{e}^{a_j}z_{ij}+\mathrm{e}^{a_{j-1}+a_j}z_{i,j-1}+\cdots+\mathrm{e}^{a_i+\cdots+a_j}z_{ii}.$$

In particular, we have

$$\sigma(z_{ii})=\mathrm{e}^{a_i}z_{ii}. \tag{4.2}$$

From (2.9) and (2.10), we have

$$\sigma(\tau_i)=\sigma_i. \tag{4.3}$$

Via the isomorphism β we consider the corresponding automorphism on $\hat{\Lambda}_{(n)}^{R(T)}$ and denote it also by σ. For example, we have

$$\sigma(\Omega(b_i|y))=\mathrm{e}^{a_i}\Omega(b_i|y). \tag{4.4}$$

In the next section, we prove

$$\sigma(g_{\rho_i}^{(k)}(y|b))=\mathrm{e}^{a_i-a_n}\tilde{g}_{\rho_i}^{(k)}(y|b). \tag{4.5}$$

Since we know $g_{\rho_i}^{(k)}(y|0)=h_i(y)$, we have $\sigma(h_i(y))=1+h_1(y)+\cdots+h_i(y)$. In fact, σ on $\hat{\Lambda}^{R(T)}$ can be defined as the $R(T)$-linear map sending $f(y_1,y_2,\dots)\in\hat{\Lambda}$ to $f(1,y_1,y_2,\dots)$. This automorphism already appeared in the study of the non-equivariant version of K-Peterson isomorphism in [2, 9].

4.2 Image of $g_\lambda^{(k)}(y|b)$ under σ for k-small λ

Let $\operatorname{diag}(\lambda)$ denote the *main diagonal* of λ, that is, the set of boxes at the (i,i)-th position for $i = 1, 2, \dots$. For $x \in \operatorname{diag}(\lambda)$, let $r(x)$ be the n-residue of the box that is furthest to the right from x, and $b(x)$ the n-residue of the box that is furthest below from x.

Proposition 4.1. *If λ is k-small, we have*

$$\sigma\big(g_\lambda^{(k)}(y|b)\big) = \left(\prod_{x\in\operatorname{diag}(\lambda)} \mathrm{e}^{a_{r(x)+1}-a_{b(x)}}\right)\tilde{g}_\lambda^{(k)}(y|b).$$

Example 4.2. When $n = 6$ and

$$\lambda = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 5 & 0 & 1 \\ \hline 4 \\ \cline{1-1}\end{array},$$

the main diagonal consists of two boxes x_1, x_2, where x_i is at the (i,i)-th position. Since $r(x_1) = 2$, $b(x_1) = 4$, $r(x_2) = 1$, and $b(x_2) = 0$, we have $\sigma(g_\lambda^{(5)}(y|b)) = \mathrm{e}^{(a_3-a_4)+(a_2-a_6)}\tilde{g}_\lambda^{(5)}(y|b)$.

Lemma 4.3. $\sigma \circ T_0 = \mathrm{e}^\theta D_0 \circ \sigma$.

Proof. Let $f(y|b) \in \hat{\Lambda}^{R(T)}$. Then by using (4.12), we have

$$\begin{aligned}\sigma T_0 f(y|b) &= \sigma\left(\frac{1}{1-\mathrm{e}^{-\theta}}\Big(\frac{\Omega(b_1|y)}{\Omega(b_n|y)}s_\theta f(y|b) - f(y|b)\Big)\right)\\ &= \frac{1}{1-\mathrm{e}^{-\theta}}\left(\mathrm{e}^\theta\frac{\Omega(b_1|y)}{\Omega(b_n|y)}\sigma((s_\theta f)(y|b)) - (\sigma f)(y|b)\right)\\ &= \frac{1}{1-\mathrm{e}^{-\theta}}\left(\mathrm{e}^\theta\frac{\Omega(b_1|y)}{\Omega(b_n|y)}s_\theta((\sigma f)(y|b)) - \mathrm{e}^\theta\sigma f(y|b) + \mathrm{e}^\theta\sigma f(y|b) - \sigma f(y|b)\right)\\ &= \mathrm{e}^\theta T_0\sigma f(y|b) + \mathrm{e}^\theta\sigma f(y|b) = \mathrm{e}^\theta D_0\sigma f(y|b),\end{aligned}$$

where we use that σ commutes with the action of s_θ. ■

Proof of Proposition 4.1. We use induction on the number of boxes of λ. The case $\lambda = \varnothing$ is obvious. Suppose $\lambda \neq \varnothing$. There is a box removable from λ, with the n-residue say i. Let μ be the partition obtained from λ by removing the box. We consider the case when $i = 0$. One easily sees that $r(x) + 1$, $b(x) \notin \{1, n\}$ for $x \in \operatorname{diag}(\mu)$. It follows that $s_\theta(e(\mu)) = e(\mu)$. We also note that $\mathrm{e}^\theta e(\mu) = e(\lambda)$,

$$\begin{aligned}\sigma\big(g_\lambda^{(k)}(y|b)\big) &= \sigma\big(T_0 g_\mu^{(k)}(y|b)\big) = \mathrm{e}^\theta D_0\sigma\big(g_\mu^{(k)}(y|b)\big) \qquad \text{by Lemma 4.3}\\ &= \mathrm{e}^\theta D_0\big(e(\mu)\tilde{g}_\mu^{(k)}(y|b)\big) = \mathrm{e}^\theta e(\mu)D_0\big(\tilde{g}_\mu^{(k)}(y|b)\big) = e(\lambda)\tilde{g}_\lambda^{(k)}(y|b).\end{aligned}$$

The case when $i \neq 0$ is left to the reader since it is similar and easier. ■

Remark 4.4. The assumption that λ is k-small in Proposition 4.1 is mandatory. Indeed, if we consider the case when $n = 3$ and $w = s_2s_1s_0$, the associated 2-bounded partition $\lambda = (2,1)$ is not 2-small, and $\sigma\big(g_w^{(2)}(y|b)\big)$ equals

$$\begin{aligned}&\mathcal{T}_2\big(\mathrm{e}^{a_2-a_3}\otimes\tilde{g}_{10}^{(2)}(y|b)\big)\\ &\quad= \big(1+\mathrm{e}^{a_3-a_2}\big)\otimes\tilde{g}_{10}^{(2)}(y|b) + \mathrm{e}^{a_3-a_2}\otimes T_2\tilde{g}_{10}^{(2)}(y|b)\\ &\quad= 1\otimes\tilde{g}_{10}^{(2)}(y|b) + \mathrm{e}^{a_3-a_2}\otimes D_2\tilde{g}_{10}^{(2)}(y|b) = 1\otimes\tilde{g}_{10}^{(2)}(y|b) + \mathrm{e}^{a_3-a_2}\otimes\tilde{g}_{210}^{(2)}(y|b).\end{aligned}$$

4.3 Regular functions c_i

For arbitrary matrix Z that commutes with A, there are unique scalars c_i satisfying

$$Z = c_0 E + c_1 A + \cdots + c_{n-1} A^{n-1}. \tag{4.6}$$

The existence of such scalars is assured by the fact that A is conjugate to a companion matrix. We will consider each c_i as an element of $\mathcal{O}(\tilde{\mathcal{Z}})$.

Example 4.5. If $n = 3$ we have expressions

$$c_0 = z_{11} + \mathrm{e}^{-a_1} z_{12} + \mathrm{e}^{-a_1-a_2} z_{13}, \qquad c_1 = -z_{12} - (\mathrm{e}^{-a_1} + \mathrm{e}^{-a_2}) z_{13}, \qquad c_2 = z_{13}$$

by comparing the 1st row of (4.6).

By comparing the diagonal entries, we have $z_{ii} = c_0 + c_1 \mathrm{e}^{-a_i} + \cdots + c_{n-1} \mathrm{e}^{-(n-1)a_i}$. We also define $c_0^{(j)}, c_1^{(j)}, \dots, c_{n-1}^{(j)} \in \mathcal{O}(\tilde{\mathcal{Z}})$ for $j \in \mathbb{Z}$ by

$$A^j Z = c_0^{(j)} E + c_1^{(j)} A + \cdots + c_{n-1}^{(j)} A^{n-1}. \tag{4.7}$$

Note that we have $c_j^{(0)} = c_j$. Comparing the diagonal entries on both sides of (4.7), we have

$$\mathrm{e}^{-ja_i} z_{ii} = c_0^{(j)} + c_1^{(j)} \mathrm{e}^{-a_i} + \cdots + c_{n-1}^{(j)} \mathrm{e}^{-(n-1)a_i}. \tag{4.8}$$

Theorem 4.6. *We have*

$$\sigma_i = \begin{vmatrix} c_0^{(0)} & c_1^{(0)} & \dots & c_{i-1}^{(0)} \\ c_0^{(1)} & c_1^{(1)} & \dots & c_{i-1}^{(1)} \\ \vdots & \vdots & \dots & \vdots \\ c_0^{(i-1)} & c_1^{(i-1)} & \dots & c_{i-1}^{(i-1)} \end{vmatrix}, \qquad \tau_i = \begin{vmatrix} c_0^{(1)} & c_1^{(1)} & \dots & c_{i-1}^{(1)} \\ c_0^{(2)} & c_1^{(2)} & \dots & c_{i-1}^{(2)} \\ \vdots & \vdots & \dots & \vdots \\ c_0^{(i)} & c_1^{(i)} & \dots & c_{i-1}^{(i)} \end{vmatrix}. \tag{4.9}$$

Proof. Since P^{-1} is lower unitriangular, we use Proposition 2.8 to have

$$\sigma_i = \det(ZP)_{[1,i]}^{[1,i]} = \det\bigl(P^{-1} Z P\bigr)_{[1,i]}^{[1,i]} = \det\bigl(c_0 E + c_1 C_A + \cdots + c_{n-1} C_A^{n-1}\bigr)_{[1,i]}^{[1,i]}.$$

By comparing the 1-st row of

$$C_A^{j-1}\bigl(c_0 E + c_1 C_A + \cdots + c_{n-1} C_A^{n-1}\bigr) = c_0^{(j-1)} E + c_1^{(j-1)} C_A + \cdots + c_{n-1}^{(j-1)} C_A^{n-1},$$

we see that the (j,l)-th entry of $c_0 E + c_1 C_A + \cdots + c_{n-1} C_A^{n-1}$ is $(-1)^{j+l} c_l^{(j-1)}$. Hence the first equation of (4.9) holds. The equation for τ_i follows from this by applying σ^{-1} because of (4.3). ■

4.4 Involution ι on $\mathcal{O}(\tilde{\mathcal{Z}})$

Let J be the permutation matrix of the longest element $w_\circ$ of S_n. Explicitly, J is $\sum_{i=1}^n E_{i,n-i+1}$. Let ι be a ring automorphism of $\mathcal{O}(\tilde{\mathcal{Z}})$ defined by $\iota(\mathrm{e}^{\gamma}) = \mathrm{e}^{-w_\circ \gamma}$ (in particular $\iota(\mathrm{e}^{a_i}) = \mathrm{e}^{-a_{n-i+1}}$), and

$$\sum_{i=0}^{n-1} \iota(c_i) A^{-i} = \left(\sum_{i=0}^{n-1} c_i A^i \right)^{-1}.$$

It is not difficult to show $\iota(C_A) = J C_A^{-1} J$. Since $C_A = P^{-1} A P$, we obtain

$$\sum_{i=0}^{n-1} \iota(c_i) C_A^{-i} = \left(\sum_{i=0}^{n-1} c_i C_A^i \right)^{-1}$$

and

$$\begin{aligned}\iota\big(P^{-1}ZP\big) &= \iota\big(c_0 + c_1 C_A + \cdots + c_{n-1} C_A^{n-1}\big)\\ &= J\big(\iota(c_0) + \iota(c_1) C_A^{-1} + \cdots + \iota(c_{n-1}) C_A^{-(n-1)}\big) J\\ &= J\big(c_0 + c_1 C_A + \cdots + c_{n-1} C_A^{n-1}\big)^{-1} J = J P^{-1} Z^{-1} P J. \end{aligned}\tag{4.10}$$

Comparing the diagonal entries on both sides of (4.10), we have

$$\iota(z_{ii}) = z^{-1}_{n-i+1,n-i+1}.\tag{4.11}$$

Identifying z_{ii} with $\Omega(b_i|y)$, we have

$$\iota(\Omega(b_i|y)) = \Omega(b_{n-i+1}|y)^{-1}.\tag{4.12}$$

Remark 4.7. The automorphism ι is induced by a diagram automorphism of the affine Dynkin diagram. See [10, Sections 2.7 and 3.3].

Proposition 4.8. *ι commutes with σ.*

Proof. Note that ι and σ are naturally extended as ring automorphisms of $\mathcal{O}\big(\tilde{\mathcal{Z}}\big)^{\Delta}$. In view of Proposition 2.6, we only need to check on the generators z_{ii} and $f \in R(T)$. From (4.11) and (4.2), we have

$$\sigma(\iota(z_{ii})) = \sigma\big(z^{-1}_{n-i+1,n-i+1}\big) = \mathrm{e}^{-a_{n-i+1}} z^{-1}_{n-i+1,n-i+1} = \iota(\mathrm{e}^{a_i} z_{ii}) = \iota(\sigma(z_{ii})).$$

Since σ is $R(T)$-linear, it is clear that $\iota(\sigma(f)) = \sigma(\iota(f)) = \iota(f)$ for $f \in R(T)$. ■

Proposition 4.9. *We have*

$$\iota(\tau_i) = \frac{\tau_{n-i}}{\tau_n}, \qquad \iota(\sigma_i) = \frac{\sigma_{n-i}}{\sigma_n}.\tag{4.13}$$

Proof. By (4.10), we have

$$\begin{aligned}\iota(\sigma_i) &= \iota\left(\det\big(P^{-1}ZP\big)^{[1,i]}_{[1,i]}\right) = \det\big(JP^{-1}Z^{-1}PJ\big)^{[1,i]}_{[1,i]} = \det\big(P^{-1}Z^{-1}P\big)^{[n-i+1,n]}_{[n-i+1,n]}\\ &= \det\big(P^{-1}Z^{-1}P\big)\cdot\det\big(P^{-1}ZP\big)^{[1,n-i]}_{[1,n-i]} = \det\big(Z^{-1}\big)\cdot\det\big(P^{-1}ZP\big)^{[1,n-i]}_{[1,n-i]}\\ &= \frac{\sigma_{n-i}}{\Omega(b_1|y)\cdots\Omega(b_n|y)} = \frac{\sigma_{n-i}}{\sigma_n},\end{aligned}$$

where for the fourth equality we use the fact $\det A^{[m+1,n]}_{[m+1,n]} = \det A \cdot \det\big(A^{-1}\big)^{[1,m]}_{[1,m]}$, which holds for any invertible A. The first equality of (4.13) is obtained from Proposition 4.8 and (4.3). ■

4.5 Basic properties of Φ_n

We collect some basic properties of Φ_n which will be used below.

Proposition 4.10. *We have*

$$\iota \circ \Phi_n = \Phi_n \circ \iota.\tag{4.14}$$

Proof. For (4.14), it suffices to show on generators z_i, and Q_i. From Proposition 4.9, we have

$$(\iota\circ\Phi_n)(z_i) = \iota\left(\frac{\tau_i\sigma_{i-1}}{\tau_{i-1}\sigma_i}\right) = \frac{\tau_{n-i}\sigma_{n-i+1}}{\sigma_{n-i}\tau_{n-i+1}} = \Phi_n\big(z^{-1}_{n-i+1}\big) = (\Phi_n\circ\iota)(z_i)$$

and

$$(\iota\circ\Phi_n)(Q_i) = \iota\left(\frac{\tau_{i-1}\tau_{i+1}}{\tau_i^2}\right) = \frac{\tau_{n-i+1}\tau_{n-i-1}}{\tau^2_{n-i}} = \Phi_n(Q_{n-i}) = (\Phi_n\circ\iota)(Q_i),$$

which concludes the proposition. ■

Let us denote the characteristic polynomial of $L^{[1,i]}_{[1,i]}$ by $\chi_i(\zeta) := \det(\zeta E - L)^{[1,i]}_{[1,i]}$. Then we have from Appendix B

$$\chi_i(\zeta) = \zeta^i - F^{(i)}_1 \zeta^{i-1} + \cdots + (-1)^i F^{(i)}_i,$$

where

$$F^{(i)}_m := \sum_{\substack{J \subset [i] \\ |J|=m}} \prod_{\substack{j \in J, \\ j+1 \notin J}} (1 - Q_j) \prod_{j \in J} z_j, \tag{4.15}$$

for $0 \leq m \leq i \leq n$, with $F^{(i)}_0 = 1$.

Lemma 4.11. *For $1 \leq m \leq i \leq n$, we have*

$$(\iota \circ \Phi_n)\big(F^{(i)}_m\big) = \frac{(-1)^m}{\tau_{n-i}} \begin{vmatrix} c^{(1)}_0 & c^{(1)}_1 & \cdots & c^{(1)}_{n-i-2} & c^{(1)}_{n-i+m-1} \\ c^{(2)}_0 & c^{(2)}_1 & \cdots & c^{(2)}_{n-i-2} & c^{(2)}_{n-i+m-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ c^{(n-i)}_0 & c^{(n-i)}_1 & \cdots & c^{(n-i)}_{n-i-2} & c^{(n-i)}_{n-i+m-1} \end{vmatrix}.$$

Proof. We first compute $\Phi_n(\chi_i(\zeta))$ as follows

$$\begin{aligned} &\det\big(U^{-1}\big)^{[1,i]}_{[1,i]} \cdot \Phi_n(\chi_i(\zeta)) \cdot \det R^{[1,i]}_{[1,i]} \\ &\quad = \det\big(U^{-1}\big)^{[1,i]}_{[1,i]} \cdot \det(\zeta E - \Phi_n(L))^{[1,i]}_{[1,i]} \cdot \det R^{[1,i]}_{[1,i]} \\ &\quad = \det\big(U^{-1}(\zeta E - \Phi_n(L))R\big)^{[1,i]}_{[1,i]} \qquad \text{by Proposition 2.8} \\ &\quad = \det\big((\zeta E - C_A)U^{-1}R\big)^{[1,i]}_{[1,i]} \qquad \text{since } \Phi_n(L) = UC_AU^{-1} \text{ from (2.12)} \\ &\quad = \det\big((\zeta E - C_A)P^{-1}ZAP\big)^{[1,i]}_{[1,i]} \qquad \text{by (2.11)}. \end{aligned}$$

Since $\det\big(U^{-1}\big)^{[1,i]}_{[1,i]} = 1$ and $\det R^{[1,i]}_{[1,i]} = \tau_i$ (see (2.13)), we deduce that

$$\Phi_n(\chi_i(\zeta)) = {\tau_i}^{-1} \det\big((\zeta E - C_A)P^{-1}ZAP\big)^{[1,i]}_{[1,i]}. \tag{4.16}$$

Let us apply ι on both sides of (4.16). By using Proposition 4.10, we can verify

$$\iota\big(P^{-1}ZAP\big) = JP^{-1}Z^{-1}A^{-1}PJ.$$

Using this together with $\iota(C_A) = JC_A^{-1}J$, we have

$$\begin{aligned} (\iota \circ \Phi_n)(\chi_i(\zeta)) &= (\tau_n/\tau_{n-i}) \cdot \det\big(J(\zeta E - C_A^{-1})P^{-1}Z^{-1}A^{-1}PJ\big)^{[1,i]}_{[1,i]} \\ &= (\tau_n/\tau_{n-i}) \cdot \det\big((\zeta E - C_A^{-1})P^{-1}Z^{-1}A^{-1}P\big)^{[n-i+1,n]}_{[n-i+1,n]} \\ &= \frac{1}{\tau_{n-i}} \det \left(\frac{\big(P^{-1}AZP\big)^{[1,n]}_{[1,n-i]}}{\big(\zeta E - C_A^{-1}\big)^{[1,n]}_{[n-i+1,n]}} \right), \end{aligned}$$

where in the last equality we use Proposition 2.9, and $\det\big(P^{-1}AZP\big) = \tau_n$. Comparing the coefficients of ζ^{i-m} on both sides of this equality, we obtain

$$(\iota \circ \Phi_n)\big(F^{(i)}_m\big) = \frac{\det\big(P^{-1}AZP\big)^{[1,n-i-1]\cup\{n-i+m\}}_{[1,n-i]}}{\tau_{n-i}}.$$

Because we know $P^{-1}AZP = \big((-1)^{j+l}c^{(j)}_l\big)_{1\leq j,l \leq n}$ (see the proof of Theorem 4.6), we obtain the desired result, after eliminating unnecessary signs. ■

5 Quantum double Grothendieck polynomials

We use notation of binary operations $x \oplus y = x + y - xy$ and $x \ominus y = (x - y)/(1 - y)$. We also denote $-x/(1 - x) = 0 \ominus x$ by $\overline{x}$.

5.1 Quantum double Grothendieck polynomials

We recall Lenart–Maeno's quantum double Grothendieck polynomials.

Let us consider the two sets of variables $z = (z_1, \dots, z_n)$ and $\eta = (\eta_1, \dots, \eta_n)$, and the polynomial ring $\mathbb{Z}[Q][z_1, \dots, z_n, \eta_1, \dots, \eta_n]$, where $Q_1, \dots, Q_{n-1}$ are the Novikov variables. For $i \in I$, define

$$T_i^Q = \frac{s_i^{(\eta)} - 1}{\eta_{i+1} \ominus \eta_i}, \qquad D_i^Q = 1 + \frac{s_i^{(\eta)} - 1}{\eta_{i+1} \ominus \eta_i}, \tag{5.1}$$

as linear endomorphisms of $\mathbb{Z}[Q][z_1, \dots, z_n, \eta_1, \dots, \eta_n]$. Define

$$\mathfrak{G}^Q_{w_\circ}(z|\eta) = \prod_{i=1}^{n-1} \psi_i, \tag{5.2}$$

$$\psi_i = \sum_{j=0}^{i} (-1)^j (1 - \eta_{n-i})^j F_j^{(i)}(z, Q). \tag{5.3}$$

Note that ψ_i can be written as

$$\psi_i = \det(E - (1 - \eta_{n-i})L)^{[1,i]}_{[1,i]} \tag{5.4}$$

if L is the matrix given as in Section 2.1.

There exists a unique collection of polynomials

$$\big\{\mathfrak{G}^Q_w(z|\eta) \in \mathbb{Z}[Q][z_1, \dots, z_n, \eta_1, \dots, \eta_n] \mid w \in S_n\big\},$$

where $\mathfrak{G}^Q_{w_\circ}$ corresponding to the longest element $w_\circ$ is given by (5.2), and each $\mathfrak{G}^Q_w$ for general w is characterized by the recursive relation

$$D_i^Q \mathfrak{G}^Q_w(z|\eta) = \begin{cases} \mathfrak{G}^Q_{s_i w}(z|\eta) & \text{if } s_i w < w, \\ \mathfrak{G}^Q_w(z|\eta) & \text{if } s_i w > w. \end{cases} \tag{5.5}$$

Remark 5.1. We follow the notation in [27] as closely as possible, however, there are some unavoidable differences as outlined below. The variables y_i used for equivariant parameters in [27] are denoted here by η_i. This change is necessary because we reserve y_i for the variables of symmetric functions. We treat the variable η_i as an element of $R(T)$ via the correspondence

$$\eta_i = 1 - \mathrm{e}^{a_{n-i+1}} = \overline{b}_{n-i+1}, \qquad 1 \le i \le n. \tag{5.6}$$

Note that our a_i corresponds to $-\epsilon_i$ in [27]. Additionally, we use $z_i = 1 - x_i$ with x_i the variable from [27].

Proposition 5.2. *For $i \in I$, we have*

$$D_{n-i} \circ \Phi_n = \Phi_n \circ D_i^Q, \qquad D_{n-i} \circ \tilde{\Phi}_n = \tilde{\Phi}_n \circ D_i^Q. \tag{5.7}$$

Proof. The operators on both sides coincide on z_i, Q_i, because τ_i and σ_i are S_n-invariant. Since Φ_n $\big($and $\tilde{\Phi}_n\big)$ is $R(T)$-linear, it suffices to see the equality on $R(T)$. In fact, we have

$$1 + \frac{s_i^{(\eta)} - 1}{\eta_{i+1} \ominus \eta_i} = 1 + \frac{s_{n-i}^{(b)} - 1}{\overline{b}_{n-i} \ominus \overline{b}_{n-i+1}} = 1 + \frac{s_{n-i}^{(b)} - 1}{b_{n-i+1} \ominus b_{n-i}},$$

under the identification (5.6), where $s_i^{(b)}$ exchanges b_i and b_{i+1}, and we use $\overline{x} \ominus \overline{y} = y \ominus x$. ■

5.2 Involution ι on the equivariant quantum K-ring

Let ι be a ring homomorphism of $R(T)[Q]\big[z_1^{\pm 1},\dots,z_n^{\pm 1}\big]$ such that

$$\iota(\mathrm{e}^{a_i})=\mathrm{e}^{-a_{n-i+1}},\qquad \iota(z_i)=z_{n-i+1}^{-1},\qquad \iota(Q_i)=Q_{n-i}.$$

Obviously, ι satisfies $\iota^2=1$. It is easy to prove that

$$\iota\circ D_i^Q=D_{n-i}^Q\circ\iota. \tag{5.8}$$

holds for $i\in I$.

Proposition 5.3. *ι preserves the ideal $\mathcal{I}_n^Q$ of $R(T)[Q]\big[z_1^{\pm 1},\dots,z_n^{\pm 1}\big]$.*

Proof. We claim that $z_1\cdots z_n\iota\big(F_i^{(n)}\big)=F_{n-i}^{(n)}$, $e_n^{(n)}\iota\big(e_i^{(n)}\big)=e_{n-i}^{(n)}$ for $1\le i\le n-1$. The second identity is obvious. For $J\subset[1,n]$, $|J|=i$, set $K:=\{i'\mid i\notin J\}$, with $i':=n-i+1$. Then $z_1\cdots z_n\prod_{j\in J}\iota(z_i)=\prod_{l\in K}z_l$. We see that $j\in J, j+1\notin J$ is equivalent to $(j+1)'\in K$, $j'\notin K$. Thus

$$\prod_{j\in J,j+1\notin J}(1-Q_j)=\prod_{l\in K,l+1\notin K}(1-Q_l).$$

Hence the claim holds and we have $z_1\cdots z_n\iota\big(F_i^{(n)}\big)=F_{n-i}^{(n)}$. Since $F_n^{(n)}=z_1\cdots z_n$, and $e_n^{(n)}=\mathrm{e}^{-(a_1+\cdots+a_n)}=1$, we are done. ■

The next result will be proved in Section 8.

Proposition 5.4. *We have $\iota\big(\mathfrak{G}_{w_\circ}^Q\big)=\mathfrak{G}_{w_\circ}^Q$.*

For $w\in S_n$, let $w^*=w_\circ w w_\circ$; w^* is obtained from any reduced expression of w by replacing s_i with s_{n-i}. Note that the map $w\mapsto w^*$ preserves the Bruhat order.

Proposition 5.5. *Let $w\in S_n$. Then*

$$\iota\big(\mathfrak{G}_w^Q(z|\eta)\big)=\mathfrak{G}_{w^*}^Q(z|\eta). \tag{5.9}$$

Proof. We use induction on $\ell(w_\circ)-\ell(w)$. Since $\mathfrak{G}_{w_\circ}^Q$ is ι-invariant (see Proposition 5.4) and $w_\circ^*=w_\circ$, (5.9) holds for $w=w_\circ$. Suppose $w\in S_n$ satisfies $s_iw>w$ for some $1\le i\le n-1$. Then we have $(s_iw)^*>w^*$. It follows that $s_{n-i}(s_iw)^*<(s_iw)^*$. Then by using (5.5) and (5.8), we have

$$\begin{aligned}\iota\big(\mathfrak{G}_w^Q(z|\eta)\big)&=\iota\big(D_i^Q\mathfrak{G}_{s_iw}^Q(z|\eta)\big)=D_{n-i}^Q\iota\big(\mathfrak{G}_{s_iw}^Q(z|\eta)\big)\\&=D_{n-i}^Q\mathfrak{G}_{(s_iw)^*}^Q(z|\eta)=\mathfrak{G}_{s_{n-i}(s_iw)^*}^Q(z|\eta)=\mathfrak{G}_{w^*}^Q(z|\eta).\end{aligned}$$ ■

5.3 Explicit formula for $\mathfrak{G}_{s_\theta}^Q(x|\eta)$

The quantum double Grothendieck polynomial for s_θ plays a fundamental role in the following.

Let $\omega^{(\eta)}$ be the cyclic permutation with respect to the variables η, that is $\omega^{(\eta)}(\eta_i)=\eta_{i+1}$ with $\eta_{n+1}=\eta_1$. We use the following notation: $[x|\eta]^i=(x\oplus\eta_1)\cdots(x\oplus\eta_i)$ for $i\ge 1$.

Proposition 5.6. *We have*

$$\mathfrak{G}_{s_\theta}^Q(z|\eta)=\mathfrak{G}_{s_1s_2\dots s_{n-1}}^Q(z|\eta)\cdot\mathfrak{G}_{s_{n-2}\dots s_2s_1}^Q\big(z|\omega^{(\eta)}(\eta)\big). \tag{5.10}$$

Proof. We first prove the classical version

$$\mathfrak{G}_{s_\theta}(z|\eta) = \mathfrak{G}_{s_1 s_2 \dots s_{n-1}}(z|\eta) \cdot \mathfrak{G}_{s_{n-2}\dots s_2 s_1}\big(x|\omega^{(\eta)}(\eta)\big) \tag{5.11}$$

of (5.10). Since $s_\theta \in S_n$ is a dominant permutation of code (see [25, Chapter I] for the definitions) $c(s_\theta) = (n-1, 1, \dots, 1, 0)$,

$$\mathfrak{G}_{s_\theta}(z|\eta) = [x_1|\eta]^{n-1}[x_2|\eta]\cdots[x_{n-1}|\eta] = [x_1|\eta][x_2|\eta]\cdots[x_{n-1}|\eta] \times \big[x_1|\omega^{(\eta)}(\eta)\big]^{n-2}.$$

For the first equality, we have used [26, Lemma B.6]. Because $s_1 s_2 \dots s_{n-1}$ and $s_{n-2}\dots s_2 s_1$ are also dominant, with the codes $(1, \dots, 1, 0)$ and $(n-2, 0, \dots, 0)$ respectively, we have by [26, Lemma B.6]

$$\mathfrak{G}_{s_1 s_2 \dots s_{n-1}}(z|\eta) = [x_1|\eta][x_2|\eta]\cdots[x_{n-1}|\eta], \qquad \mathfrak{G}_{s_{n-2}\dots s_2 s_1}(z|\eta) = [x_1|\eta]^{n-2}.$$

Therefore, (5.11) holds.

We will apply the quantization map $\hat{Q}$ of Lenart–Maeno (see [23, Section 3]) to (5.11) with respect to the variables $x_1, \dots, x_n$, where $x_i = 1 - z_i$. Let $f_j^{(i)} = e_j(1-x_1, \dots, 1-x_i)$. Then $\hat{Q}$ is a linear map such that

$$\hat{Q}\big(f_{p_1}^{(1)} f_{p_2}^{(2)} \cdots f_{p_{n-2}}^{(n-2)} f_{p_{n-1}}^{(n-1)}\big) = F_{p_1}^{(1)} F_{p_2}^{(2)} \cdots F_{p_{n-2}}^{(n-2)} F_{p_{n-1}}^{(n-1)}$$

for $0 \le p_i \le i$ (see [23, Proposition 3.16]). As a polynomial in x_1, $[x_1|\eta]^{n-2}$ has degree $n-2$. Hence it is easy to see that $\mathfrak{G}_{s_{n-2}\dots s_2 s_1}(z|\eta) = [x_1|\eta]^{n-2}$ is a linear combination of $f_{p_1}^{(1)} f_{p_2}^{(2)} \cdots f_{p_{n-2}}^{(n-2)}$ with $0 \le p_i \le i$. Also,

$$\mathfrak{G}_{s_1 s_2 \dots s_{n-1}}(z|\eta) = [x_1|\eta][x_2|\eta]\cdots[x_{n-1}|\eta] = 1 + \sum_{j=1}^{n-1} (\eta_1 - 1)^j f_j^{(n-1)}. \tag{5.12}$$

Therefore, $\hat{Q}$ preserves the product of (5.11), so we have (5.10). ■

By the proof of the previous proposition, we have the following.

Corollary 5.7. *We have*

$$\mathfrak{G}^Q_{s_1 s_2 \dots s_{n-1}}(z|\eta) = \sum_{j=0}^{n-1} (-1)^j (1-\eta_1)^j F_j^{(n-1)}.$$

Proof. This identity is obtained by applying $\hat{Q}$ to (5.12). ■

5.4 Affine K-nil-Hecke action on the equivariant quantum K-ring

Here we explain an action of the 0-th Demazure operator on $QK_T(\mathrm{SL}_n(\mathbb{C})/B)_Q$. We define an operator D_0^Q on $QK_T(\mathrm{SL}_n(\mathbb{C})/B)_Q$ by

$$D_0^Q = T_\theta^Q + Q^{-\theta^\vee} \mathcal{O}^{s_\theta} \cdot s_\theta, \tag{5.13}$$

where an element T_θ^Q is defined by

$$T_\theta^Q := \big(\omega^{(\eta)}\big)^{-i} \circ T_i^Q \circ \big(\omega^{(\eta)}\big)^i = \frac{s_\theta^{(\eta)} - 1}{\eta_1 \ominus \eta_n} = \frac{s_\theta^{(\eta)} - 1}{1 - \mathrm{e}^{-\theta}}.$$

If we present $QK_T(\mathrm{SL}_n(\mathbb{C})/B)$ as a quotient ring by [26], we can replace $Q^{-\theta^\vee}\mathcal{O}^{s_\theta}$, acting as a multiplication operator, with $Q^{-\theta^\vee}\mathfrak{G}^Q_{s_\theta}$.

Together with D_i^Q, $i \in \tilde{I}$, and the left multiplication by $R(T)$, $QK_T(\mathrm{SL}_n(\mathbb{C})/B)_Q$ has a structure of K-nil-DAHA-module (see [14, 17, 28]).

Proposition 5.8. *Let $x = wt_\xi \in \hat{W}_G^0$. Then, with the presentation* [26]*, we have*

$$D_x^Q(1) = Q^\xi \mathfrak{G}_w^Q \mod \mathcal{J}_n^Q.$$

See Appendix C.

5.5 Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1

Remark 5.9. The image of $\tilde{\Phi}_n$ is $\hat{\Lambda}_{(n)}^{R(T)}\big[\sigma_i^{-1}, (\sigma(\sigma_i))^{-1} | 1 \le i \le n\big]$.

The following result is an important special case of Theorem 1.1. This proposition in turn implies the commutativity of Φ_n with the Demazure operators (see Corollary 5.15).

Proposition 5.10. *We have*

$$\tilde{\Phi}_n\big(Q^{-\theta^\vee} \mathfrak{G}_{s_\theta}^Q\big) = \tilde{g}_{s_0}^{(k)}(y|b).$$

Remark 5.11. The proof of this result is purely combinatorial. We do not have to resort to the deep geometric fact of Corollary C.3.

We first compute the image of the factor $\mathfrak{G}_{s_1 s_2 \dots s_{n-1}}^Q$ of $\mathfrak{G}_{s_\theta}^Q$ in the factorized form (5.10) under Φ_n.

Lemma 5.12. *We have*

$$(\Phi_n \circ \iota)\big(\mathfrak{G}_{s_1 s_2 \dots s_{n-1}}^Q\big) = \frac{\mathrm{e}^{-a_1} z_{11}}{\tau_1}, \tag{5.14}$$

$$\Phi_n\big(\mathfrak{G}_{s_1 s_2 \dots s_{n-1}}^Q\big) = \frac{\tau_n}{\tau_{n-1}} \frac{\mathrm{e}^{a_n}}{z_{nn}}. \tag{5.15}$$

Proof. We will show (5.14). Then (5.15) is obtained from this by Proposition 4.9. Since $\iota(1-\eta_1) = \mathrm{e}^{-a_1}$, we have by Corollary 5.7

$$\iota\big(\mathfrak{G}_{s_1 s_2 \dots s_{n-1}}^Q\big) = \sum_{j=0}^{n-1} (-1)^j \mathrm{e}^{-j a_1} \iota\big(F_j^{(n-1)}\big).$$

By Lemma 4.11, we have $(\Phi_n \circ \iota)\big(F_j^{(n-1)}\big) = (\iota \circ \Phi_n)\big(F_j^{(n-1)}\big) = (-1)^j c_j^{(1)}/\tau_1$. Note that we have $\tau_1 = c_0^{(1)}$ from (4.9). Therefore, we have

$$(\Phi_n \circ \iota)\big(\mathfrak{G}_{s_1 s_2 \dots s_{n-1}}^Q\big) = \frac{1}{\tau_1} \sum_{j=0}^{n-1} \mathrm{e}^{-j a_1} c_j^{(1)} = \frac{\mathrm{e}^{-a_1} z_{11}}{\tau_1} \qquad \text{by (4.8)}. \qquad \blacksquare$$

In order to compute the image of $\mathfrak{G}_{s_{n-2} \dots s_2 s_1}^Q\big(z|\omega^{(\eta)}\eta\big)$ under Φ_n, we prepare the following lemma.

Proposition 5.13. *We have*

$$\Phi_n\big(\mathfrak{G}_{s_{n-2} \dots s_2 s_1}^Q(z|\eta)\big) = \frac{\mathrm{e}^{-a_1 - a_2} z_{12}}{\tau_1}.$$

Proof. By (5.5) and Proposition 5.5, we have

$$\mathfrak{G}_{s_{n-2} \dots s_2 s_1}^Q(z|\eta) = \iota\big(D_1^Q \mathfrak{G}_{s_1 s_2 \dots s_{n-1}}^Q(z|\eta)\big). \tag{5.16}$$

Applying Φ_n to both sides of (5.16), we have

$$\begin{aligned}\Phi_n\big(\mathfrak{G}^Q_{s_{n-1}\dots s_2 s_1}(z|\eta)\big) &= (\Phi_n\circ\iota)\big(D^Q_1\mathfrak{G}^Q_{s_1 s_2\dots s_{n-1}}(z|\eta)\big)\\ &= D_1\big((\Phi_n\circ\iota)\big(\mathfrak{G}^Q_{s_1 s_2\dots s_{n-1}}(z|\eta)\big)\big) \qquad \text{by (5.8) and (5.7)}\\ &= D_1\left(\frac{\mathrm{e}^{-a_1}z_{11}}{\tau_1}\right) \qquad \text{by (5.14)}\\ &= \frac{1}{\tau_1}D_1(\mathrm{e}^{-a_1}z_{11}) \qquad \text{since } \tau_1 \text{ is } S_n\text{-invariant.}\end{aligned}$$

Finally, it is easy to show $D_1(\mathrm{e}^{-a_1}z_{11}) = \mathrm{e}^{-a_1-a_2}z_{12}$ by using (2.6). ■

Proof of Proposition 5.10. Note that $\omega^{(\eta)}$ corresponds to ω^{-1} under the identification (5.6). By using Proposition 5.6, (5.15), and Proposition 5.13, we have

$$\begin{aligned}\Phi_n\big(\mathfrak{G}^Q_{s_\theta}(z|\eta)\big) &= \Phi_n\big(\mathfrak{G}^Q_{s_1 s_2\dots s_{n-1}}(z|\eta)\cdot\mathfrak{G}^Q_{s_{n-2}\dots s_2 s_1}\big(x|\omega^{(\eta)}(\eta)\big)\big)\\ &= \frac{\tau_n}{\tau_{n-1}}\frac{\mathrm{e}^{a_n}}{z_{nn}}\cdot\omega^{-1}\left(\frac{\mathrm{e}^{-a_1-a_2}z_{12}}{\tau_1}\right)\\ &= \frac{\tau_n}{\tau_{n-1}}\frac{\mathrm{e}^{a_n}}{z_{nn}}\cdot\frac{\mathrm{e}^{-a_n-a_1}z_{n,n+1}}{\tau_1} \qquad \text{since } \omega(z_{ij}) = z_{i+1,j+1}\\ &= \frac{\tau_n}{\tau_{n-1}\tau_1}\frac{\mathrm{e}^{-a_1}z_{n,n+1}}{z_{nn}}.\end{aligned}$$

From (3.5), we have $z_{n,n+1} = \omega^{-1}(z_{12}) = \mathrm{e}^{a_n}z_{nn}g^{(k)}_{s_0}(y|b)$. Thus, together with (4.5), we have

$$\frac{\mathrm{e}^{-a_1}z_{n,n+1}}{z_{nn}} = \mathrm{e}^{-a_1+a_n}g^{(k)}_{s_0}(y|b) = \sigma^{-1}\big(\tilde{g}^{(k)}_{s_0}(y|b)\big).$$

In view of $\tilde{\Phi}_n\big(Q^{\theta^\vee}\big) = \frac{\tau_n}{\tau_{n-1}\tau_1}$, the proof is complete. ■

Now we can prove a crucial property of $\tilde{\Phi}_n$.

Proposition 5.14. *We have* $\tilde{\Phi}_n\circ D^Q_0 = D_0\circ\tilde{\Phi}_n$.

Proof. Since $s^*_\theta = s_\theta$, we have $\tilde{\Phi}_n\circ T^Q_\theta = T_\theta\circ\tilde{\Phi}_n$ and $\tilde{\Phi}_n\circ s_\theta = s_\theta\circ\tilde{\Phi}_n$. From Proposition 5.10, we have

$$\begin{aligned}\tilde{\Phi}_n\circ D^Q_0 &= \tilde{\Phi}_n\circ\big(T^Q_\theta + Q^{-\theta^\vee}\mathfrak{G}^Q_{s_\theta}s_\theta\big) = \tilde{\Phi}_n\circ T^Q_\theta + \tilde{\Phi}_n\circ\big(Q^{-\theta^\vee}\mathfrak{G}^Q_{s_\theta}(x,y)s_\theta\big)\\ &= T_\theta\circ\tilde{\Phi}_n + \tilde{\Phi}_n\big(Q^{-\theta^\vee}\mathfrak{G}^Q_{s_\theta}(x,y)\big)\tilde{\Phi}_n\circ s_\theta = T_\theta\circ\tilde{\Phi}_n + \tilde{g}^{(k)}_{s_0}(y|b)\cdot\tilde{\Phi}_n\circ s_\theta\\ &= T_\theta\circ\tilde{\Phi}_n + D_0(1)\cdot s_\theta\circ\tilde{\Phi}_n = D_0\circ\tilde{\Phi}_n,\end{aligned}$$

where the last equality follows from the simply identity $D_0 = T_\theta + D_0(1)s_\theta$. ■

Corollary 5.15. *Let* $x\in\hat{W}^0_G$. *Then* $\tilde{\Phi}_n\circ D^Q_x = D_{x^{\omega_k}}\circ\tilde{\Phi}_n$.

Proof. This is an immediate consequence of Proposition 5.14 and (5.7). ■

Proof of Theorem 1.1.

$$\begin{aligned}\tilde{\Phi}_n\big(Q^\xi\mathfrak{G}^Q_w\big) &= \tilde{\Phi}_n\big(D^Q_x(1)\big) \qquad \text{by Proposition 5.8}\\ &= D_{x^{\omega_k}}\big(\tilde{\Phi}_n(1)\big) \qquad \text{by Corollary 5.15}\\ &= D_{x^{\omega_k}}(1) = \tilde{g}^{(k)}_{x^{\omega_k}}(y|b).\end{aligned}$$

■

6 Determinantal formulas for the k-small $\tilde{g}^{(k)}_{\lambda}(y|b)$

6.1 S_n-and T_i-actions on $\mathcal{O}(\mathcal{Z})$

Let s_i $(1 \le i \le n-1)$ be the generators of the symmetric group S_n, which acts on $\mathcal{O}\big(\tilde{\mathcal{Z}}\big)^{\Delta}$ as a $\mathbb{C}$-algebra morphism by

$$s_i\big(\mathrm{e}^{\pm a_j}\big) = \mathrm{e}^{\pm a_{s_i(j)}}, \qquad s_i(Z) = \mathfrak{s}_i Z \mathfrak{s}_i^{-1}, \qquad \text{where} \quad \mathfrak{s}_i = E - \big(\mathrm{e}^{-a_i} - \mathrm{e}^{-a_{i+1}}\big) E_{i+1,i}.$$

The S_n-action preserves the subalgebra $\mathcal{O}(\mathcal{Z})$. The isomorphism $\beta\colon \mathcal{O}(\mathcal{Z}) \to \hat{\Lambda}^{R(T)}_{(n)}$ given in Theorem 3.1 satisfies $s_i \circ \beta = \beta \circ s_i$ for all i because $s_i(Z)$ is contained in the centralizer of $s_i(A) := A|_{a_i \leftrightarrow a_{i+1}}$.

On the coordinate functions z_{ij}, S_n acts as

$$s_i(z_{jj}) = z_{s_i(j), s_i(j)}, \tag{6.1}$$

$$s_i(z_{ki}) = z_{ki} + \big(\mathrm{e}^{-a_i} - \mathrm{e}^{-a_{i+1}}\big) z_{k,i+1}, \qquad k < i, \tag{6.2}$$

$$s_i(z_{i+1,j}) = z_{i+1,j} - \big(\mathrm{e}^{-a_i} - \mathrm{e}^{-a_{i+1}}\big) z_{ij}, \qquad j > i, \tag{6.3}$$

$$s_i(z_{kj}) = z_{kj}, \qquad k \ne i+1 \quad \text{or} \quad j \ne i. \tag{6.4}$$

It should be noted that (6.1) is a consequence of (6.2), and z_{ij} is defined by (2.6).

Proposition 6.1. *The action variables $c_0, c_1, \dots, c_{n-1}$ are S_n-invariant.*

Proof. From (4.6) and the definition of s_i, we have

$$\begin{aligned} s_i(Z) = \mathfrak{s}_i Z \mathfrak{s}_i^{-1} &= \mathfrak{s}_i \left(\sum_{j=0}^{n-1} c_j A^j \right) \mathfrak{s}_i^{-1} \\ &= \sum_{j=0}^{n-1} c_j \mathfrak{s}_i A^j \mathfrak{s}_i^{-1} = \sum_{j=0}^{n-1} c_j \big(\mathfrak{s}_i A \mathfrak{s}_i^{-1}\big)^j = \sum_{j=0}^{n-1} c_j s_i(A)^j. \end{aligned}$$

Since $s_i(A)$ is also regular, this equation uniquely determines c_j's. On the other hand, because s_i is a $\mathbb{C}$-algebra homomorphism we have $s_i(Z) = \sum_{j=0}^{n-1} s_i(c_j) s_i(A)^j$. Hence we have $s_i(c_j) = c_j$. ■

Theorem 4.6 implies the following important consequence.

Corollary 6.2. *τ_i and σ_i are S_n-invariant.*

Remark 6.3. We have

$$P^{-1} Z P = \sum_{j=0}^{n-1} c_j C_A^j.$$

Since the entries of C_A are S_n-invariant, the entries of $P^{-1}ZP$ are S_n-invariant by Proposition 6.1.

Proposition 6.4. *For $1 \le i \le n-1$,*

$$T_i(z_{ki}) = \mathrm{e}^{-a_i} z_{k,i+1}, \tag{6.5}$$

$$T_i(z_{i+1,j}) = -\mathrm{e}^{-a_i} z_{ij}, \tag{6.6}$$

$$T_i(z_{kj}) = 0, \qquad k \ne i+1 \quad \textit{or} \quad j \ne i. \tag{6.7}$$

Proof. Equations (6.5), (6.6), and (6.7) are immediate consequences of (6.2), (6.3), and (6.4), respectively. ■

For convenience, we extend the definition of z_{ij} to any $1 \le i \le j$ by letting $z_{ii} = z_{i+n,i+n} = z_{i+2n,i+2n} = \cdots$ and

$$z_{ij} = -\frac{z_{i,j-1} - z_{i+1,j}}{\mathrm{e}^{-a_{i(\operatorname{mod} n)}} - \mathrm{e}^{-a_{j(\operatorname{mod} n)}}}.$$

By definition, z_{ij} is an element of $\mathcal{O}\big(\tilde{\mathcal{Z}}\big)^{\Delta}$. Let $\omega := s_1 s_2 \dots s_{n-1}$. It is shown by induction on $j - i \ge 0$ that

$$\omega(z_{ij}) = z_{i+1,j+1}, \tag{6.8}$$

which implies $z_{ij} \in \mathcal{O}(\mathcal{Z})$. In particular, we have $z_{ij} = z_{i+n,j+n}$.

Remark 6.5. T_θ can be seen as the n-th divided difference operator. Proposition 6.4 is naturally extended to the case for $i = n$ as $T_\theta(z_{k,n}) = \mathrm{e}^{-a_n} z_{k,n+1}$ and $T_\theta(z_{n+1,j}) = -\mathrm{e}^{-a_n} z_{nj}$. Moreover, the expressions (6.1)–(6.6) are also valid for arbitrarily $i, k, j \in \mathbb{Z}$ under the identification $T_{i+n} = T_i$ and $T_n = T_\theta$.

6.2 Determinantal formula for $\tilde{g}^{(k)}_{\lambda}$

A k-bounded partition λ is *k-small* if λ is contained in at least one of $R_1, R_2, \dots, R_{n-1}$. This is equivalent to $\ell(\lambda) + \lambda_1 \le n$.

6.2.1 Notation

We identify the set $\tilde{I} = \{0, 1, \dots, n-1\}$ of the type $A^{(1)}_{n-1}$ affine Dynkin nodes with $\mathbb{Z}/n\mathbb{Z}$. The *n-residue* is the map $\operatorname{res}\colon \mathbb{N} \times \mathbb{N} \to \tilde{I} \cong \mathbb{Z}/n\mathbb{Z}$, $(i,j) \to j - i \mod n$. For $x = (i,j) \in \lambda$, let $\mathfrak{d}(x) := \{\mathrm{e}^{-a_{\operatorname{res}(s,j)}} \mid i \le s \le \lambda'_j\}$.

For any subset X of $\{\mathrm{e}^{-a_1}, \dots, \mathrm{e}^{-a_n}\}$ we denote by $h_m(X)$ the m-th complete symmetric polynomial in X. We use the abbreviation $f_m^{\langle i,j\rangle;\lambda} := h_m(\mathfrak{d}(i,j))$, for $(i,j) \in \lambda$. For k-small partition λ, define

$$\xi_\lambda(y) = \prod_{i=1}^{\lambda'_1} \Omega(b_{\operatorname{res}(i,1)}|y). \tag{6.9}$$

Example 6.6. When $n = 6$ and $\lambda = (3,3,1)$. If we fill the boxes of λ with the n-residue, we have

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 5 & 0 & 1 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}.$$

For example, $\mathfrak{d}(2,1) = \big\{\mathrm{e}^{-a_5}, \mathrm{e}^{-a_4}\big\}$ so $f_m^{\langle 2,1\rangle;\lambda} = h_m(\mathrm{e}^{-a_5}, \mathrm{e}^{-a_4})$. We also have $\xi_\lambda(y) = \Omega(b_6|y) \times \Omega(b_5|y)\Omega(b_4|y)$.

6.2.2 Determinantal formula for $\tilde{g}_\lambda(y|b)$ for k-small λ

Theorem 6.7. *Let λ be a k-small partition. Set $l = n - \lambda'_1 + 1$ and $r = \lambda_1$. Define the following square matrix of size $n - l + 1 + r$*

$$M_\lambda = \left(\begin{array}{cccc|cccc}
z_{ll} & \cdots & \cdots & z_{ln} & z_{l,n+1} & \cdots & \cdots & z_{l,n+r} \\
 & \ddots & & \vdots & \vdots & \cdots & \cdots & \vdots \\
 & & \ddots & \vdots & \vdots & \cdots & \cdots & \vdots \\
 & & & z_{nn} & z_{n,n+1} & \cdots & \cdots & z_{n,n+r} \\ \hline
 & \cdots & f_2^{\langle 2,1\rangle;\lambda} & -f_1^{\langle 1,1\rangle;\lambda} & 1 & & & \\
 & & \cdots & f_2^{\langle 2,2\rangle;\lambda} & -f_1^{\langle 1,2\rangle;\lambda} & \ddots & & \\
 & & \cdots & \vdots & \vdots & \ddots & \ddots & \\
 & & & \pm f_r^{\langle r,r\rangle;\lambda} & \mp f_{r-1}^{\langle r-1,r\rangle;\lambda} & \cdots & -f_1^{\langle 1,r\rangle;\lambda} & 1
\end{array}\right). \tag{6.10}$$

Then we have

$$\tilde{g}_\lambda^{(k)}(y|b) = \frac{\det(M_\lambda)}{\xi_\lambda(y)}. \tag{6.11}$$

Remark 6.8.

(1) l is the n-residue of the bottom box of the first column of λ.

(2) We have $\lambda_1 < l$ because λ is k-small. If we take r such that $\lambda_1 \le r < l$ and consider the matrix M_λ by the same formula (6.10), then (6.11) also holds.

Example 6.9. Let $n = 6$ and λ be as in Example 6.6. We have $l = 4$. The formula for $\tilde{g}_\lambda^{(5)}(y|b)$ by taking $r = 3$ is

$$\frac{1}{\Omega(b_4|y)\Omega(b_5|y)\Omega(b_6|y)}\begin{vmatrix}
z_{44} & z_{45} & z_{46} & z_{47} & z_{48} & z_{49} \\
0 & z_{55} & z_{56} & z_{57} & z_{58} & z_{59} \\
0 & 0 & z_{66} & z_{67} & z_{68} & z_{69} \\
-f_3^{\langle 3,1\rangle;\lambda} & f_2^{\langle 2,1\rangle;\lambda} & -f_1^{\langle 1,1\rangle;\lambda} & 1 & 0 & 0 \\
0 & 0 & f_2^{\langle 2,2\rangle;\lambda} & -f_1^{\langle 1,2\rangle;\lambda} & 1 & 0 \\
0 & 0 & 0 & f_2^{\langle 2,3\rangle;\lambda} & -f_1^{\langle 1,3\rangle;\lambda} & 1
\end{vmatrix}.$$

Remark 6.10. In the non-equivariant case $a_i = 0$, (6.11) reduces to

$$\tilde{g}_\lambda^{(k)}(y|0) = \begin{vmatrix}
h_0(y) & h_1(y) & h_2(y) & \cdots & \cdots & \cdots & \cdots & h_{n-1}(y) \\
 & h_0(y) & h_1(y) & h_2(y) & \cdots & \cdots & \cdots & h_{n-2}(y) \\
 & & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\
 & & & h_0(y) & h_1(y) & h_2(y) & \cdots & h_r(y) \\
\genfrac{[}{]}{0pt}{}{\lambda'_1}{\lambda'_1} & \cdots & \genfrac{[}{]}{0pt}{}{\lambda'_1}{2} & \genfrac{[}{]}{0pt}{}{\lambda'_1}{1} & \genfrac{[}{]}{0pt}{}{\lambda'_1}{0} & & & \\
 & \genfrac{[}{]}{0pt}{}{\lambda'_2}{\lambda'_2} & \cdots & \genfrac{[}{]}{0pt}{}{\lambda'_2}{2} & \genfrac{[}{]}{0pt}{}{\lambda'_2}{1} & \genfrac{[}{]}{0pt}{}{\lambda'_2}{0} & & \\
 & & \ddots & \cdots & \cdots & \ddots & \ddots & \\
 & & & & \genfrac{[}{]}{0pt}{}{\lambda'_r}{\lambda'_r} & \cdots & \genfrac{[}{]}{0pt}{}{\lambda'_r}{1} & \genfrac{[}{]}{0pt}{}{\lambda'_r}{0}
\end{vmatrix},$$

where $h_i = h_i(y)$ and $\genfrac{[}{]}{0pt}{}{a}{b} = (-1)^b\binom{a}{b}$.

The proof of Theorem 6.7 is divided into two cases: (i) the case when λ is a one-column partition and (ii) the case when λ is a general k-small partition.

6.2.3 Case: $\boldsymbol{\lambda} = (\mathbf{1}^{\boldsymbol{i}})$

Let λ is a one-column partition $\lambda = (1^i)$. We have

$$M_{(1^i)} = \left(\begin{array}{cccc|c} z_{n-i+1,n-i+1} & \cdots & \cdots & z_{n-i+1,n} & z_{n-i+1,n+1} \\ & \ddots & & \vdots & \vdots \\ & & \ddots & \vdots & \vdots \\ & & & z_{nn} & z_{n,n+1} \\ \hline (-1)^i f_i^{\langle i,1\rangle;(1^i)} & \cdots & f_2^{\langle 2,1\rangle;(1^i)} & -f_1^{\langle 1,1\rangle;(1^i)} & 1 \end{array}\right).$$

In this case, $\det(M_{(1^i)})$ given in (6.11) is expanded as

$$\det(M_{(1^i)}) = \sum_{m=0}^{i} f_m^{\langle m,1\rangle;(1^i)} d_m^{(i)}$$

by the expansion along the bottom row, where $d_m^{(i)} = \det Z_{[n-i+1,n]}^{[n-i+1,n+1]\setminus\{n-m+1\}}$. For $0 \le m \le i < n$, define

$$f_m^{(i)} = f_m^{\langle m,1\rangle;(1^i)} \cdot d_m^{(i)}. \tag{6.12}$$

Then (6.11) reads

$$\tilde{g}_{(1^i)}^{(k)}(y|b) = \frac{\sum_{m=0}^{i} \varphi_m^{(i)}}{\xi_{(1^i)}(y)}. \tag{6.13}$$

We consider the minor $d_m^{(i+1)}$ of size $i+1$. Note that we have

$$d_m^{(i+1)} = z_{n-i,n-i} d_m^{(i)} \tag{6.14}$$

since Z is upper triangular. It follows that

$$z_{n-i,n-i}\varphi_m^{(i)} = f_m^{\langle m,1\rangle;(1^i)} d_m^{(i+1)}. \tag{6.15}$$

Lemma 6.11. *For $1 \le i \le n-1$, we have $T_{n-i}\big(d_i^{(i+1)}\big) = \mathrm{e}^{-a_{n-i}} d_{i+1}^{(i+1)}$.*

Proof. From (6.6) and (6.7), we have

$$T_{n-i}\big(d_i^{(i)}\big) = T_{n-i}\left(\det Z_{[n-i+1,n]}^{[n-i+2,n+1]}\right) = -\mathrm{e}^{-a_{n-i}} \cdot \det Z_{[n-i,n]\setminus\{n-i+1\}}^{[n-i+2,n+1]} \tag{6.16}$$

because every entry of the rows of $Z_{[n-i+1,n]}^{[n-i+2,n+1]}$ except for the first row is invariant under s_{n-i}. Then by using this, we have

$$\begin{aligned} T_{n-i}\big(d_i^{(i+1)}\big) &= T_{n-i}\big(z_{n-i,n-i} d_i^{(i)}\big) \qquad \text{by (6.14)} \\ &= T_{n-i}(z_{n-i,n-i}) \cdot d_i^{(i)} + s_{n-i}(z_{n-i,n-i}) \cdot T_{n-i}\big(d_i^{(i)}\big) \\ &= \mathrm{e}^{-a_{n-i}} z_{n-i,n-i+1} \cdot d_i^{(i)} \\ &\quad - z_{n-i+1,n-i+1} \cdot \mathrm{e}^{-a_{n-i}} \det Z_{[n-i,n]\setminus\{n-i+1\}}^{[n-i+2,n+1]} \qquad \text{by (6.5) and (6.16)} \\ &= \mathrm{e}^{-a_{n-i}} d_{i+1}^{(i+1)}, \end{aligned}$$

where the last equality follows by the expansion along the 1st column of $d_{i+1}^{(i+1)}$. ■

Lemma 6.12. *For $1 \le i \le n-1$, and $0 \le m \le i$, we have*

$$D_{n-i}\big(z_{n-i,n-i}\varphi_m^{(i)}\big) = \begin{cases} \varphi_m^{(i+1)}, & m < i, \\ \varphi_i^{(i+1)} + \varphi_{i+1}^{(i+1)}, & m = i. \end{cases}$$

Proof. Suppose $m < i$. We claim that $d_m^{(i+1)}$ is invariant under s_{n-i}. In fact, we have

$$d_m^{(i+1)} = z_{n-i,n-i} z_{n-i+1,n-i+1} d_m^{(i-1)}$$

by (6.14), which is s_{n-1}-invariant by (6.1) and (6.7). We have

$$\begin{aligned} D_{n-i}\big(z_{n-i,n-i}\varphi_m^{(i)}\big) &= D_{n-i}\big(f_m^{\langle m,1\rangle;\left(1^i\right)} \cdot d_m^{(i+1)}\big) && \text{by (6.15)} \\ &= D_{n-i}\big(f_m^{\langle m,1\rangle;\left(1^i\right)}\big) \cdot d_m^{(i+1)} && \text{since } d_m^{(i+1)} \text{ is } s_{n-i}\text{-invariant} \\ &= f_m^{\langle m,1\rangle;(1^{i+1})} \cdot d_m^{(i+1)} && \text{by Lemma 3.9} \\ &= \varphi_m^{(i+1)}. \end{aligned}$$

When $m = i$, noting that $f_i^{\langle i,1\rangle;(1^i)} = \mathrm{e}^{-ia_{n-i+1}}$, we have

$$\begin{aligned} D_{n-i}\big(z_{n-i,n-i}\varphi_i^{(i)}\big) &= D_{n-i}\big(f_i^{\langle i,1\rangle;(1^i)} d_i^{(i+1)}\big) && \text{by (6.15)} \\ &= D_{n-i}\big(f_i^{\langle i,1\rangle;(1^i)}\big) \cdot d_i^{(i+1)} + \mathrm{e}^{-ia_{n-i}} \cdot T_{n-i}\big(d_i^{(i+1)}\big) \\ &= h_i(\mathrm{e}^{-a_{n-i}}, \mathrm{e}^{-a_{n-i+1}}) \cdot d_i^{(i+1)} \\ &\quad + \mathrm{e}^{-(i+1)a_{n-i}} d_{i+1}^{(i+1)} && \text{by Lemmas 3.9 and 6.11} \\ &= f_i^{\langle i,1\rangle;(1^{i+1})} \cdot d_i^{(i+1)} + f_{i+1}^{\langle i+1,1\rangle;(1^{i+1})} \cdot d_{i+1}^{(i+1)} \\ &= \varphi_i^{(i+1)} + \varphi_{i+1}^{(i+1)} && \text{by (6.12).} \end{aligned}$$

■

Proof of Theorem 6.7 for $\boldsymbol{\lambda = (1)^i}$. We show (6.13) by induction on $i \ge 1$. When $i = 1$, the desired equation is shown directly as follows:

$$\begin{aligned} \tilde{g}_{(1)}^{(k)}(y|b) = D_0(1) &= 1 + \Omega(b_1|y) T_\theta\left(\frac{1}{\Omega(b_1|y)}\right) = 1 + \Omega(b_1|y) T_\theta\left(\frac{z_{nn}}{\Omega(b_1|y)\Omega(b_n|y)}\right) \\ &= 1 + \frac{T_\theta(z_{nn})}{\Omega(b_n|y)} = \frac{z_{nn} + \mathrm{e}^{-a_n} z_{n,n+1}}{\Omega(b_n|y)} = \frac{\varphi_0^{(1)} + \varphi_1^{(1)}}{\Omega(b_n|y)} = \frac{\varphi_0^{(1)} + \varphi_1^{(1)}}{\xi_{(1)}(y)}. \end{aligned}$$

Suppose (6.13) holds for some $1 \le i \le n-2$. By induction hypothesis,

$$\begin{aligned} \tilde{g}_{(1^{i+1})}^{(k)}(y|b) = D_{n-i}\left(\frac{\sum_{m=0}^{i} \varphi_m^{(i)}}{\xi_{(1^i)}(y)}\right) &= D_{n-i}\left(\frac{z_{n-i,n-i}\sum_{m=0}^{i} \varphi_m^{(i)}}{\Omega(b_{n-i}|y)\xi_{(1^i)}(y)}\right) \\ &= \frac{\sum_{m=0}^{i} D_{n-i}\left(z_{n-i,n-i}\varphi_m^{(i)}\right)}{\Omega(b_{n-i}|y)\xi_{(1^i)}(y)} = \frac{\sum_{m=0}^{i+1} \varphi_m^{(i+1)}}{\xi_{(1^{i+1})}(y)} \qquad \text{by Lemma 6.12,} \end{aligned}$$

where in the third equality we use fact that $\Omega(b_{n-i}|y)\xi_{(1^i)}(y)$ is s_{n-i} invariant. ■

6.2.4 Case: general k-small λ

Before proving Theorem 6.7 for general k-small λ, we list a few important properties of M_λ.

Lemma 6.13. *Let λ be a k-small partition. Assume λ is i-addable. Let κ be a partition obtained from λ by adding a box of n-residue i. We also assume κ is k-small. Then $D_i(\det M_\lambda) = \det M_\kappa$ if $i \neq 0$, and $D_\theta(\det M_\lambda) = \det M_\kappa$.*

Proof. We may assume $\lambda \neq \varnothing$. The integers l, r are defined in Theorem 6.7. Let $\hat{x} := x - l + 1$ and $x^\dagger := x - l + 1 + n$ for $x \in \mathbb{Z}$. In M_λ, $z_{q,p}$ is at the $(\hat{q}, \hat{p})$-th position (if exists). For $1 \leq s \leq r$, the $\big(s^\dagger, \hat{p}\big)$-th entry of M_λ is either a constant or a complete symmetric polynomial in $\mathrm{e}^{-a_b}, \mathrm{e}^{-a_{b+1}}, \dots, \mathrm{e}^{-a_p}$ where b is the n-residue of the bottom box of the s-th column.

Let $M_\lambda^{(i)}$ be the matrix obtained from M_λ by applying the following column or row operations:

(a) Add e^{-a_i} times the $\big(\hat{i}+1\big)$-th column to the $\hat{i}$-th column.

(b) If $l \leq i < n$, subtract e^{-a_i} times the $\hat{i}$-th row from the $\big(\hat{i}+1\big)$-th row.

Since λ is k-small and λ is i-addable, there is a unique integer j such that the n-residue of the bottom box of the j-th column is $i+1$. In $M_\lambda^{(i)}$, all the entries in rows except for the $j^\dagger$-th one are s_i-invariant because:

- the $(\hat{q}, \hat{p})$-th entry is
$$\begin{aligned} &-\mathrm{e}^{a_i - a_{i+1}} z_{i,i+1} && \text{if } q = i+1, \quad p = i, \\ &z_{qi} + \mathrm{e}^{-a_i} z_{q,i+1} && \text{if } q \neq i+1, \quad p = i, \\ &z_{i+1,p} - \mathrm{e}^{-a_i} z_{ip} && \text{if } q = i+1, \quad p \neq i, \\ &z_{qp} \quad \text{if } q \neq i+1, \quad p \neq i, \end{aligned}$$
One can verify that these entries are s_i-invariant from (6.1)–(6.4).
- the $\big(s^\dagger, \hat{i}\big)$-th entry $(s \neq j)$ is either a constant or a polynomial of the form
$$h_t(\mathrm{e}^{-a_b}, \dots, \mathrm{e}^{-a_i}) - \mathrm{e}^{-a_i} h_{t-1}(\mathrm{e}^{-a_b}, \dots, \mathrm{e}^{-a_i}, \mathrm{e}^{-a_{i+1}}),$$
up to a sign, where b is the n-residue of the bottom box of the s-th column, and $t > 0$. It is not difficult to show the polynomial above is s_i-invariant.
- the $\big(s^\dagger, \hat{p}\big)$-th entry $(s \neq j, \ p \neq i)$ is either a constant or a symmetric polynomial in $\mathrm{e}^{-a_b}, \mathrm{e}^{-a_{b+1}}, \dots, \mathrm{e}^{-a_p}$ with b the n-residue of the bottom box of the s-th column. Note that we have $b \neq i+1$. Clearly, these are s_i-invariant.

Therefore, it suffices to compute the D_i-actions on the $j^\dagger$-th row of $M_\lambda^{(i)}$ only. Set $m = \lambda'_j$. Then $m = j - i + n - 1$, and the $\big(j^\dagger, \hat{p}\big)$-th entry of $M_\lambda^{(i)}$ is
$$\begin{aligned} &(-1)^{m-p+i+1} h_{m-p+i+1}\big(\mathrm{e}^{-a_{i+1}}, \dots, \mathrm{e}^{-a_p}\big) && \text{if } \hat{p} < \hat{i}, \\ &(-1)^{m+1} \mathrm{e}^{-a_i} h_m\big(\mathrm{e}^{-a_{i+1}}\big) && \text{if } \hat{p} = \hat{i}, \\ &0 \quad \text{if } \hat{p} > \hat{i} \end{aligned}$$
and the $\big(j^\dagger, \hat{p}\big)$-th entry of $M_\kappa^{(i)}$ is
$$\begin{aligned} &(-1)^{m-p+i+1} h_{m-p+i+1}(\mathrm{e}^{-a_i}, \mathrm{e}^{-a_{i+1}}, \dots, \mathrm{e}^{-a_p}) && \text{if } \hat{p} < \hat{i}, \\ &(-1)^{m+1} (h_{m+1}(\mathrm{e}^{-a_i}) - \mathrm{e}^{-a_i} h_m(\mathrm{e}^{-a_i}, \mathrm{e}^{-a_{i+1}})) && \text{if } \hat{p} = \hat{i}, \\ &0 \quad \text{if } \hat{p} > \hat{i}. \end{aligned}$$
By Lemma 3.9, the image of the $j^\dagger$-th row of $M_\lambda^{(i)}$ under D_i coincides with the $j^\dagger$-th row of $M_\kappa^{(i)}$. ∎

Proof of Theorem 6.7. Let w be the element of $\hat{W}_G^0$ associated with λ. Since λ is k-small, there exists a reduced expression $w = vs_l s_{l+1} \dots s_{n-1}s_0$ for some $v \in \hat{W}_G$, where l is the n-residue at the bottom of the first column of λ.

We show the theorem by induction on $\ell(v)$. The case $\ell(v) = 0$ is done, so suppose $\ell(v) > 0$. In this case, λ contains at least one removable box not included in the first column. Let i be the n-residue of the box. Let μ the partition obtained from λ by removing the box of n-residue i.

(i) If $i \neq 0$, by induction hypothesis, we have

$$\tilde{g}_\lambda^{(k)}(y|b) = D_i\big(\tilde{g}_\mu^{(k)}(y|b)\big) = D_i\left(\frac{\det(M_\mu)}{\xi_\lambda(y)}\right).$$

Since λ is k-small, we easily show that $\xi_\lambda(y)$ is invariant under s_i. Therefore, from Lemma 6.13, we have

$$D_i\left(\frac{\det M_\mu}{\xi_\lambda(y)}\right) = \frac{D_i(\det M_\mu)}{\xi_\lambda(y)} = \frac{\det M_\lambda}{\xi_\lambda(y)}.$$

(ii) For $i = 0$, because $\Omega(b_1|y)\xi_\lambda(y)$ is s_θ-invariant, we have

$$\begin{aligned}\tilde{g}_\lambda^{(k)}(y|b) &= \Omega(b_1|y) D_\theta\left(\frac{\det M_\mu}{\Omega(b_1|y)\xi_\lambda(y)}\right)\\ &= \Omega(b_1|y)\frac{D_\theta(\det M_\mu)}{\Omega(b_1|y)\xi_\lambda(y)} = \frac{\det M_\lambda}{\xi_\lambda(y)}\end{aligned}$$

from Lemma 6.13. ■

Corollary 6.14. *Suppose $i + j \leq n$. Then we have*

$$\tilde{g}_{(i^j)}^{(k)}(y|b) = \frac{\det\big(M'_{(i^j)}\big)}{\xi_{(i^j)}(y)}, \qquad M'_{(i^j)} = \omega^{-j}\left(\frac{Z_{[1,j]}^{[1,i+j]}}{\big(P^{-1}\big)_{[j+1,i+j]}^{[1,i+j]}}\right).$$

Proof. Note also that the size of the square matrix $M_{(i^j)}$ is $i+j$. Let W be the matrix consisting of the last i rows of $M_{(i^j)}$. Then we can write

$$M_{(i^j)} = \left(\frac{\omega^{-j} Z_{[1,j]}^{[1,i+j]}}{W}\right). \tag{6.17}$$

There is a lower unitriangular $i \times i$ matrix N such that

$$N\omega^j(W) = \big(P^{-1}\big)_{[j+1,i+j]}^{[1,i+j]}. \tag{6.18}$$

In fact, if we define $N := N_1 N_2 \cdots N_{i-1}$, where $N_m = 1 - \sum_{j=1}^m \mathrm{e}^{-a_j} E_{r+j-m,r+j-m-1}$, then by using the explicit formula for P^{-1}, it is straightforward to check (6.18).

Combined with (6.17) and (6.18), we have

$$\begin{pmatrix} E_j & O \\ O & N \end{pmatrix} \omega^j M_{(i^j)} = \left(\frac{Z_{[1,j]}^{[1,i+j]}}{N\omega^j W}\right) = \left(\frac{Z_{[1,j]}^{[1,i+j]}}{\big(P^{-1}\big)_{[j+1,i+j]}^{[1,i+j]}}\right) = \omega^j M'_{(i^j)}.$$

Therefore, by taking determinants we obtain the desired result. ■

Corollary 6.15. *We have*

$$\tilde{g}_{R_i}^{(k)}(y|b) = \frac{\sigma_{n-i}}{\xi_{R_i}(y)}, \tag{6.19}$$

$$g_{R_i}^{(k)}(y|b) = \mathrm{e}^{\sum_{s=0}^{n-i-1} a_{i-s}} \frac{\tau_{n-i}}{\xi_{R_i}(y)}. \tag{6.20}$$

Proof. Let M'_{R_i} be the $n \times n$ matrix in Corollary 6.14. We have

$$\omega^i M'_{R_i} = \left(\frac{Z^{[1,n]}_{[1,n-i]}}{(P^{-1})^{[1,n]}_{[n-i+1,n]}} \right).$$

So we have

$$\left(\omega^i M'_{R_i}\right) P = \left(\frac{(ZP)^{[1,n]}_{[1,n-i]}}{O_{n-i}|E_i} \right).$$

By taking determinants, we have

$$\omega^i \det(M'_{R_i}) = \det\left(\omega^i M'_{R_i}\right) = \det\left((ZP)^{[1,n-i]}_{[1,n-i]} \right) = \sigma_{n-i}.$$

Since σ_{n-i} is S_n-invariant (see Corollary 6.2), we obtain (6.19). It is straightforward to obtain (6.20) from (6.19) by using (4.4) and Proposition 4.1. ■

6.3 Determinantal formula for $g^{(k)}_\lambda$

The determinantal formula for the K-k-Schur function $g^{(k)}_\lambda(y|b)$ is also obtained for a k-small λ.

Theorem 6.16. *Let λ be a k-small partition. Let l, r be as in Theorem 6.7. Let W_λ be the matrix consisting of the last i rows of M_λ. Then*

$$g^{(k)}_\lambda(y|b) = \mathrm{e}^{\sum_{x \in \mathrm{diag}(\lambda)}(a_{r(x)+1} - a_{b(x)})} \frac{\det(N_\lambda)}{\xi_\lambda(y)}, \qquad N_\lambda = \left(\frac{(ZA)^{[l,n+r]}_{[l,n]}}{W_\lambda} \right). \tag{6.21}$$

Proof. Since $\sigma^{-1}(Z) = ZA$ and σ is $R(T)$-linear, we have $\sigma^{-1}(M_\lambda) = N_\lambda$. In view of this, (6.21) is obtained from (6.11) by applying (4.4) and Proposition 4.1. ■

Example 6.17. For λ in Example 6.9, $g^{(5)}_\lambda(y|b)$ is

$$\frac{\mathrm{e}^{a_2+a_3+a_5}}{\Omega(b_4|y)\Omega(b_5|y)\Omega(b_6|y)} \begin{vmatrix} z'_{44} & z'_{45} & z'_{46} & z'_{47} & z'_{48} & z'_{49} \\ 0 & z'_{55} & z'_{56} & z'_{57} & z'_{58} & z'_{59} \\ 0 & 0 & z'_{66} & z'_{67} & z'_{68} & z'_{69} \\ -f^{\langle 3,1\rangle;\lambda}_3 & f^{\langle 2,1\rangle;\lambda}_2 & -f^{\langle 1,1\rangle;\lambda}_1 & 1 & 0 & 0 \\ 0 & 0 & f^{\langle 2,2\rangle;\lambda}_2 & -f^{\langle 1,2\rangle;\lambda}_1 & 1 & 0 \end{vmatrix},$$

where $z'_{ij} = \mathrm{e}^{-a_i} z_{ij} - z_{i,j-1}$. Note that z'_{ij} is specialized to $h_{j-i}(y) - h_{j-i-1}(y)$ at $a_i = 0$.

7 k-rectangle factorization property

Define for $1 \le j \le n$

$$\varrho_j := \Omega(b_{n-j+1}|y)^{-1} \cdots \Omega(b_n|y)^{-1}. \tag{7.1}$$

Note that for $1 \le i \le n-1$, we have

$$\varrho_{n-i} = \xi_{R_i}(y)^{-1}. \tag{7.2}$$

Lemma 7.1. *Let $1 \leq j \leq n$. We consider ϱ_j a linear transformation of $\hat{\Lambda}^{R(T)}$ by multiplication. For any $i \in \tilde{I}$, we have*

$$D_i \circ \varrho_j = \varrho_j \circ \left(\omega^{-j} D_{i+j} \omega^j\right), \tag{7.3}$$

where the subscripts are taken modulo n.

Proof. Let us consider the case $i \neq 0$ and $i + j = 0 \mod n$. We have

$$\begin{aligned} D_i \circ \varrho_j &= D_i \circ \Omega(b_{i+1}|y)^{-1} \dots \Omega(b_n|y)^{-1} \qquad \text{by assumption } i + j \equiv 0 \mod n \\ &= D_i \circ \Omega(b_i|y)\Omega(b_i|y)^{-1}\Omega(b_{i+1}|y)^{-1}\varrho_{j-1} \\ &= \Omega(b_i|y)^{-1}\Omega(b_{i+1}|y)^{-1}\varrho_{j-1} \cdot D_i \circ \Omega(b_i|y) = \varrho_j \circ \Omega(b_i|y)^{-1} \circ D_i \circ \Omega(b_i|y). \end{aligned}$$

In the third equality, we used that $\Omega(b_i|y)^{-1}\Omega(b_{i+1}|y)^{-1}\varrho_{j-1}$ is s_i-invariant. It is straightforward to show $\Omega(b_i|y)^{-1} \circ D_i \circ \Omega(b_i|y) = \omega^i \circ D_0 \circ \omega^{-i}$. Hence we have (7.3) in this case. The other cases are left to the reader. ■

Theorem 7.2. *Let λ be an arbitrary k-bounded partition. Then we have*

$$\tilde{g}^{(k)}_{\lambda \cup R_i}(y|b) = \tilde{g}^{(k)}_{\lambda}\left(y|\omega^i b\right) \cdot \tilde{g}^{(k)}_{R_i}(y|b).$$

Proof. Let $w_\lambda = s_{j_1} s_{j_2} \cdots s_{j_l}$ be a reduced expression. Let $w'_\lambda = s_{j_1+i} s_{j_2+i} \cdots s_{j_l+i}$. It is shown in [32, Lemma 2.15] that we have a length additive decomposition $w_{\lambda \cup R_i} = w'_\lambda w_{R_i}$.

Therefore, we have

$$\begin{aligned} \tilde{g}^{(k)}_{\lambda \cup R_i}(y|b) &= D_{w'_\lambda} D_{R_i}(1) = D_{w'_\lambda}\left(\tilde{g}^{(k)}_{R_i}(y|b)\right) \\ &= D_{w'_\lambda}\left(\frac{\sigma_{n-i}}{\xi_{R_i}(y)}\right) \qquad \text{by (6.19)} \\ &= \sigma_{n-i} \cdot D_{w'_\lambda} \xi_{R_i}(y)^{-1} \qquad \text{by Proposition 6.1} \\ &= \sigma_{n-i} \cdot D_{w'_\lambda} \varrho_{n-i} \qquad \text{by (7.2)} \\ &= \sigma_{n-i} \varrho_{n-i} \cdot \omega^i D_{w_\lambda} \omega^{-i}(1) \qquad \text{by Lemma 7.1} \\ &= \tilde{g}^{(k)}_{R_i}(u|b) \cdot \omega^i(D_{w_\lambda}(1)) = \tilde{g}^{(k)}_{R_i}(u|b) \cdot \tilde{g}^{(k)}_{\lambda}\left(u|\omega^i b\right). \end{aligned}$$

■

8 Proof of Theorem 1.2

This is obtained by computing the image of $\mathfrak{G}^Q_{w_\circ}$ under $\tilde{\Phi}_n$ in two different ways.

8.1 $\tilde{\Phi}_n(\psi_i)$

Recall that ψ_i $(1 \leq i \leq n-1)$ is defined by (5.3).

Proposition 8.1. *For $1 \leq i \leq n-1$,*

$$\tilde{\Phi}_n(\psi_i) = \frac{\prod_{l=1}^{i} \Omega(b_{i+l+1}|y)}{\sigma_i} \cdot \tilde{g}^{(k)}_{(n-i-1)^i}\left(y|\omega^{2i+1} b\right).$$

Note that for $i = n-1$, the partition appearing on the right-hand side is $\varnothing$. Since $\mathfrak{G}^Q_{w_\circ}$ is the product of ψ_i $(1 \leq i \leq n-1)$, we immediately obtain the following formula.

Corollary 8.2. *We have*

$$\tilde{\Phi}_n\left(\mathfrak{G}^Q_{w_\circ}\right) = \frac{\prod_{i=1}^{n-1} \prod_{l=1}^{i} \Omega(b_{i+l+1}|y)}{\sigma_1 \sigma_2 \cdots \sigma_{n-1}} \cdot \prod_{i=1}^{n-2} \tilde{g}^{(k)}_{(n-i-1)^i}\left(y|\omega^{2i+1} b\right). \tag{8.1}$$

Here we note that the factor $\prod_{i=1}^{n-1}\prod_{l=1}^{i}\Omega(b_{i+l+1}|y)$ in (8.1) can be expressed as follows:

$$\prod_{i=1}^{n}\Omega(b_i|y)^m \qquad \text{for } n=2m+1, \qquad \prod_{i=1}^{m}\Omega(b_{2i-1}|y)^m\Omega(b_{2i}|y)^{m-1} \qquad \text{for } n=2m. \tag{8.2}$$

As an application of Corollary 8.2, we have a proof of Proposition 5.4.

Proof of Proposition 5.4. The product $\prod_{i=1}^{n-2}\tilde{g}^{(k)}_{(n-i-1)^i}\big(y|\omega^{2i+1}b\big)$ is ι-invariant because

$$\iota\big(\tilde{g}^{(k)}_{(n-i-1)^i}\big(y|\omega^{2i+1}b\big)\big)=\tilde{g}^{(k)}_{i^{(n-i-1)}}\big(y|\omega^{-2i-1}b\big)=\tilde{g}^{(k)}_{i^{(n-i-1)}}\big(y|\omega^{2(n-i-1)+1}b\big)$$

by Proposition 3.5. Let Ω_n denote the factor (8.2). We will show the factor $\Omega_n\cdot(\sigma_1\sigma_2\cdots\sigma_{n-1})^{-1}$ is also ι-invariant. In fact, from (8.2) and (4.12) we have $\iota(\Omega_n)=\Omega_n/\sigma_n^{n-1}$. From (4.9), we have $\iota(\sigma_1\cdots\sigma_{n-1})=\sigma_1\cdots\sigma_{n-1}\cdot\sigma_n^{n-1}$.

Hence, from Corollary 8.2, $\tilde{\Phi}_n\big(\mathfrak{G}^{Q}_{w_\circ}\big)$ is invariant under the action of ι. ■

The rest of this subsection is devoted to the proof of Proposition 8.1.

We start with the expression

$$\tilde{\Phi}_n(\psi_i)=\frac{\det\big((E-\mathrm{e}^{a_{i+1}}C_A)P^{-1}ZP\big)^{[1,i]}_{[1,i]}}{\sigma_i}.$$

This is obtained from (5.4) and (4.16).

Let us study the matrix $\Xi:=(E-\mathrm{e}^{a_1}C_A)P^{-1}ZP$. Because the entries of C_A and $P^{-1}ZP$ are S_n-invariant (see Remark 6.3), we have $\omega^i\Xi=(E-\mathrm{e}^{a_{i+1}}C_A)P^{-1}ZP$.

In order to finish the proof of Proposition 8.1, it suffices to prove the following.

Lemma 8.3. *Let $M'_{(n-i-1)^i}$ be as defined in Corollary 6.14. We have*

$$\det\Xi^{[1,i]}_{[1,i]}=\omega^{i+1}\det M'_{(n-i-1)^i}. \tag{8.3}$$

Proof. Recall $C_A=P^{-1}AP$. Because P^{-1} is lower unitriangular, we have

$$\det\Xi^{[1,i]}_{[1,i]}=\det((E-\mathrm{e}^{a_1}A)ZP)^{[1,i]}_{[1,i]}.$$

As all elements in the 1st column of $E-\mathrm{e}^{a_1}A$ are zero and $(E-\mathrm{e}^{a_1}A)^{[2,n]}_{[1,n-1]}$ is lower triangular, the determinant $\det((E-\mathrm{e}^{a_1}A)ZP)^{[1,i]}_{[1,i]}$ decomposes as

$$\det(E-\mathrm{e}^{a_1}A)^{[2,i+1]}_{[1,i]}\det(ZP)^{[1,i]}_{[2,i+1]}=\mathrm{e}^{ia_1}\det(ZP)^{[1,i]}_{[2,i+1]}.$$

Therefore, we obtain

$$\det\Xi^{[1,i]}_{[1,i]}=\mathrm{e}^{ia_1}\det(ZP)^{[1,i]}_{[2,i+1]}=\mathrm{e}^{ia_1}\det\begin{pmatrix}Z^{[1,n]}_{[2,i+1]}\\ \hline (P^{-1})^{[1,n]}_{[i+1,n]}\end{pmatrix}, \tag{8.4}$$

where for the last equality we used $\det(P)=1$ (cf. proof of Corollary 6.15). By row reduction, we can replace $(P^{-1})^{[1,n]}_{[i+1,n]}$ in (8.4) by

$$\left(\begin{array}{c|c}(-\mathrm{e}^{-a_1})^i & * \ \cdots \ * \\ \hline \mathbf{0}_{n-i-1} & \omega\big(P^{-1}\big)^{[1,n-1]}_{[i+1,n-1]}\end{array}\right). \tag{8.5}$$

In fact we sweep out the first column of $\left(P^{-1}\right)^{[1,n]}_{[i+1,n]}$ taking $(1,1)$ entry $(-\mathrm{e}^{-a_1})^i$ as the pivot. By using the following obvious identity

$$h_m(x_1,\dots,x_n) - h_{m-1}(x_1,\dots,x_n)x_n = h_m(x_1,\dots,x_{n-1})$$

repeatedly, we obtain (8.5). Thus by expanding along the first column, we have

$$\det \Xi^{[1,i]}_{[1,i]} = \det \left(\frac{Z^{[2,n]}_{[2,i+1]}}{\omega\left(P^{-1}\right)^{[1,n-1]}_{[i+1,n-1]}} \right).$$

If we compare this with

$$M'_{(n-i-1)^i} = \omega^{-i} \left(\frac{Z^{[1,n-1]}_{[1,i]}}{\left(P^{-1}\right)^{[1,n-1]}_{[i+1,n-1]}} \right),$$

and use (6.8), we obtain (8.3). ■

Proof of Proposition 8.1.

$$\begin{aligned}
\tilde{\Phi}_n(\psi_i) &= {\sigma_i}^{-1}\det\left(\omega^i\Xi\right)^{[1,i]}_{[1,i]} = {\sigma_i}^{-1}\det\left(\omega^{2i+1}M'_{(n-i-1)^i}\right)^{[1,i]}_{[1,i]} && \text{by Lemma 8.3}\\
&= {\sigma_i}^{-1}\omega^{2i+1}\left(\xi_{(n-i-1)^i}(y)\cdot \tilde{g}^{(k)}_{(n-i-1)^i}(y|b)\right) && \text{by Corollary 6.14}\\
&= {\sigma_i}^{-1}\prod_{l=1}^{i}\Omega(b_{i+l+1}|y)\cdot \tilde{g}^{(k)}_{(n-i-1)^i}\left(y|\omega^{2i+1}b\right).
\end{aligned}$$ ■

8.2 Proof of Theorem 1.2

Lemma 8.4. *Let $\rho^\vee = \sum_{i=1}^{n-1}\varpi_i^\vee$. Suppose n is even and $n=2m$. Then $\rho^\vee + \varpi_m^\vee \in Q^\vee$ and $w_\circ t_{-\rho^\vee-\varpi_m^\vee} = x_{R_m\cup\nu_n}$. Suppose n is odd. Then $\rho^\vee\in Q^\vee$, and $w_\circ t_{-\rho^\vee} = x_{\nu_n}$.*

Proof. It is well known that

$$\rho^\vee = \frac{1}{2}\sum_{i=1}^{n-1} i(n-i)\alpha_i^\vee. \tag{8.6}$$

We see that if n is odd then $\rho^\vee\in Q^\vee$ and if n is even then $\rho^\vee+\varpi_m^\vee\in Q^\vee$.

We work in the extended affine Weyl group $\tilde{W}_G := \langle\pi\rangle \ltimes \hat{W}_G$ where $\pi^n = \mathrm{id}$ and $\pi s_i = s_{i+1}\pi$ for $i\in\tilde{I} = \mathbb{Z}/n\mathbb{Z}$. By using [9, Lemma 4.6], we have

$$w_\circ t_{-\rho^\vee} = w_\circ t_{-\sum_{i=1}^{n-1}\varpi_i^\vee} = \pi^{-\sum_{i=1}^{n-1} i} x_{\nu_n}. \tag{8.7}$$

Suppose $n=2m$ is even. Then $\pi^{-\sum_{i=1}^{n-1} i} = \pi^m$. From this, together with $t_{-\varpi_m^\vee} = \pi^{-m}x_{R_m}$, we have

$$w_\circ t_{-\rho^\vee-\varpi_m^\vee} = \pi^m x_{\nu_n}\pi^{-m}x_{R_m} = x_{\nu_n\cup R_m},$$

where, for the last equality, we use [32, Lemma 2.15]. If n is odd, then $\pi^{-\sum_{i=1}^{n-1} i} = \mathrm{id}$ and the desired equality is nothing but (8.7). ■

Proof of Theorem 1.2. We first note that ν_n and R_m are invariant under ω_k.

Suppose n is odd. From Theorem 1.1,

$$\tilde{\Phi}_n\big(Q^{-\rho^\vee}\mathfrak{G}^Q_{w_\circ}\big) = \tilde{g}^{(k)}_{\nu_n^{\omega_k}}(y|b) = \tilde{g}^{(k)}_{\nu_n}(y|b).$$

From (8.6), it is straightforward to show

$$\tilde{\Phi}_n\big(Q^{\rho^\vee}\big) = \frac{\sigma_n^{(n-1)/2}}{\sigma_1\sigma_2\cdots\sigma_{n-1}}.$$

Comparing these equations with Proposition 8.1 and using $\sigma_n = \Omega(b_1|y)\cdots\Omega(b_n|y)$, we obtain (1.6).

Next we consider the case $n = 2m$. From Theorem 1.1, we have

$$\begin{aligned}\tilde{\Phi}_n\big(Q^{-\rho^\vee-\varpi_m^\vee}\mathfrak{G}^Q_{w_\circ}\big) &= \tilde{g}^{(k)}_{(\nu_n\cup R_m)^{\omega_k}}(y|b) \qquad \text{by Lemma 8.4}\\ &= \tilde{g}^{(k)}_{\nu_n\cup R_m}(y|b) = \tilde{g}^{(k)}_{R_m}(y|b)\tilde{g}^{(k)}_{\nu_n}(y|\omega^m b) \qquad \text{by Theorem 7.2.}\end{aligned}$$

From (8.6), it is straightforward to show

$$\tilde{\Phi}_n\big(Q^{\rho^\vee+\varpi_m^\vee}\big) = \frac{\sigma_n^m}{\sigma_m(\sigma_1\sigma_2\cdots\sigma_{n-1})}.$$

Comparing these equations with Proposition 8.1, and using $\sigma_n = \Omega(b_1|y)\cdots\Omega(b_n|y)$ together with $\tilde{g}^{(k)}_{R_m}(y|b) = \sigma_m/\xi_{R_m}(y)$, we obtain (1.5). ■

A Discrete relativistic Toda lattice

Let $L = MN^{-1}$ be the Lax matrix of the relativistic Toda equation (2.2). Define the new matrix $L^+ = N^{-1}M$ by switching the position of M and N^{-1}. By the Gauss decomposition $L^+ = M^+\big(N^+\big)^{-1}$, we can define the rational map $(N, M) \mapsto \big(N^+, M^+\big)$, where N^+ and M^+ are matrices of the form

$$N^+ = \begin{pmatrix} 1 & & & & \\ -Q_1^+z_1^+ & 1 & & & \\ & -Q_2^+z_2^+ & \ddots & & \\ & & \ddots & 1 & \\ & & & -Q_{n-1}^+z_{n-1}^+ & 1\end{pmatrix}, \qquad M^+ = \begin{pmatrix} z_1^+ & -1 & & \\ & z_2^+ & \ddots & \\ & & \ddots & -1 \\ & & & z_n^+\end{pmatrix}.$$

Indeed, this correspondence defines an automorphism over the quotient field of $R[T][z, Q]$, written as dToda: $\mathbb{C}(z_i, Q_i) \to \mathbb{C}(z_i, Q_i)$, $z_i \mapsto z_i^+$, $Q_i \mapsto Q_i^+$. This birational map is explicitly written as

$$Q_i^+ = \frac{z_i}{z_{i+1}}Q_i, \qquad z_i^+ = \frac{1 - Q_{i-1}^+}{1 - Q_i^+}z_i, \qquad Q_0 := 0,$$

which is known as *the discrete Relativistic Toda lattice* [31].

As $NL^+ = LN$, the two matrices L and L^+ are similar. Following the same argument as (2.11) and (2.12), we have the pair of matrices U^+, R^+ satisfying

$$P^{-1}Z^+AP = \big(U^+\big)^{-1}R^+ \qquad \text{and} \qquad L^+ = U^+C_A\big(U^+\big)^{-1} = R^+C_A(R^+)^{-1}.$$

Comparing them with $L^+ = N^{-1}LN = M^{-1}LM$, we obtain the equations $U^+ = N^{-1}U$, $R^+ = c\cdot M^{-1}R$ with some nonzero constant c. From these expressions, we can show $P^{-1}Z^+AP =$

$c \cdot P^{-1}ZP$, and then, $Z^+ = cZA^{-1}$. The correspondence $Z \mapsto Z^+$ defines the same map as σ (see (4.1)) on the centralizer space $\tilde{\mathcal{Z}}$. Symbolically, we have $\sigma = \mathrm{dToda}$.

Let $\tau_i^+ := \sigma_i = \sigma(\tau_i)$ and $\tau_i^- := \sigma^{-1}(\tau_i)$ be the tau-functions at the next and previous time step, respectively. Substituting (2.14) and (2.15) to

$$z_i^+ = \frac{1 - Q_{i-1}^+}{1 - Q_i^+} z_i,$$

we obtain

$$\frac{\tau_i^2 - \tau_{i+1}\tau_{i-1}}{\tau_i^+\tau_i^-} = \frac{\tau_{i-1}^2 - \tau_i\tau_{i-2}}{\tau_{i-1}^+\tau_{i-1}^-} = \cdots = \frac{\tau_1^2 - \tau_2\tau_0}{\tau_1^+\tau_1^-} = 1.$$

Finally, we obtain $\tau_i^2 - \tau_{i+1}\tau_{i-1} = \tau_i^+\tau_i^-$, which is known as *Hirota's bilinear form of the discrete Toda equation.*

B Proof of (2.3)

Note that for any square matrix A of size n, we have

$$\det(\zeta E - A) = \sum_{i=0}^{n} (-1)^i \cdot \zeta^{n-i} \sum_{\substack{J \subset [1,n] \\ |J| = i}} \det A_J^J.$$

Therefore, it suffices to show

$$\det L_J^J = \prod_{j \in J, j+1 \notin J} (1 - Q_j) \prod_{j \in J} z_j.$$

Because the matrix $L_{[a,b]}^{[a,b]}$ decomposes as

$$\begin{pmatrix} z_a & -1 & & & \\ & z_{a+1} & \ddots & & \\ & & \ddots & -1 & \\ & & & z_{b-1} & -1 \\ & & & & (1-Q_b)z_b \end{pmatrix} \begin{pmatrix} 1 & & & & \\ -Q_a z_a & 1 & & & \\ & -Q_{a+1}z_{a+1} & \ddots & & \\ & & \ddots & 1 & \\ & & & -Q_{b-1}z_{b-1} & 1 \end{pmatrix}^{-1},$$

we have

$$\det L_{[a,b]}^{[a,b]} = (1 - Q_b) z_a \cdots z_b \qquad \text{for} \quad 1 \le a \le b \le n.$$

If we decompose J as $J = J_1 \sqcup \cdots \sqcup J_r$ in such a way that $J_i = [a_i, b_i]$ with $a_{i+1} - b_i \ge 2$, then L_J^J is blockwise lower triangular with respect to $J = J_1 \sqcup \cdots \sqcup J_r$. So we have $\det L_J^J = \prod_{i=1}^r \det L_{J_i}^{J_i}$ which concludes (2.3).

C Affine K-nil-Hecke action on quantum K-ring

The aim of this appendix is to give a proof of Proposition 5.8. We use the theory of semi-infinite flag manifolds.

Let G be a simple, simply-connected complex algebraic group. Let B be a Borel subgroup of G and T a maximal torus contained in B. Let Q, P be the root lattice and weight lattice, respectively. Let $W_G = \langle s_i \mid i \in I\rangle$ be the Weyl group of G, and $\hat{W}_G = W_G \ltimes Q^\vee$ the affine Weyl group with $\hat{W}_G = \langle s_i \mid i \in \tilde{I}\rangle$. The *semi-infinite flag manifold* associated with G is the (reduced) ind-scheme of ind-infinite type $\mathbf{Q}_G^{\mathrm{rat}} = G(\mathbb{F})/(T(\mathbb{C})N(\mathbb{F}))$ (see [12, 14]), where $\mathbb{F} = \mathbb{C}((t))$ and N is the unipotent radical of B. For each $x \in \hat{W}_G$, there is the corresponding Schubert variety $\mathbf{Q}_G(x) \subset \mathbf{Q}_G^{\mathrm{rat}}$. Let $\mathbf{Q}_G = \mathbf{Q}_G(e)$, where e is the identity. Let $Q^{\vee,+}$ denote the positive part of the coroot lattice. The T-equivariant K-group $K_T(\mathbf{Q}_G)$ is, as an $R(T)$-module, the direct product $\prod_{x\in\hat{W}_G^+} R(T)\mathcal{O}_{\mathbf{Q}_G(x)}$, where $\hat{W}_G^+ = \{wt_\xi \mid w \in W_G,\, \xi \in Q^{\vee,+}\}$. Kato [13] established an $R(T)$-module isomorphism from $QK_T(G/B)$ to $K_T(\mathbf{Q}_G)$. Explicitly, $\mathrm{e}^\mu\mathcal{O}^wQ^\xi$ in $QK_T(G/B)$, with $\mu \in P$, $w \in W_G$, $\xi \in Q^\vee$, corresponds to $\mathrm{e}^{-\mu}\mathcal{O}_{\mathbf{Q}_G(wt_\xi)}$ in $K_T(\mathbf{Q}_G)$; here we follow the convention of [24, Section 6.1] and [26, Section 5].

The *semi-infinite length* of $x = wt_\xi \in \hat{W}_G$ is defined by $\ell^{\frac{\infty}{2}}(w) = \ell(w) + 2\langle\rho,\xi\rangle$, where ρ is the half sum of the positive roots. Let $\le_{\frac{\infty}{2}}$ denote the *semi-infinite Bruhat order* (see [11, Section 2.4]). We note that

$$s_ix >_{\frac{\infty}{2}} x \Longleftrightarrow \ell^{\frac{\infty}{2}}(s_ix) = \ell^{\frac{\infty}{2}}(x) + 1, \qquad i \in \tilde{I}$$

(see [11, Lemma 4.1.2]). Let $\hat{W}_G^0$ denote the minimal-length coset representatives for the coset space $\hat{W}_G/W_G$ with respect to the ordinary length function.

The following fact is due to Peterson [29].

Lemma C.1. *For $x \in \hat{W}_G^0$, we have $\ell(x) = -\ell^{\frac{\infty}{2}}(x)$.*

Proof. Let $x = wt_\lambda \in \hat{W}_G^0$ $(w \in W_G,\, \lambda \in Q^\vee)$. Note that $\lambda \in Q^\vee$ is anti-dominant according to [20, Lemma 3.3]. We have

$$\ell(x) = \ell(t_\lambda) - \ell(w) = \langle e\cdot\lambda, -2\rho\rangle - \ell(w) = -\ell^{\frac{\infty}{2}}(x),$$

where the first equality uses [20, Lemma 3.3], and the second equality applies [20, Lemma 3.2] with $w = e$. ■

Motivated by the nil-DAHA action on $K_T(\mathbf{Q}_G^{\mathrm{rat}})$ [14, Theorem 6.5] (see also [17, Section 3.1], [28, Section 2.6]), we define an endomorphism of $QK_T(G/B)_Q$,

$$D_0^Q = T_\theta + Q^{-\theta^\vee}\mathcal{O}^{s_\theta}\cdot s_\theta,$$

where θ is the highest root and T_θ is defined in (3.8). Then D_i $(i \in \tilde{I})$ satisfy the braid relations, and hence D_x^Q is defined for $x \in \hat{W}_G$.

The next result is transported from the corresponding one for semi-infinite flag manifolds (see [28, Section 2.6] and [17, Section 3.1]).

Proposition C.2. *Let $w \in W$. For $i \in I$, we have*

$$D_i^Q(\mathcal{O}^w) = \begin{cases} \mathcal{O}^{s_iw} & \text{if } s_iw < w,\\ \mathcal{O}^w & \text{if } s_iw > w.\end{cases} \tag{C.1}$$

Moreover, we have

$$D_0^Q(\mathcal{O}^w) = \begin{cases} Q^{-w^{-1}(\theta^\vee)}\mathcal{O}^{s_\theta w} & \text{if } s_\theta w > w,\\ \mathcal{O}^w & \text{if } s_\theta w < w.\end{cases} \tag{C.2}$$

Proof. Let $x = wt_\xi$ with $w \in W_G$ and $\xi \in Q^\vee$. We use the following fact (see [11, Appendix A])

$$\text{for}\quad i \in I, \qquad s_ix <_{\frac{\infty}{2}} x \iff s_iw < w, \qquad s_0x <_{\frac{\infty}{2}} x \iff s_\theta w > w.$$

By the isomorphism $QK_T(G/B) \cong K_T(\mathbf{Q}_G)$ described above, (C.1) and (C.2) are obtained from [28, equation (2.28)] or [17, equation (3.19)]. Note that $s_0w = s_\theta wt_{-w^{-1}(\theta^\vee)}$ for $w \in W_G$. ■

Corollary C.3. *Let* $x = wt_\xi \in \hat{W}_G^0$*. Then* $D_x^Q(1) = Q^\xi \mathcal{O}^w$*.*

Proof. Let $x = s_{i_1} \cdots s_{i_l}$ be a reduced expression. Then we have a saturated decreasing chain of elements in $\hat{W}_G^0$

$$x = s_{i_1} \cdots s_{i_l} > s_{i_2} \cdots s_{i_l} > \cdots > s_{i_l} > e.$$

By Lemma C.1, we obtain a saturated increasing chain of elements with respect to $<_{\frac{\infty}{2}}$

$$x = s_{i_1} \cdots s_{i_l} <_{\frac{\infty}{2}} s_{i_2} \cdots s_{i_l} <_{\frac{\infty}{2}} \cdots <_{\frac{\infty}{2}} s_{i_l} <_{\frac{\infty}{2}} e.$$

Thus from Proposition C.2, we deduce the corollary. ■

Proof of Proposition 5.8. Apply Corollary C.3 to the T-equivariant quantum K-ring of $\mathrm{SL}_n(\mathbb{C})$ presented in [26] and use the fact that $\mathfrak{G}_w^Q(z|\eta)$ represents $\mathcal{O}^w$. ■

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