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1-Point Functions for $\mathbb{Z}_2$-Orbifolds of Lattice VOAs

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Abstract. In this paper, we compute the 1-point correlation functions of all states for the $\mathbb{Z}_2$-orbifolds of lattice vertex operator algebras.

Key words: one-point functions; trace functions; modular invariance; lattice vertex operator algebra; orbifold theory; vertex operator algebra

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1 Introduction

The first example of an orbifold in the theory of vertex operator algebras is the moonshine module $V^\natural$. It was constructed by Igor Frenkel, James Lepowsky and Arne Meurman [5] as a $\mathbb{Z}_2$-orbifold of the vertex operator algebra associated with the Leech lattice. The construction of the moonshine module can be generalized by replacing the Leech lattice with any even, positive-definite lattice L of rank $k = 8l$, where $l \in \mathbb{Z}^+$ (the set of positive integers) such that $\sqrt{2}L^*$ is also even, where L^* denotes the dual lattice of L, to obtain the $\mathbb{Z}_2$-orbifold of the lattice vertex operator algebra V_L with respect to the involutive automorphism θ, where θ is the lift of the (-1)-involution of the lattice L. This was first done explicitly by Dolan, Goddard and Montague [1]. This condition on L is more general than unimodularity. When L is unimodular, the orbifold that is obtained through this construction indeed coincides with that of the holomorphic vertex operator algebra obtained through the cyclic orbifold theory given by Ekeren, Möller and Scheithauer in [9].

One of the most intriguing features of the moonshine module as it was initially constructed was its character (1-point function corresponding to the vacuum state) being the modular function $j(\tau) - 744$. The modular invariance of the character function was later explained by Zhu [10] as a consequence of the axioms of the vertex operator algebras together with some finiteness conditions. Thus modular invariance of characters of vertex operator algebras (VOAs) became a subject of interest further. In [4], Dong, Mason and Nagatomo investigated the modular properties of 1-point functions corresponding to free bosonic VOAs and lattice VOAs and found that the trace functions in these two theories have the shape $f(q)/\eta(q)^d$, where $f(q)$ is quasi-modular in the case of d free bosons and modular in the latter case. In analogy with these works, one may study the modular properties of 1-point functions associated with the $\mathbb{Z}_2$-orbifolds of lattice VOAs. This requires, as a first step, explicit computations of such 1-point functions, which is the focus of the present paper.

In this paper, we focus on the computation of 1-point correlation functions for the $\mathbb{Z}_2$-orbifolds of vertex operator algebras associated with unimodular, even, positive definite lattices L of rank $k = 8l$ where $l \in \mathbb{Z}^+$. The approach in this paper is inspired by the techniques developed

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This paper is a contribution to the Special Issue on Recent Advances in Vertex Operator Algebras in honor of James Lepowsky. The full collection is available at https://sigma-journal.com/Lepowsky.html by Geoffrey Mason and Michael Mertens in their paper on 1-point functions for symmetrized Heisenberg and lattice vertex operator algebras [7]. We adopt and extend those techniques to the $\mathbb{Z}_2$-orbifold setting to perform computations on the twisted sector and obtain results for the 1-point functions for the $\mathbb{Z}_2$-orbifold. While the work by Mason and Mertens sheds light on how the 1-point functions traced over V_L^+ exhibit modular invariance with respect to a congruence subgroup, our work finds the trace contributed by $\left(V_L^T\right)^+$ (which is the $+1$-eigenspace of the action of the (-1)-involution (θ) on V_L^T defined in Section 2.3). Combining the two traces, we obtain the 1-point functions for the $\mathbb{Z}_2$-orbifold V [9], which are expected to exhibit modular invariance under the full modular group up to a character. This modular invariance is a property that can be realized as a special case of [10, Theorem 5.1.1] when $n = 1$ and it is verified in Proposition 6.2. Further, we also observe that due to the structure of the twisted module, the 1-point functions corresponding to the lattice states of the form $h_{i_1}[-n_1]\cdots h_{i_p}[-n_p]e_\alpha$ when $\alpha \in L\backslash 2L$ vanish.

The main contributions of this paper are as follows:

1. We derive explicit formulas for the 1-point correlation functions of the $\mathbb{Z}_2$-orbifold of lattice vertex operator algebras, as in the following theorems.

Theorem 1.1. *Let L be a positive-definite even unimodular lattice of rank $k = 8l$ (where $l \in \mathbb{Z}^+$). Let V be the $\mathbb{Z}_2$-orbifold of the VOA V_L associated with the lattice L formed by the construction of Dolan, Goddard and Montague [1] (also by [9]). For positive integers $n_i \geq 1$, corresponding to the Heisenberg state $u := h_{i_1}[-n_1]\cdots h_{i_p}[-n_p]\mathbf{1}$, we have the 1-point function given by*

$$\begin{aligned} Z_V(u,\tau) = &\frac{1}{2}\sum_{\Delta\subseteq\Lambda}\frac{\theta_L(\tau,P_\Delta)}{\eta(\tau)^k}\Bigg(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p}\backslash\Delta)}\prod_{(rs)}\delta_{i_r,i_s}\hat{E}_{n_r+n_s}(\tau)\Bigg)\\ &+\frac{1}{2}\eta(\tau)^{k/2}\left(\frac{\Theta_1(\tau)}{2}\right)^{-k/2}\Bigg(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p})}\prod_{(rs)}\delta_{i_r,i_s}\hat{F}_{n_r+n_s}(\tau)\Bigg)\\ &+\frac{1}{2}\eta(\tau)^{k/2}\left(\frac{\Theta_2(\tau)}{2}\right)^{-k/2}\Bigg(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p})}\prod_{(rs)}\delta_{i_r,i_s}\overline{E}_{n_r+n_s}(\tau)\Bigg)\\ &+\frac{(-1)^l}{2}\eta(\tau)^{k/2}\left(\frac{\Theta_3(\tau)}{2}\right)^{-k/2}\Bigg(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p})}\prod_{(rs)}\delta_{i_r,i_s}\overline{F}_{n_r+n_s}(\tau)\Bigg), \end{aligned}$$

where $\Lambda = \{j \in \underline{p} \mid n_j = 1\}$, Δ is a subset of Λ of even cardinality, $\underline{p} := \{1, 2, \ldots, p\}$, for a set S, $\mathrm{Inv}_0(S)$ is the set of all fixed-point-free involutions of the set S and $P_\Delta(\alpha) = \prod_{j\in\Delta}\langle h_{i_j}, \alpha\rangle$. More details about the notation can be found in Section 2.

Theorem 1.2. *In the same setting as the above theorem, for positive integers $n_i \geq 1$, $\alpha \in 2L$, corresponding to the state $u := h_{i_1}[-n_1]\cdots h_{i_p}[-n_p]e_\alpha$, where e_α is $f_\alpha = \mathrm{e}^\alpha + \mathrm{e}^{-\alpha}$ or $g_\alpha = \mathrm{e}^\alpha - \mathrm{e}^{-\alpha}$ accordingly as p is even or odd, respectively, the 1-point function is given by*

$$\begin{aligned} Z_V(u,\tau) = &\,\eta(\tau)^{k/2}\left(\frac{\Theta_1(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle-k/2}\Bigg(\sum_{\sigma}\prod_{(rs)(t)}\delta_{i_r,i_s}\tilde{F}_{n_t}(\tau)\hat{F}_{n_r+n_s}(\tau)\Bigg)\\ &+\eta(\tau)^{k/2}\left(\frac{\Theta_2(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle-k/2}\Bigg(\sum_{\sigma}\prod_{(rs)(t)}\delta_{i_r,i_s}\overline{\tilde{E}}_{n_t}\overline{E}_{n_r+n_s}(\tau)\Bigg)\\ &+(-1)^l\eta(\tau)^{k/2}\left(\frac{\Theta_3(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle-k/2}\Bigg(\sum_{\sigma}\prod_{(rs)(t)}\delta_{i_r,i_s}\overline{\tilde{F}}_{n_t}\overline{F}_{n_r+n_s}(\tau)\Bigg), \end{aligned}$$

where σ ranges over all involutions of the index set $S_p = \{1, 2, \ldots, p\}$ and (rs), (t) range over the 2-cycles and 1-cycles respectively in the decomposition of σ in S_p as a product of disjoint 2-cycles and 1-cycles. For $\alpha \in L \backslash 2L$, we have $Z_V(u, \tau) = 0$.

2. We analyse the modular properties of these 1-point functions, confirming that they indeed exhibit modular invariance under the full modular group (up to a character).

This paper is organized as follows: In Section 2, we provide a brief overview of the necessary background and notation on modular forms, elliptic-type functions, theta functions, lattice vertex operator algebras, the corresponding $\mathbb{Z}_2$-orbifold, and the square bracket formalism. Section 3 outlines the methods used by Mason and Mertens, which we adapt for our purposes to develop $\mathbb{Z}_2$-twisted Zhu theory. Here, we produce a reduction theorem for 2-point functions traced over the twisted module V_L^T and similar formulas were first produced in [2, Theorem 8.4]. In Section 4, we present some computations in the twisted space. In Section 5, we present our main results, including the explicit computation of 1-point functions for the $\mathbb{Z}_2$-orbifolds of lattice vertex operator algebras. In Section 6, we verify that these 1-point functions exhibit modular invariance under the full modular group up to a character.

2 Background and notation

2.1 Modular forms and elliptic-type functions

Let $\mathbb{H}$ denote the complex upper-half plane, τ a typical element in $\mathbb{H}$ and $q = \mathrm{e}^{2\pi \mathrm{i} \tau}$. The Dedekind eta function is defined as

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

The Bernoulli numbers B_k are defined by

$$\sum_{k \geq 0} \frac{B_k}{k!} z^k \coloneqq \frac{z}{(\mathrm{e}^z - 1)}.$$

The Eisenstein series for even positive integers k are defined by

$$E_k(\tau) \coloneqq -\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where $\sigma_{k-1}(m) \coloneqq \sum_{d|m} d^{k-1}$ denotes the usual divisor sum function. We denote the level 2 Eisenstein series defined for positive k, by F_k, where $F_k(\tau) \coloneqq 2E_k(2\tau) - E_k(\tau)$. Note that $E_k(\tau)$ is a modular form of weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ as long as $k \neq 2$ and the functions F_k are modular forms of weight k for

$$\Gamma_0(2) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{2} \right\}.$$

We use the same notation for renormalised Eisenstein series as in [7]

$$\hat{E}_{m+n}(\tau) \coloneqq (-1)^{n+1} n \binom{m+n-1}{n} E_{m+n}(\tau),$$
$$\hat{F}_{m+n}(\tau) \coloneqq (-1)^{n+1} n \binom{m+n-1}{n} F_{m+n}(\tau).$$

Additionally, we introduce a new notation for the following level 2 and level 4 modular forms

$$\overline{E}_{m+n}(\tau) := 2^{-(m+n-1)}\hat{E}_{m+n}(\tau/2) - \hat{E}_{m+n}(\tau),$$
$$\overline{F}_{m+n}(\tau) := 2^{-(m+n-1)}\hat{F}_{m+n}(\tau/2) - \hat{F}_{m+n}(\tau).$$

We also use the same notation for the following elliptic-type functions used in [7] as

$$P_1(z,\tau) := \frac{1}{2} + \sum_{0\neq n\in\mathbb{Z}} \frac{\mathrm{e}^{nz}}{(1-q^n)} = -\frac{1}{z} + \sum_{k\geq 1} E_{2k}(\tau)z^{2k-1} = E_2(\tau)z - \zeta(z,\tau), \tag{2.1}$$

where $\zeta(z,\tau)$ is the Weirstrass zeta function for the lattice $\Lambda = 2\pi\mathrm{i}(\mathbb{Z}\oplus\mathbb{Z}\tau)$, i.e.,

$$\zeta(z,\tau) = \frac{1}{z} + \sum_{\omega\in\Lambda}\left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right), \qquad z\in\mathbb{C}\backslash\Lambda.$$

The z-derivatives of the above Lambert series are thus

$$P_1^{(m)}(z,\tau) = m!\left(\frac{(-1)^{m+1}}{z^{m+1}} + \sum_{k\geq 1}\binom{2k-1}{m}E_{2k}(\tau)z^{2k-m-1}\right), \qquad m\geq 0.$$

We also use the function

$$Q_1(z,\tau) := \sum_{n\in\mathbb{Z}} \frac{\mathrm{e}^{nz}}{1+q^n},$$

whose higher z-derivatives have series expansions as follows:

$$Q_1^{(m)}(z,\tau) := \sum_{n\in\mathbb{Z}} \frac{n^m\mathrm{e}^{nz}}{1+q^n} = m!\left(\frac{(-1)^{m+1}}{z^{m+1}} + \sum_{k\geq 1}\binom{2k-1}{m}F_{2k}(\tau)z^{2k-m-1}\right) \tag{2.2}$$

for $m\geq 0$.

2.2 Theta functions

The theta function of an even lattice L of rank k with a positive definite bilinear form $\langle\cdot,\cdot\rangle$ is defined by

$$\theta_L(\tau) = \sum_{\alpha\in L} q^{\langle\alpha,\alpha\rangle/2}.$$

We use the same notation as [7] for the following variant of the theta function:

$$\theta_L(\tau,v,m) = \sum_{\alpha\in L}\langle v,\alpha\rangle^m q^{\langle\alpha,\alpha\rangle/2}$$

for any vector $v\in L\otimes\mathbb{C}$ and non-negative integer $m\geq 0$. We also require the following variant of the theta function of a lattice L. For any function $P\colon L\to\mathbb{C}$, set

$$\theta_L(\tau,P) := \sum_{\alpha\in L} P(\alpha)q^{\langle\alpha,\alpha\rangle/2}.$$

We follow the notation of [3] to denote the Jacobi theta functions by Θ_1, Θ_2, Θ_3, where

$$\Theta_1(\tau) = 2\frac{\eta(2\tau)^2}{\eta(\tau)}, \qquad \Theta_2(\tau) = \frac{\eta(\tau/2)^2}{\eta(\tau)}, \qquad \Theta_3(\tau) = \frac{\eta(\tau)^5}{\eta(\tau/2)^2\eta(2\tau)^2}.$$

2.3 Lattice vertex operator algebra V_L and the $\mathbb{Z}_2$-twisted module V_L^T

Let L be an even, positive definite, unimodular lattice of rank $k = 8l$ (for some $l \in \mathbb{Z}^+$) with the bilinear form $\langle \cdot, \cdot \rangle \colon L \times L \to \mathbb{Z}$. The $\mathbb{C}$-linear extension defines a bilinear form on $\mathfrak{h} := L \otimes \mathbb{C}$. Let $\left(\hat{L}, {}^{-}\right)$ be a central extension of L by a cyclic group of order 2 $\langle \kappa \rangle = \left\langle \kappa \mid \kappa^2 = 1 \right\rangle$ such that

$$\begin{tikzcd} 1 \arrow[r] & \langle \kappa \rangle \arrow[r] & \hat{L} \arrow[r, "-"] & L \arrow[r] & 1. \end{tikzcd}$$

Let $e \colon L \to \hat{L}$, $\alpha \to e_\alpha$ be a section of $\hat{L}$, that is, a map e such that ${}^{-} \circ e = 1$ such that $e_0 = 1$, and denote by $\epsilon_0 \colon L \times L \to \mathbb{Z}/2\mathbb{Z}$ the corresponding 2-cocycle, which is defined by the condition

$$e_\alpha e_\beta = \kappa^{\epsilon_0(\alpha,\beta)} e_{\alpha+\beta} \qquad \text{for } \alpha, \beta \in L.$$

Further, define

$$\epsilon\colon \ L \times L \to \mathbb{C}^\times, \qquad (\alpha, \beta) \to (-1)^{\epsilon_0(\alpha,\beta)}.$$

We can choose ϵ [8] (and hence ϵ_0) so that it satisfies

$$\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{\langle \alpha,\beta\rangle + \langle \alpha,\alpha\rangle\langle \beta,\beta\rangle}, \qquad \epsilon(\alpha, \alpha) = (-1)^{(\langle \alpha,\alpha\rangle + \langle \alpha,\alpha\rangle^2)/2}.$$

Define a faithful character $\chi \colon \langle \kappa \rangle \to \mathbb{C}^\times$ by the condition $\chi(\kappa) = -1$. Denote by $\mathbb{C}_\chi$ the one-dimensional space $\mathbb{C}$ viewed as a $\langle \kappa \rangle$-module on which $\langle \kappa \rangle$ acts according to χ: $\kappa.1 = -1$. Denote by $\mathbb{C}\{L\} = \mathrm{Ind}_{\langle \kappa \rangle}^{\hat{L}} \mathbb{C}_\chi = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\langle \kappa \rangle]} \mathbb{C}_\chi = \mathbb{C}[\hat{L}]/(\kappa + 1)\mathbb{C}[\hat{L}]$. For $a \in \hat{L}$, set $\iota(a) = a \otimes 1 \in \mathbb{C}\{L\}$. The choice of the section allows us to identify $\mathbb{C}\{L\}$ with the group algebra $\mathbb{C}[L]$ twisted by the 2-cocycle ϵ, viewed as a vector space, by the linear isomorphism [6]

$$\mathbb{C}[L] \to \mathbb{C}\{L\}, \qquad \mathrm{e}^\alpha \to \iota(e_\alpha) \qquad \text{for } \alpha \in L,$$

where e^α is the image of 1 through e_α (realised through the action of $\hat{L}$ on $\mathbb{C}[L]$). Recall that for $\alpha, \beta, \gamma \in L$, the action of $\hat{L}$ on $\mathbb{C}[L]$ is given by

$$e_\alpha.\mathrm{e}^\beta = \epsilon(\alpha, \beta)\mathrm{e}^{\alpha+\beta}, \qquad \kappa.\mathrm{e}^\beta = -\mathrm{e}^\beta \qquad \text{for } \alpha, \beta \in L.$$

Thus following the notation as in [7], we denote the Fock space of the corresponding Heisenberg VOA of rank k with M and that of the corresponding lattice theory VOA with

$$V_L = M \otimes \mathbb{C}\{L\} = M \otimes \mathbb{C}[L] = \oplus_{\alpha \in L} M \otimes \mathrm{e}^\alpha.$$

We further use the same notation as in [7, Section 2.4] for representing different Fock states of V_L after considering an orthonormal basis $\{h_1, h_2, \ldots, h_k\}$ of $\mathfrak{h}$. However, we use θ to denote the involutive automorphism of V_L which is a lift of the negating automorphism of L. To form the $\mathbb{Z}_2$-twisted module V_L^T, consider the twisted affine algebra

$$\hat{\mathfrak{h}}[-1] = \bigoplus_{n \in \mathbb{Z}+1/2} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c.$$

Let $S\big(\hat{\mathfrak{h}}[-1]^-\big)$ denote the symmetric algebra generated by the $\mathbb{Z}^-$-graded subalgebra of the twisted affine algebra. Let the objects associated with $\hat{L}$ be defined as in [5, equations (7.1.6)–(7.1.27)] and let T be any irreducible $\hat{L}$-module such that $\kappa.v = -v$ for $v \in T$. Observe that we set $s = 2$ and thus $\omega = -1$ in the notation of [5]. Since L is unimodular, there exists a unique such T. The $\mathbb{Z}_2$-twisted module V_L^T is the space

$$V_L^T = S(\mathfrak{h}[-1]^-) \otimes T.$$

Note that the elements of T are all graded by the $L[0]$-weight $k/16$ [5]. Further, for $h_i \in \mathfrak{h}$, $n_i - 1/2 \in \mathbb{Z}^+ \cup \{0\}$, $t \in T$, the action of θ on the twisted module V_L^T is defined by

$$\theta(h_{i_1}[-n_1] \cdots h_{i_j}[-n_j] \otimes t) = (-1)^{j+k/8} h_{i_1}[-n_1] \cdots h_{i_j}[-n_j] \otimes t.$$

2.4 $\mathbb{Z}_2$-orbifold of the lattice VOA V_L

The holomorphic $\mathbb{Z}_2$-orbifold V of V_L is thus formed from the fixed subspaces of the involutive automorphism θ, $V_L^\theta \subseteq V_L$ and $\left(V_L^T\right)^\theta \subseteq V_L^T$ as [1, 9] $V = V_L^\theta \oplus \left(V_L^T\right)^\theta$. Moving forward, we shall denote V_L^θ with V_L^+ and $\left(V_L^T\right)^\theta$ with $\left(V_L^T\right)^+$ and the -1-eigenspaces of θ in V_L and V_L^T with V_L^- and $\left(V_L^T\right)^-$, respectively. We further use the same notation as in [7] for states in V_L^+, where for $\alpha \in L$ $f_\alpha \coloneqq \mathrm{e}^\alpha + \mathrm{e}^{-\alpha}$, $g_\alpha \coloneqq \mathrm{e}^\alpha - \mathrm{e}^{-\alpha}$.

2.5 1-point functions

Also similarly as in [7], we denote the zero mode of $v \in V_k$ with $o(v) \coloneqq v(k-1)$ and extend the definition linearly to every state $u \in V$. Define the 1-point function for $u \in V$ by

$$Z_V(u,\tau) \coloneqq \mathrm{Tr}_V\, o(u) q^{L(0)-c/24}.$$

Further, for a module or a twisted module M of a vertex operator algebra V, corresponding to a homogeneous state $u \in V_k$, the trace function for $u \in V$ over M can be defined as

$$Z_M(u,\tau) \coloneqq \mathrm{Tr}_M\, o(u) q^{L(0)-c/24}.$$

2.6 Square bracket formalism

Suppose that $V = \bigoplus_k V_k$ is a vertex operator algebra of central charge c. We are going to define some new endomorphisms $v[n]$ for states $v \in V$, called the square bracket modes. The round and square bracket modes are related as follows: for $m \geq 0$ and $v \in V_k$, we have

$$Y[v,z] = \sum_{n\in\mathbb{Z}} v[n] z^{-n-1} \coloneqq Y\left(\mathrm{e}^{zL(0)}v, \mathrm{e}^z - 1\right),$$
$$v[m] = m! \sum_{i \geq m} c(k,i,m) v(i), \qquad m \geq 0,$$

where

$$\sum_{m=0}^{i} c(k,i,m) x^m \coloneqq \binom{k-1+x}{i}, \qquad \text{where } i \geq 0. \tag{2.3}$$

3 Z_2-twisted Zhu theory

3.1 Reduction theorems

For the twisted vertex operator defined as in [5], $Y \colon V_L \to \mathrm{End}\left(V_L^T\right)\left[\left[z^{1/2}, z^{-1/2}\right]\right]$, $z_1, z_2 \in \mathbb{C}$, $q_i = \mathrm{e}^{z_i}$, $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, define the twisted 2-point function on the torus as

$$F_{V_L^T}((u,z_1),(v,z_2),\tau) \coloneqq \mathrm{Tr}_{V_L^T}\, Y\left(q_1^{L(0)}u, q_1\right) Y\left(q_2^{L(0)}v, q_2\right) q^{L(0)-c/24}.$$

In this subsection, we obtain reduction theorems where we express the above twisted 2-point function on the torus in terms of modular data and 1-point functions.

Theorem 3.1. *For $u, v \in V_L$,*

$$F_{V_L^T}((u,z_1),(v,z_2),\tau) = \sum_{p=0,1} \sum_{m \geq 0} \frac{2^{-m-1}}{m!} Z_{V_L^T}((\theta^p u)[m]v,\tau) P_1^{(m)}(z_{21}/2 + p\pi\mathrm{i}, \tau/2),$$

where $z_{21} \coloneqq z_2 - z_1 = -z_{12}$.

Proof. Since L is even, for $m, n \in \frac{1}{2}\mathbb{Z}$, we have the following twisted commutator formula from [5]:

$$[u(m), v(n)] = \frac{1}{2} \sum_{p=0,1} (-1)^{2pm} \sum_{i \in \mathbb{Z}^+} \binom{m}{i} ((\theta^p u)(i)v)(m+n-i),$$

and the above equation could further be rewritten as

$$[u(m), Y(v,z)] = \frac{1}{2} \sum_{p=0,1} (-1)^{2pm} \sum_{i \in \mathbb{Z}^+} \binom{m}{i} Y((\theta^p u)(i)v, z) z^{m-i}.$$

For $n \in \frac{1}{2}\mathbb{Z}$, $k \in \mathbb{Z}$, $u \in V_{L_{(k)}}$ (weight k $L(0)$-eigenspace of V_L), $v \in V_L$,

$$\left[u(n), Y\left(q_2^{L(0)} v, q_2\right)\right] = \frac{1}{2} \sum_{p=0,1} (-1)^{2pn} \sum_{i \in \mathbb{Z}^+} \binom{n}{i} Y\left((\theta^p u)(i) q_2^{L(0)} v, q_2\right) q_2^{n-i}, \tag{3.1}$$

and we have

$$\sum_{i \in \mathbb{Z}^+} \binom{n}{i} Y\left((\theta^p u)(i) q_2^{L(0)} v, q_2\right) q_2^{n-i} = q_2^{n-k+1} Y\left(q_2^{L(0)} \sum_{i \in \mathbb{Z}^+} \binom{n}{i} (\theta^p u)(i) v, q_2 \right).$$

From (2.3), we have

$$\sum_{i \in \mathbb{Z}^+} \binom{n}{i} Y\left((\theta^p u)(i) q_2^{L(0)} v, q_2\right) q_2^{n-i} = q_2^{n-k+1} \sum_{m \geq 0} \frac{(n-k+1)^m}{m!} Y\left(q_2^{L(0)} (\theta^p(u))[m]v, q_2\right).$$

We set $r := n-k+1$, we have

$$\begin{aligned}
&\operatorname{Tr}_{V_L^T} \left\{ u(n) Y\left(q_2^{L(0)} v, q_2\right) \right\} q^{L(0)-c/24} \\
&\quad = \frac{1}{2} \sum_{p=0,1} (-1)^{2pn} \operatorname{Tr}_{V_L^T} q_2^r \sum_{m \geq 0} \frac{r^m}{m!} Y\left(q_2^{L(0)} (\theta^p(u))[m]v, q_2\right) q^{L(0)-c/24} \\
&\qquad + \operatorname{Tr}_{V_L^T} \left\{ Y\left(q_2^{L(0)} v, q_2\right) u(n) \right\} q^{L(0)-c/24}, \\
&\operatorname{Tr}_{V_L^T} \left\{ u(n) Y\left(q_2^{L(0)} v, q_2\right) \right\} q^{L(0)-c/24} \\
&\quad = \frac{1}{2} \sum_{p=0,1} (-1)^{2pn} \operatorname{Tr}_{V_L^T} q_2^r \sum_{m \geq 0} \frac{r^m}{m!} Y\left(q_2^{L(0)} (\theta^p(u))[m]v, q_2\right) q^{L(0)-c/24} \\
&\qquad + q^r \operatorname{Tr}_{V_L^T} \left\{ Y\left(q_2^{L(0)} v, q_2\right) q^{L(0)-c/24} \right\} u(n).
\end{aligned} \tag{3.2}$$

Since $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$, the last term above can be rewritten as follows:

$$\begin{aligned}
&\operatorname{Tr}_{V_L^T} \left\{ u(n) Y\left(q_2^{L(0)} v, q_2\right) \right\} q^{L(0)-c/24} \\
&\quad = \frac{1}{2} \sum_{p=0,1} (-1)^{2pn} \operatorname{Tr}_{V_L^T} q_2^r \sum_{m \geq 0} \frac{r^m}{m!} Y\left(q_2^{L(0)} (\theta^p(u))[m]v, q_2\right) q^{L(0)-c/24} \\
&\qquad + q^r \operatorname{Tr}_{V_L^T} \left\{ u(n) Y\left(q_2^{L(0)} v, q_2\right) q^{L(0)-c/24} \right\}, \\
&\operatorname{Tr}_{V_L^T} \left\{ u(n) Y\left(q_2^{L(0)} v, q_2\right) \right\} q^{L(0)-c/24} \\
&\quad = \frac{1}{2} q_2^r \sum_{p=0,1} (-1)^{2pn} \sum_{m \geq 0} \frac{r^m}{m!} \operatorname{Tr}_{V_L^T} Y\left(q_2^{L(0)} (\theta^p(u))[m]v, q_2\right) q^{L(0)-c/24}
\end{aligned}$$

$$
\begin{aligned}
&\qquad + q^r \operatorname{Tr}_{V_L^T}\big\{u(n)Y\big(q_2^{L(0)}v, q_2\big)q^{L(0)-c/24}\big\},\\
&\operatorname{Tr}_{V_L^T}\big\{u(n)Y\big(q_2^{L(0)}v, q_2\big)\big\}q^{L(0)-c/24}\\
&\quad = \frac{1}{2}q_2^r \sum_{p=0,1}(-1)^{2pn}\sum_{m\geq 0}\frac{r^m}{m!}Z_{V_L^T}((\theta^p(u))[m]v,\tau)\\
&\qquad + q^r \operatorname{Tr}_{V_L^T}\big\{u(n)Y\big(q_2^{L(0)}v, q_2\big)q^{L(0)-c/24}\big\}.
\end{aligned}
$$

For $r=0$, the above equation says that

$$\sum_{p=0,1}(-1)^{2pn}Z_{V_L^T}((\theta^p(u))[0]v,\tau) = 0.$$

For $r \neq 0$, the above equation gives

$$
\begin{aligned}
&\operatorname{Tr}_{V_L^T}\big\{u(n)Y(q_2^{L(0)}v, q_2)\big\}q^{L(0)-c/24}\\
&\quad = \frac{q_2^r}{2(1-q^r)}\sum_{p=0,1}(-1)^{2pn}\sum_{m\geq 0}\frac{r^m}{m!}Z_{V_L^T}((\theta^p(u))[m]v,\tau).
\end{aligned}
\tag{3.3}
$$

However, observe that the commutator formula (3.1) is non-zero only when $n \notin \mathbb{Z}$ (i.e., $r \neq 0$) and hence the 2-point function we defined above can now be written as

$$
\begin{aligned}
F_{V_L^T}((u,z_1),(v,z_2),\tau) &= \sum_{n\in\frac{1}{2}\mathbb{Z}} q_1^{k-n-1}\operatorname{Tr}_{V_L^T}\big\{u(n)Y\big(q_2^{L(0)}v, q_2\big)\big\}q^{L(0)-c/24}\\
&= \sum_{n\in\frac{1}{2}\mathbb{Z}, r\neq 0} q_1^{-r}\frac{q_2^r}{2(1-q^r)}\sum_{m\geq 0}\sum_{p=0,1}(-1)^{2pn}\frac{r^m}{m!}Z_{V_L^T}((\theta^p u)[m]v,\tau)\\
&= \frac{1}{2}\sum_{m\geq 0}\sum_{n\in\frac{1}{2}\mathbb{Z}, r\neq 0}\sum_{p=0,1}\frac{(-1)^{2pn}}{m!}Z_{V_L^T}((\theta^p u)[m]v,\tau)r^m\frac{q_2^r q_1^{-r}}{1-q^r}.
\end{aligned}
$$

We set $q_{21} := \frac{q_2}{q_1}$, then

$$= \frac{1}{2}\sum_{m\geq 0}\sum_{r\in\frac{1}{2}\mathbb{Z}, r\neq 0}\sum_{p=0,1}\frac{(-1)^{2pr}}{m!}Z_{V_L^T}((\theta^p u)[m]v,\tau)r^m\frac{q_{21}^r}{(1-q^r)}. \tag{3.4}$$

The above expression can be written as

$$\sum_{p=0,1}\sum_{r\in\frac{1}{2}\mathbb{Z}, r\neq 0}(-1)^{2pr}r^m\frac{q_{21}^r}{(1-q^r)} = 2^{-m}\sum_{p=0,1}P_1^{(m)}(z_{21}/2 + p\pi\mathrm{i}, \tau/2).$$

Thus expression in (3.4) can be rewritten as

$$F_{V_L^T}((u,z_1),(v,z_2),\tau) = \sum_{p=0,1}\sum_{m\geq 0}\frac{2^{-m-1}}{m!}Z_{V_L^T}((\theta^p u)[m]v,\tau)P_1^{(m)}(z_{21}/2 + p\pi\mathrm{i}, \tau/2). \qquad \blacksquare$$

Theorem 3.2. *The twisted* 2*-point function can be rewritten as*

$$\operatorname{Tr}_{V_L^T} Y\big(q_1^{L(0)}u, q_1\big)Y\big(q_2^{L(0)}v, q_2\big)q^{L(0)-c/24} = Z_{V_L^T}(Y_{\mathbb{Z}}[u, z_{12}]v,\tau),$$

where $Y_{\mathbb{Z}}[v,z] := \sum_{n\in\mathbb{Z}} v[n]z^{-n-1}$*, i.e., the restriction of the twisted vertex operator* Y (*which is also referred to as* $Y_{V_L^T}$ *occasionally in the paper*) *to integer-modes.*

Proof. From the associativity [5, formula (9.3.52)], we have

$$Y_{V_L^T}(u, z_1+z_2)Y_{V_L^T}(v, z_2) = Y_{V_L^T}(Y_{V_L}(u, z_1)v, z_2),$$

using which we have

$$\mathrm{Tr}_{V_L^T}\, Y_{V_L^T}(u, z_1+z_2)Y_{V_L^T}(v, z_2)q^{L(0)-c/24} = \mathrm{Tr}_{V_L^T}\, Y_{V_L^T}(Y_{V_L}(u, z_1)v, z_2)q^{L(0)-c/24}.$$

Further, in [7, Theorem 10] we have the 2-point function $F_V((u, z_1), (v, z_2), \tau)$ rewritten as $Z_V(Y[u, z_{12}]v, \tau)$. Using a similar argument, we have

$$\begin{aligned}&\mathrm{Tr}_{V_L^T}\, Y\big(q_1^{L(0)}u, q_1\big)Y\big(q_2^{L(0)}v, q_2\big)q^{L(0)-c/24}\\&\quad = \mathrm{Tr}_{V_L^T}\, Y\big(Y_{\mathbb{Z}}\big(q_1^{L(0)}u, q_1-q_2\big)q_2^{L(0)}v, q_2\big)q^{L(0)-c/24},\end{aligned}$$

and thus we obtain the required result. ■

3.2 Recursion formula

In this subsection, we obtain a recursion formula where we express 1-point functions in terms of modular functions and 1-point functions of states of lower weight.

Theorem 3.3. *For $u, v \in V_L^-$, $n \geq 1$,*

$$Z_{V_L^T}(u[-n]v, \tau) = \sum_{m\geq 1}\binom{1}{m}\overline{E}_{m+n}(\tau)Z_{V_L^T}(u[m]v, \tau).$$

Proof. From Theorems 3.1 and 3.2, we have

$$Z_{V_L^T}(Y_{\mathbb{Z}}[u, z_{12}]v, \tau) = \sum_{p=0,1}\sum_{m\geq 1}\frac{2^{-m-1}}{m!}Z_{V_L^T}((\theta^p u)[m]v, \tau)P_1^{(m)}\left(\frac{z_{21}}{2}+p\pi\mathrm{i}, \frac{\tau}{2}\right).$$

Since

$$P_1^{(m)}(z/2+\pi\mathrm{i}, \tau/2) = 2^{m+1}P_1^{(m)}(z, \tau) - P_1^{(m)}(z/2, \tau/2), \tag{3.5}$$

we have

$$\begin{aligned}&Z_{V_L^T}(Y_{\mathbb{Z}}[u, z_{12}]v, \tau)\\&\quad = \sum_{m\geq 1}\frac{2^{-m-1}}{m!}Z_{V_L^T}((u)[m]v, \tau)P_1^{(m)}(z_{21}/2, \tau/2)\\&\qquad + \sum_{m\geq 1}\frac{2^{-m-1}}{m!}Z_{V_L^T}((\theta u)[m]v, \tau)P_1^{(m)}(z_{21}/2+\pi\mathrm{i}, \tau/2)\\&\quad = \sum_{m\geq 1}\frac{2^{-m-1}}{m!}Z_{V_L^T}((u)[m]v, \tau)P_1^{(m)}(z_{21}/2, \tau/2)\\&\qquad + \sum_{m\geq 1}\frac{2^{-m-1}}{m!}Z_{V_L^T}((\theta u)[m]v, \tau)\big\{2^{m+1}P_1^{(m)}(z_{21}, \tau) - P_1^{(m)}(z_{21}/2, \tau/2)\big\}\\&\quad = \sum_{m\geq 1}\frac{1}{m!}P_1^{(m)}(z_{21}, \tau)Z_{V_L^T}((\theta u)[m]v, \tau)\\&\qquad + \sum_{m\geq 1}\frac{2^{-m-1}}{m!}P_1^{(m)}(z_{21}/2, \tau/2)(Z_{V_L^T}((u)[m]v, \tau) - Z_{V_L^T}((\theta u)[m]v, \tau)).\end{aligned}$$

For $n \in \mathbb{Z}^+$, comparing coefficients of z_{12}^{n-1} above gives us

$$\begin{aligned} Z_{V_L^T}(u[-n]v,\tau) = {} & \sum_{m\geq 1}(-1)^{m+1}\binom{m+n-1}{m}E_{m+n}(\tau)Z_{V_L^T}((\theta u)[m]v,\tau) \\ & + \sum_{m\geq 1}(-1)^{m+1}2^{-(m+n)}\binom{m+n-1}{m}E_{m+n}(\tau/2) \\ & \times \big(Z_{V_L^T}(u[m]v,\tau) - Z_{V_L^T}((\theta u)[m]v,\tau)\big). \qquad \blacksquare \end{aligned}$$

3.3 Twisted reduction theorems

For $u, v \in V_L^-$, we can define modified twisted 2-point functions on the torus

$$F_{(V_L^T)^+}((u,z_1),(v,z_2),\tau) := \operatorname{Tr}_{(V_L^T)^+} Y\big(q_1^{L(0)}u,q_1\big)Y\big(q_2^{L(0)}v,q_2\big)q^{L(0)-c/24}.$$

In this subsection, we obtain expressions for the above modified twisted 2-point functions on the torus in the form of theorems analogous to the reduction theorem in Section 3.1. Before we prove the main theorems, we shall recall the following lemma from [7, Lemma 12].

Lemma 3.4. *Let the finite-dimensional linear space $X = X_1 \oplus X_2$ decompose as indicated, and let $f, g \in \operatorname{End}(X)$ be a pair of endomorphisms mapping $X_1 \to X_2$ and $X_2 \to X_1$. Then the following statements hold:*

1. *We have $\operatorname{Tr}_{X_2} fg = \operatorname{Tr}_{X_1} gf$.*
2. *We have $\operatorname{Tr}_{X_1}(fg + gf) = \operatorname{Tr}_X fg$.*
3. *If $\operatorname{Tr}_X fg = 0$ then we have $\operatorname{Tr}_{X_1} fg = -\operatorname{Tr}_{X_1} gf$.*

Theorem 3.5. *For $u, v \in V_L^-$,*

$$\begin{aligned} & F_{(V_L^T)^+}((u,z_1),(v,z_2),\tau) \\ & \quad = \frac{1}{2}\sum_{p=0,1}\sum_{m\geq 0}\frac{2^{-m}}{m!}Q_1^{(m)}(z_{21}/2 + p\pi \mathrm{i},\tau/2)Z_{(V_L^T)^+}((\theta^p u)[m]v,\tau) \\ & \qquad + \frac{1}{2}\sum_{p=0,1}\sum_{m\geq 1}\frac{2^{-m}}{2(m!)}\big(P_1^{(m)}(z_{21}/2 + p\pi \mathrm{i},\tau/2) - Q_1^{(m)}(z_{21}/2 + p\pi \mathrm{i},\tau/2)\big) \\ & \qquad \times Z_{V_L^T}((\theta^p(u))[m]v,\tau). \end{aligned}$$

Proof. For $u, v \in V_L^-$ and $m, n \in \frac{1}{2}\mathbb{Z}$,

$$u(m)\colon\ \big(V_L^T\big)^- \to \big(V_L^T\big)^+, \qquad v(n)\colon\ \big(V_L^T\big)^+ \to \big(V_L^T\big)^-,$$

and when $u \in V_{L_{(k)}}$,

$$F_{(V_L^T)^+}((u,z_1),(v,z_2),\tau) := \sum_{n\in\frac{1}{2}\mathbb{Z}} q_1^{k-n-1}\operatorname{Tr}_{(V_L^T)^+}\big\{u(n)Y\big(q_2^{L(0)}v,q_2\big)\big\}q^{L(0)-c/24}.$$

Using the commutator formula (3.1), and a similar argument as before, where $r := n-k+1$, we have

$$\operatorname{Tr}_{(V_L^T)^+}\big\{u(n)Y\big(q_2^{L(0)}v,q_2\big)\big\}q^{L(0)-c/24}$$

$$
\begin{aligned}
&= \frac{1}{2} q_2^r \sum_{p=0,1} \sum_{m\geq 0} (-1)^{2pr} \frac{r^m}{m!} Z_{(V_L^T)^+}((\theta^p(u))[m]v,\tau) \\
&\quad + \mathrm{Tr}_{(V_L^T)^+} \{ Y(q_2^{L(0)}v, q_2) u(n) \} q^{L(0)-c/24} \\
&= \frac{1}{2} q_2^r \sum_{p=0,1} \sum_{m\geq 0} (-1)^{2pr} \frac{r^m}{m!} Z_{(V_L^T)^+}((\theta^p(u))[m]v,\tau) \\
&\quad + q^r \, \mathrm{Tr}_{(V_L^T)^+} \{ Y(q_2^{L(0)}v, q_2) q^{L(0)-c/24} u(n) \}.
\end{aligned}
$$

Using Lemma 3.4, we have

$$
\begin{aligned}
&= \frac{1}{2} q_2^r \sum_{p=0,1} \sum_{m\geq 0} (-1)^{2pr} \frac{r^m}{m!} Z_{(V_L^T)^+}((\theta^p(u))[m]v,\tau) \\
&\quad - q^r \, \mathrm{Tr}_{(V_L^T)^+} \{ u(n) Y(q_2^{L(0)}v, q_2) q^{L(0)-c/24} \} \\
&\quad + q^r \, \mathrm{Tr}_{(V_L^T)} \{ Y(q_2^{L(0)}v, q_2) q^{L(0)-c/24} u(n) \}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\mathrm{Tr}_{(V_L^T)^+} \{ u(n) Y(q_2^{L(0)}v, q_2) q^{L(0)-c/24} \} \\
&\quad = \frac{1}{2} \sum_{p=0,1} \frac{q_2^r}{1+q^r} \sum_{m\geq 0} (-1)^{2pr} \frac{r^m}{m!} Z_{(V_L^T)^+}((\theta^p u)[m]v,\tau) \\
&\quad + \frac{q^r}{1+q^r} \mathrm{Tr}_{(V_L^T)} \{ Y(q_2^{L(0)}v, q_2) q^{L(0)-c/24} u(n) \}.
\end{aligned}
$$

Using (3.2),

$$
\begin{aligned}
&= \frac{1}{2} \sum_{p=0,1} \frac{q_2^r}{1+q^r} \sum_{m\geq 0} (-1)^{2pr} \frac{r^m}{m!} Z_{(V_L^T)^+}((\theta^p(u))[m]v,\tau) \\
&\quad + \frac{1}{1+q^r} \Bigg\{ \mathrm{Tr}_{V_L^T} \big(u(n) Y(q_2^{L(0)}v, q_2) q^{L(0)-c/24} \big) \\
&\quad - \frac{1}{2} q_2^r \sum_{p=0,1} \sum_{m\geq 0} (-1)^{2pr} \frac{r^m}{m!} Z_{V_L^T}((\theta^p(u))[m]v,\tau) \Bigg\}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\mathrm{Tr}_{(V_L^T)^+} \{ u(n) Y(q_2^{L(0)}v, q_2) q^{L(0)-c/24} \} \\
&\quad = \frac{1}{2} \sum_{p=0,1} \frac{q_2^r}{1+q^r} \sum_{m\geq 0} (-1)^{2pr} \frac{r^m}{m!} \{ Z_{(V_L^T)^+}((\theta^p(u))[m]v,\tau) - Z_{V_L^T}((\theta^p(u))[m]v,\tau) \} \\
&\quad + \frac{1}{1+q^r} \mathrm{Tr}_{V_L^T} \big(u(n) Y(q_2^{L(0)}v, q_2) q^{L(0)-c/24} \big).
\end{aligned}
$$

Note that when $u, v \in V_L^-$, $\mathrm{Tr}_{V_L^T} o(u)o(v)q^{L(0)-c/24} = 0$ since both $o(u)$ and $o(v)$ are zero as we set for $\alpha \in \mathfrak{h}$, $\alpha(n) = 0$ for $n \in \mathbb{Z}$ (see [5, equation (9.1.13)]). Since $u, v \in V_L^-$, we have

$$
\mathrm{Tr}_{(V_L^T)^+} o(u)o(v)q^{L(0)-c/24} = \mathrm{Tr}_{V_L^T} o(u)o(v)q^{L(0)-c/24} = 0. \tag{3.6}
$$

When $r \neq 0$, using (3.3) we have

$$
\mathrm{Tr}_{(V_L^T)^+} \{ u(n) Y(q_2^{L(0)}v, q_2) q^{L(0)-c/24} \}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{p=0,1}\frac{q_2^r}{1+q^r}\left(\frac{1-q^r}{1-q^r}\right)\sum_{m\geq 0}(-1)^{2pr}\frac{r^m}{m!}\\
&\quad\times\left\{Z_{(V_L^T)^+}((\theta^p(u))[m]v,\tau)-Z_{V_L^T}((\theta^p(u))[m]v,\tau)\right\}\\
&\qquad+\frac{1}{1+q^r}\left(\frac{q_2^r}{2(1-q^r)}\right)\sum_{p=0,1}\sum_{m\geq 1}(-1)^{2pr}\frac{r^m}{m!}Z_{V_L^T}((\theta^p(u))[m]v,\tau).
\end{aligned}
$$

That is, we have

$$
\begin{aligned}
&\operatorname{Tr}_{(V_L^T)^+}\left\{u(n)Y\left(q_2^{L(0)}v,q_2\right)q^{L(0)-c/24}\right\}\\
&\quad=\frac{1}{2}\sum_{p=0,1}\frac{q_2^r}{1+q^r}\sum_{m\geq 0}(-1)^{2pr}\frac{r^m}{m!}Z_{(V_L^T)^+}((\theta^p u)[m]v,\tau)\\
&\qquad+\frac{1}{2}\sum_{p=0,1}\frac{q_2^rq^r}{1-q^{2r}}\sum_{m\geq 1}(-1)^{2pr}\frac{r^m}{m!}Z_{V_L^T}((\theta^p u)[m]v,\tau).
\end{aligned}
$$

Returning to the 2-point function we defined above,

$$
\begin{aligned}
&F_{(V_L^T)^+}((u,z_1),(v,z_2),\tau)\\
&\quad=\sum_{n\in\frac{1}{2}\mathbb{Z}}q_1^{k-n-1}\operatorname{Tr}_{(V_L^T)^+}\left\{u(n)Y\left(q_2^{L(0)}v,q_2\right)\right\}q^{L(0)-c/24}\\
&\quad=\frac{1}{2}\sum_{p=0,1}\sum_{r\neq 0,r\in\frac{1}{2}\mathbb{Z}}(-1)^{2pr}\frac{q_{21}^r}{1+q^r}\sum_{m\geq 0}\frac{r^m}{m!}Z_{(V_L^T)^+}((\theta^p u)[m]v,\tau)\\
&\qquad+\frac{1}{2}\sum_{p=0,1}\sum_{r\neq 0,r\in\frac{1}{2}\mathbb{Z}}(-1)^{2pr}\frac{q_{21}^rq^r}{1-q^{2r}}\sum_{m\geq 1}\frac{r^m}{m!}Z_{V_L^T}((\theta^p(u))[m]v,\tau)\\
&\qquad+\operatorname{Tr}_{(V_L^T)^+}o(u)o(v)q^{L(0)-c/24}.
\end{aligned}
$$

Using (3.6), we have

$$
\begin{aligned}
&=\frac{1}{2}\sum_{p=0,1}\sum_{r\in\frac{1}{2}\mathbb{Z}}(-1)^{2pr}\frac{q_{21}^r}{1+q^r}\sum_{m\geq 0}\frac{r^m}{m!}Z_{(V_L^T)^+}((\theta^p u)[m]v,\tau)-\frac{1}{2}\sum_{p=0,1}\frac{1}{2}Z_{(V_L^T)^+}((\theta^p u)[0]v,\tau)\\
&\quad+\frac{1}{2}\sum_{p=0,1}\sum_{r\neq 0,r\in\frac{1}{2}\mathbb{Z}}(-1)^{2pr}\frac{q_{21}^rq^r}{1-q^{2r}}\sum_{m\geq 1}\frac{r^m}{m!}Z_{V_L^T}((\theta^p(u))[m]v,\tau)\\
&=\frac{1}{2}\sum_{p=0,1}\sum_{m\geq 0}\frac{2^{-m}}{m!}Q_1^{(m)}(z_{21}/2+p\pi\mathrm{i},\tau/2)Z_{(V_L^T)^+}((\theta^p u)[m]v,\tau)\\
&\quad-\frac{1}{2}\sum_{p=0,1}\frac{1}{2}Z_{(V_L^T)^+}((\theta^p u)[0]v,\tau)\\
&\quad+\frac{1}{2}\sum_{p=0,1}\frac{1}{2}\sum_{r\neq 0,r\in\frac{1}{2}\mathbb{Z}}(-1)^{2pr}r^mq_{21}^r\left(\frac{1}{1-q^r}-\frac{1}{1+q^r}\right)\sum_{m\geq 1}\frac{1}{m!}Z_{V_L^T}((\theta^p(u))[m]v,\tau)\\
&=\frac{1}{2}\sum_{p=0,1}\sum_{m\geq 0}\frac{2^{-m}}{m!}Q_1^{(m)}(z_{21}/2+p\pi\mathrm{i},\tau/2)Z_{(V_L^T)^+}((\theta^p u)[m]v,\tau)\\
&\quad+\frac{1}{2}\sum_{p=0,1}\sum_{m\geq 1}\frac{2^{-m}}{2(m!)}\left(P_1^{(m)}(z_{21}/2+p\pi\mathrm{i},\tau/2)-Q_1^{(m)}(z_{21}/2+p\pi\mathrm{i},\tau/2)\right)\\
&\qquad\times Z_{V_L^T}((\theta^p(u))[m]v,\tau).
\end{aligned}
$$

Now, using (3.6), we have

$$\begin{aligned}
&= \frac{1}{2}\sum_{p=0,1}\sum_{m\geq 0}\frac{2^{-m}}{m!}Q_1^{(m)}(z_{21}/2+p\pi\mathrm{i},\tau/2)Z_{(V_L^T)^+}((\theta^p u)[m]v,\tau)\\
&\quad+\frac{1}{2}\sum_{p=0,1}\sum_{m\geq 1}\frac{2^{-m}}{2(m!)}\big(P_1^{(m)}(z_{21}/2+p\pi\mathrm{i},\tau/2)-Q_1^{(m)}(z_{21}/2+p\pi\mathrm{i},\tau/2)\big)\\
&\quad\times Z_{V_L^T}((\theta^p(u))[m]v,\tau). \qquad \blacksquare
\end{aligned}$$

Theorem 3.6. *For $u,v\in V_L^-$, the modified twisted 2-point functions can be rewritten as*

$$\mathrm{Tr}_{(V_L^T)^+}Y\big(q_1^{L(0)}u,q_1\big)Y\big(q_2^{L(0)}v,q_2\big)q^{L(0)-c/24}=Z_{(V_L^T)^+}(Y_{\mathbb{Z}}[u,z_{12}]v,\tau).$$

Proof. Similar to the proof of Theorem 3.2. ■

3.4 Twisted recursion formulas

In this subsection, we obtain recursion formulas for trace functions of states in V_L^+ traced over $\left(V_L^T\right)^+$ analogous to the recursion formula obtained in Section 3.2. Before we prove the main theorem, we shall recall the following lemma from [7, Lemma 3].

Lemma 3.7. *The following hold:*

(1) *Laurent series expansions for $P_1(z,\tau)$ and $Q_1(z,\tau)$ are as in (2.1) and (2.2), respectively.*

(2) *With respect to the variable z, $P_1^{(m)}(z,\tau)$ and $Q_1^{(m)}(z,\tau)$ for $m\geq 0$ are odd functions if m is even and even functions if m is odd.*

(3) *We have $Q_1(z,\tau)=2P_1(z,2\tau)-P_1(z,\tau)$ and $Q_1(z,\tau)$ is an elliptic function for the lattice $2\pi\mathrm{i}(\mathbb{Z}\oplus 2\tau\mathbb{Z})$.*

Theorem 3.8. *For $u,v\in V_L^-$, $n\geq 1$,*

$$\begin{aligned}
Z_{(V_L^T)^+}(u[-n]v,\tau)&=\overline{F}_{0+n}(\tau)Z_{(V_L^T)^+}(u[0]v,\tau)+\sum_{m\geq 1}\frac{1}{m}\overline{F}_{m+n}(\tau)Z_{(V_L^T)^+}(u[m]v,\tau)\\
&\quad+\frac{1}{2}\sum_{m\geq 1}\frac{1}{m}\left(\overline{E}_{m+n}(\tau)-\overline{F}_{m+n}(\tau)\right)Z_{V_L^T}(u[m]v,\tau).
\end{aligned}$$

Proof. We have

$$Q_1^{(m)}(z/2+\pi\mathrm{i},\tau/2)=2^{m+1}Q_1^{(m)}(z,\tau)-Q_1^{(m)}(z/2,\tau/2). \tag{3.7}$$

Using Theorem 3.6, we have

$$F_{(V_L^T)^+}((u,z_1),(v,z_2),\tau)=Z_{(V_L^T)^+}(Y_{\mathbb{Z}}[u,z_{12}]v,\tau)$$

and from Theorem 3.5,

$$\begin{aligned}
&Z_{(V_L^T)^+}(Y_{\mathbb{Z}}[u,z_{12}]v,\tau)\\
&\quad=\frac{1}{2}\sum_{p=0,1}\sum_{m\geq 0}\frac{2^{-m}}{m!}Q_1^{(m)}(z_{21}/2+p\pi\mathrm{i},\tau/2)Z_{(V_L^T)^+}((\theta^p u)[m]v,\tau)\\
&\quad\quad+\frac{1}{2}\sum_{p=0,1}\sum_{m\geq 1}\frac{2^{-m}}{2(m!)}\big(P_1^{(m)}(z_{21}/2+p\pi\mathrm{i},\tau/2)-Q_1^{(m)}(z_{21}/2+p\pi\mathrm{i},\tau/2)\big)
\end{aligned}$$

$$\times Z_{V_L^T}((\theta^p(u))[m]v,\tau).$$

Using equations (3.5), (3.7), we shall rewrite the above equation as

$$\begin{aligned}
&Z_{(V_L^T)^+}(Y_{\mathbb{Z}}[u,z_{12}]v,\tau)\\
&\quad=\frac{1}{2}\sum_{m\geq 0}\frac{2^{-m}}{m!}Q_1^{(m)}(z_{21}/2,\tau/2)Z_{(V_L^T)^+}(u[m]v,\tau)\\
&\qquad+\frac{1}{2}\sum_{m\geq 0}\frac{2^{-m}}{m!}Q_1^{(m)}(z_{21}/2+\pi\mathrm{i},\tau/2)Z_{(V_L^T)^+}((\theta u)[m]v,\tau)\\
&\qquad+\frac{1}{2}\sum_{m\geq 1}\frac{2^{-m}}{2(m!)}\big(P_1^{(m)}(z_{21}/2,\tau/2)-Q_1^{(m)}(z_{21}/2,\tau/2)\big)Z_{V_L^T}(u[m]v,\tau)\\
&\qquad+\frac{1}{2}\sum_{m\geq 1}\frac{2^{-m}}{2(m!)}\big(P_1^{(m)}(z_{21}/2+\pi\mathrm{i},\tau/2)-Q_1^{(m)}(z_{21}/2+\pi\mathrm{i},\tau/2)\big)Z_{V_L^T}((\theta u)[m]v,\tau).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&Z_{(V_L^T)^+}(Y_{\mathbb{Z}}[u,z_{12}]v,\tau)\\
&\quad=\frac{1}{2}\sum_{m\geq 0}\frac{2^{-m}}{m!}Q_1^{(m)}(z_{21}/2,\tau/2)\{Z_{(V_L^T)^+}(u[m]v,\tau)-Z_{(V_L^T)^+}((\theta u)[m]v,\tau)\}\\
&\qquad+\frac{1}{2}\sum_{m\geq 0}\frac{2}{m!}Q_1^{(m)}(z_{21},\tau)Z_{(V_L^T)^+}((\theta u)[m]v,\tau)\\
&\qquad+\frac{1}{2}\sum_{m\geq 1}\frac{2^{-m}}{2(m!)}\big(P_1^{(m)}(z_{21}/2,\tau/2)-Q_1^{(m)}(z_{21}/2,\tau/2)\big)\\
&\qquad\times\{Z_{V_L^T}(u[m]v,\tau)-Z_{V_L^T}((\theta u)[m]v,\tau)\}\\
&\qquad+\frac{1}{2}\sum_{m\geq 1}\frac{2}{2(m!)}\big(P_1^{(m)}(z_{21},\tau)-Q_1^{(m)}(z_{21},\tau)\big)Z_{V_L^T}((\theta u)[m]v,\tau).
\end{aligned}$$

Now using Lemma 3.7, we shall rewrite the above as

$$\begin{aligned}
&Z_{(V_L^T)^+}(Y_{\mathbb{Z}}[u,z_{12}]v,\tau)\\
&\quad=\frac{1}{2}\sum_{m\geq 0}\frac{(-1)^{m+1}2^{-m}}{m!}Q_1^{(m)}(z_{12}/2,\tau/2)\{Z_{(V_L^T)^+}(u[m]v,\tau)-Z_{(V_L^T)^+}((\theta u)[m]v,\tau)\}\\
&\qquad+\frac{1}{2}\sum_{m\geq 0}\frac{(-1)^{m+1}2}{m!}Q_1^{(m)}(z_{12},\tau)Z_{(V_L^T)^+}((\theta u)[m]v,\tau)\\
&\qquad+\frac{1}{2}\sum_{m\geq 1}\frac{(-1)^{m+1}2^{-m}}{2(m!)}\big(P_1^{(m)}(z_{12}/2,\tau/2)-Q_1^{(m)}(z_{12}/2,\tau/2)\big)\\
&\qquad\times\{Z_{V_L^T}(u[m]v,\tau)-Z_{V_L^T}((\theta u)[m]v,\tau)\}\\
&\qquad+\frac{1}{2}\sum_{m\geq 1}\frac{(-1)^{m+1}2}{2(m!)}\big(P_1^{(m)}(z_{12},\tau)-Q_1^{(m)}(z_{12},\tau)\big)Z_{V_L^T}((\theta u)[m]v,\tau).
\end{aligned}$$

Comparing the coefficients of z_{12}^{n-1} above (where $n\in\mathbb{Z}^+$), we have

$$Z_{(V_L^T)^+}(u[-n]v,\tau)$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{m\geq 0}(-1)^{m+1}2^{-m-n+1}\binom{m+n-1}{m}F_{m+n}(\tau/2)\\
&\quad \times \{Z_{(V_L^T)^+}(u[m]v,\tau) - Z_{(V_L^T)^+}((\theta u)[m]v,\tau)\}\\
&\quad + \frac{1}{2}\sum_{m\geq 0}(-1)^{m+1}2\binom{m+n-1}{m}F_{m+n}(\tau)Z_{(V_L^T)^+}((\theta u)[m]v,\tau)\\
&\quad + \frac{1}{2}\sum_{m\geq 1}\frac{(-1)^{m+1}2^{-m-n+1}}{2}\binom{m+n-1}{m}(E_{m+n}(\tau/2) - F_{m+n}(\tau/2))\\
&\quad \times \{Z_{V_L^T}((u)[m]v,\tau) - Z_{V_L^T}((\theta u)[m]v,\tau)\}\\
&\quad + \frac{1}{2}\sum_{m\geq 1}\frac{(-1)^{m+1}2}{2}\binom{m+n-1}{m}(E_{m+n}(\tau) - F_{m+n}(\tau))Z_{V_L^T}((\theta u)[m]v,\tau).
\end{aligned}
$$

Since $u \in V_L^-$, we have $\theta u = -u$, and hence

$$
\begin{aligned}
&Z_{(V_L^T)^+}(u[-n]v,\tau)\\
&\quad = \sum_{m\geq 0}(-1)^{m+1}2^{-m-n+1}\binom{m+n-1}{m}F_{m+n}(\tau/2)Z_{(V_L^T)^+}(u[m]v,\tau)\\
&\quad - \sum_{m\geq 0}(-1)^{m+1}\binom{m+n-1}{m}F_{m+n}(\tau)Z_{(V_L^T)^+}(u[m]v,\tau)\\
&\quad + \frac{1}{2}\sum_{m\geq 1}(-1)^{m+1}2^{-m-n+1}\binom{m+n-1}{m}(E_{m+n}(\tau/2) - F_{m+n}(\tau/2))Z_{V_L^T}(u[m]v,\tau)\\
&\quad - \frac{1}{2}\sum_{m\geq 1}(-1)^{m+1}\binom{m+n-1}{m}(E_{m+n}(\tau) - F_{m+n}(\tau))Z_{V_L^T}(u[m]v,\tau).
\end{aligned}
$$

Using the notation for renormalized Eisenstein series, we have

$$
\begin{aligned}
Z_{(V_L^T)^+}(u[-n]v,\tau) = &\sum_{m\geq 1}\frac{1}{m}\big\{2^{-m-n+1}\hat{F}_{m+n}(\tau/2) - \hat{F}_{m+n}(\tau)\big\}Z_{(V_L^T)^+}(u[m]v,\tau)\\
&+ \big\{2^{-n+1}\hat{F}_{0+n}(\tau/2) - \hat{F}_{0+n}(\tau)\big\}Z_{(V_L^T)^+}(u[0]v,\tau)\\
&+ \frac{1}{2}\sum_{m\geq 1}\frac{1}{m}\big\{2^{-m-n+1}\hat{E}_{m+n}(\tau/2) - \hat{E}_{m+n}(\tau)\big\}Z_{V_L^T}(u[m]v,\tau)\\
&- \frac{1}{2}\sum_{m\geq 1}\frac{1}{m}\big\{2^{-m-n+1}\hat{F}_{m+n}(\tau/2) - \hat{F}_{m+n}(\tau)\big\}Z_{V_L^T}(u[m]v,\tau).
\end{aligned}
$$

Thus, the above can rewritten as

$$
\begin{aligned}
Z_{(V_L^T)^+}(u[-n]v,\tau) = &\overline{F}_{0+n}(\tau)Z_{(V_L^T)^+}(u[0]v,\tau) + \sum_{m\geq 1}\frac{1}{m}\overline{F}_{m+n}(\tau)Z_{(V_L^T)^+}(u[m]v,\tau)\\
&+ \frac{1}{2}\sum_{m\geq 1}\frac{1}{m}\big(\overline{E}_{m+n}(\tau) - \overline{F}_{m+n}(\tau)\big)Z_{V_L^T}(u[m]v,\tau). \qquad \blacksquare
\end{aligned}
$$

4 Calculations in the twisted space

In this section, we perform certain computations as applications of the theorems we obtained in the previous section.

4.1 In the twisted module

Here we derive formulas corresponding to the twisted space using the recursion formula we obtained in Section 3.2.

Theorem 4.1. *For positive integers $n_j \geq 1$, corresponding to the state $u := h_{i_1}[-n_1] \cdots h_{i_p}[-n_p]e_\alpha$, where $e_\alpha = f_\alpha$ or g_α according as p is even or odd respectively, we have*

$$Z_{(V_L^T)}(u,\tau) = \left\{ \sum_{\sigma} \prod_{(rs)(t)} \delta_{i_r,i_s} \tilde{\overline{E}}_{n_t} \overline{E}_{n_r+n_s}(\tau) \right\} Z_{(V_L^T)}(f_\alpha,\tau),$$

where $\tilde{\overline{E}}_{n_k}(\tau) := \langle h_{i_k}, \alpha\rangle \overline{E}_{n_k}(\tau)$ where $\langle\cdot,\cdot\rangle$ is naturally extended from L to $\mathfrak{h} = L \otimes \mathbb{C}$ and σ ranges over all involutions of the index set $S_p = \{1,2,\dots,p\}$ and (rs), (t) range over the 2-cycles and 1-cycles respectively in the decomposition of σ in S_p as a product of disjoint 2-cycles and 1-cycles.

Proof. Let $v := h_{i_2}[-n_2] \cdots h_{i_p}[-n_p]e_\alpha$, then Theorem 3.3 gives

$$Z_{(V_L^T)}(u,\tau) = \sum_{m\geq 1} \left(\frac{1}{m}\right) \overline{E}_{m+n}(\tau) Z_{V_L^T}(h_{i_1}[m]v,\tau).$$

Since

$$\sum_{m\geq 1} h_{i_1}[m]v = \sum_{j=2}^{p} \delta_{i_1,i_j} n_j (v\backslash j),$$

where $v\backslash j$ denotes the state obtained from v by deleting the operator $h_{i_j}[-n_j]$, we have

$$\begin{aligned} Z_{V_L^T}(u,\tau) = {} & \sum_{j=2}^{p} \delta_{i_1,i_j} \overline{E}_{n_j+n_1}(\tau) Z_{V_L^T}(v\backslash j,\tau) \\ & + \langle h_{i_1},\alpha\rangle \left\{ 2^{-(n_1-1)} E_{n_1}(\tau/2) - E_{n_1}(\tau) \right\} Z_{V_L^T}(v',\tau), \end{aligned}$$

where v' denotes the state $v' := h_{i_2}[-n_2] \cdots h_{i_p}[-n_p]e'_\alpha$ and $e'_\alpha = g_\alpha$ or f_α according as $e_\alpha = f_\alpha$ or g_α, respectively. Thus similarly as in [7] one can easily obtain using induction that

$$Z_{(V_L^T)}(u,\tau) = \left(\sum_{\sigma} \prod_{(rs)(t)} \delta_{i_r,i_s} \tilde{\overline{E}}_{n_t} \overline{E}_{n_r+n_s}(\tau) \right) Z_{V_L^T}(f_\alpha,\tau). \qquad \blacksquare$$

An immediate corollary of the above theorem is the following.

Corollary 4.2. *For positive integers $n_j \geq 1$, and even p, corresponding to the state $u := h_{i_1}[-n_1] \cdots h_{i_p}[-n_p]\mathbf{1}$, we have*

$$Z_{(V_L^T)}(u,\tau) = \left\{ \sum_{\sigma} \prod_{(rs)} \delta_{i_r,i_s} \overline{E}_{n_r+n_s}(\tau) \right\} Z_{(V_L^T)}(\mathbf{1},\tau),$$

where σ ranges over all fixed point free involutions of S_p and (rs) ranges over all 2-cycles in the decomposition of σ in S_p.

Proof. Evaluate the above theorem at $\alpha = 0$. $\blacksquare$

4.2 Outside 2L

Here we obtain formulas corresponding to certain states in V_L^+ whose lattice part comes from an element outside $2L$.

Theorem 4.3. *If $\alpha \in L\backslash 2L$, for positive integers $n_j \geq 1$, corresponding to the state*

$$u \coloneqq h_{i_1}[-n_1]\cdots h_{i_p}[-n_p]e_\alpha,$$

where $e_\alpha = f_\alpha$ or g_α according as p is even or odd, respectively, we have

$$Z_{(V_L^T)^+}(u,\tau) = \Big\{ \sum_{\sigma} \prod_{(rs)(t)} \delta_{i_r,i_s} \tilde{\overline{F}}_{n_t}(\tau)\overline{F}_{n_r+n_s}(\tau) \Big\} Z_{(V_L^T)^+}(f_\alpha,\tau)$$

where $\tilde{\overline{F}}_{n_k}(\tau) \coloneqq \langle h_{i_k},\alpha\rangle \overline{F}_{n_k}(\tau)$ where $\langle\cdot,\cdot\rangle$ is naturally extended from L to $\mathfrak{h} = L\otimes\mathbb{C}$ and σ ranges over all involutions of the index set $S_p = \{1,2,\ldots,p\}$ and (rs), (t) range over the 2-cycles and 1-cycles respectively in the decomposition of σ in S_p as a product of disjoint 2-cycles and 1-cycles.

Proof. Let $v \coloneqq h_{i_2}[-n_2]\cdots h_{i_p}[-n_p]e_\alpha$, $v' \coloneqq h_{i_2}[-n_2]\cdots h_{i_p}[-n_p]e'_\alpha$, where $e'_\alpha \coloneqq g_\alpha$ or f_α according as $e_\alpha = f_\alpha$ or g_α, respectively. From Theorem 3.8, we have

$$\begin{aligned} Z_{(V_L^T)^+}(u[-n]v,\tau) &= \overline{F}_{0+n}(\tau)Z_{(V_L^T)^+}(u[0]v,\tau) + \sum_{m\geq 1}\frac{1}{m}\overline{F}_{m+n}(\tau)Z_{(V_L^T)^+}(u[m]v,\tau) \\ &\quad + \frac{1}{2}\sum_{m\geq 1}\frac{1}{m}\left(\overline{E}_{m+n}(\tau) - \overline{F}_{m+n}(\tau)\right) Z_{V_L^T}(u[m]v,\tau). \end{aligned}$$

Thus,

$$\begin{aligned} Z_{(V_L^T)^+}(u,\tau) &= \overline{F}_{n_1+0}(\tau)Z_{(V_L^T)^+}(h_{i_1}[0]v,\tau) + \sum_{m\geq 1}\left(\frac{1}{m}\right)\overline{F}_{m+n_1}(\tau)Z_{(V_L^T)^+}(h_{i_1}[m]v,\tau) \\ &\quad + \frac{1}{2}\sum_{m\geq 1}\left(\frac{1}{m}\right)\left\{\overline{E}_{m+n_1}(\tau) - \overline{F}_{m+n_1}(\tau)\right\} Z_{V_L^T}(h_{i_1}[m]v,\tau). \end{aligned}$$

If $\alpha \notin 2L$, since $Z_{V_L^T}(h_{i_1}[m]v,\tau) = 0$ due to the structure of T, we have

$$\begin{aligned} Z_{(V_L^T)^+}(u,\tau) &= \overline{F}_{n_1+0}(\tau)Z_{(V_L^T)^+}(h_{i_1}[0]v,\tau) + \sum_{m\geq 1}\left(\frac{1}{m}\right)\overline{F}_{m+n_1}(\tau)Z_{(V_L^T)^+}(h_{i_1}[m]v,\tau), \\ Z_{(V_L^T)^+}(u,\tau) &= \left\{2^{-(n_1-1)}F_{n_1}(\tau/2) - F_{n_1}(\tau)\right\}Z_{(V_L^T)^+}(h_{i_1}[0]v,\tau) \\ &\quad + \sum_{j=2}^{p}\left(\frac{1}{n_j}\right)\left\{2^{-(n_j+n_1-1)}\hat{F}_{n_j+n_1}(\tau/2) - \hat{F}_{n_j+n_1}(\tau)\right\}Z_{(V_L^T)^+}(h_{i_1}[n_j]v,\tau), \\ Z_{(V_L^T)^+}(u,\tau) &= \langle h_{i_1},\alpha\rangle\left\{2^{-(n_1-1)}F_{n_1}(\tau/2) - F_{n_1}(\tau)\right\}Z_{(V_L^T)^+}(v',\tau) \\ &\quad + \sum_{j=2}^{p}\delta_{i_1,i_j}\left\{2^{-(n_1+n_j-1)}\hat{F}_{n_j+n_1}(\tau/2) - \hat{F}_{n_j+n_1}(\tau)\right\}Z_{(V_L^T)^+}(v\backslash j,\tau), \end{aligned}$$

which can further be rewritten as

$$Z_{(V_L^T)^+}(u,\tau) = \tilde{\overline{F}}_{n_1}(\tau)Z_{(V_L^T)^+}(v',\tau) + \sum_{j=2}^{p}\delta_{i_1,i_j}\overline{F}_{n_j+n_1}(\tau)Z_{(V_L^T)^+}(v\backslash j,\tau).$$

Thus, we have

$$Z_{(V_L^T)^+}(u,\tau) = \Big\{ \sum_{\sigma} \prod_{(rs)(t)} \delta_{i_r,i_s} \tilde{\overline{F}}_{n_t}(\tau)\overline{F}_{n_r+n_s}(\tau) \Big\} Z_{(V_L^T)^+}(f_\alpha,\tau). \qquad \blacksquare$$

4.3 At Heisenberg states

Here we obtain formulas corresponding to the Heisenberg states in V_L^+.

Theorem 4.4. *For positive integers $n_j \geq 1$ and p even, corresponding to the state $u \coloneqq h_{i_1}[-n_1] \cdots h_{i_p}[-n_p]\mathbf{1}$, we have*

$$Z_{(V_L^T)^+}(u,\tau) = \frac{1}{2} Z_{V_L^T}(u,\tau) + \sum_{\sigma\in \mathrm{Inv}_0(\underline{p})} \prod_{(rs)} \delta_{i_r,i_s} \overline{F}_{n_r+n_s}(\tau) \left\{ Z_{(V_L^T)^+}(\mathbf{1},\tau) - \frac{1}{2} Z_{V_L^T}(\mathbf{1},\tau) \right\},$$

where $\underline{p} \coloneqq \{1,2,\ldots,p\}$ and $\mathrm{Inv}_0(\underline{p})$ is the set of all fixed-point-free involutions of $\underline{p}$.

Proof. From Theorem 3.8, if $v \coloneqq h_{i_2}[-n_2] \cdots h_{i_p}[-n_p]\mathbf{1}$, then for $n = n_1$, we have

$$\begin{aligned} Z_{(V_L^T)^+}(u,\tau) = {}& \sum_{m\geq 1} \frac{1}{m} \overline{F}_{m+n}(\tau) Z_{(V_L^T)^+}(h_{i_1}[m]v,\tau) + \frac{1}{2} \sum_{m\geq 1} \left(\overline{E}_{m+n}(\tau) - \overline{F}_{m+n}(\tau)\right) \\ & \times Z_{V_L^T}(h_{i_1}[m]v,\tau). \end{aligned}$$

Using Heisenberg relations, we have

$$\begin{aligned} Z_{(V_L^T)^+}(u,\tau) &= \sum_{j=2}^{p} \delta_{i_1,i_j} \overline{F}_{n_1+n_j}(\tau) \left\{ Z_{(V_L^T)^+}(v\backslash j,\tau) - \frac{1}{2} Z_{V_L^T}(v\backslash j,\tau) \right\} \\ &\quad + \frac{1}{2} \sum_{j=2}^{p} \delta_{i_1,i_j} \overline{E}_{n_1+n_j}(\tau) Z_{V_L^T}(v\backslash j,\tau), \\ Z_{(V_L^T)^+}(u,\tau) - \frac{1}{2} Z_{V_L^T}(u,\tau) &= \sum_{j=2}^{p} \delta_{i_1,i_j} \overline{F}_{n_1+n_j}(\tau) \left\{ Z_{(V_L^T)^+}(v\backslash j,\tau) - \frac{1}{2} Z_{V_L^T}(v\backslash j,\tau) \right\}. \end{aligned}$$

Using induction further, we obtain

$$Z_{(V_L^T)^+}(u,\tau) - \frac{1}{2} Z_{V_L^T}(u,\tau) = \sum_{\sigma\in \mathrm{Inv}_0(\underline{p})} \prod_{(rs)} \delta_{i_r,i_s} \overline{F}_{n_r+n_s}(\tau) \left\{ Z_{(V_L^T)^+}(\mathbf{1},\tau) - \frac{1}{2} Z_{V_L^T}(\mathbf{1},\tau) \right\}. \qquad \blacksquare$$

4.4 Inside 2L

Here we obtain formulas corresponding to certain states in V_L^+ whose lattice part comes from an element inside $2L$.

Theorem 4.5. *For positive integers $n_j \geq 1$, $\alpha \in 2L$, corresponding to the state $u \coloneqq h_{i_1}[-n_1] \cdots h_{i_p}[-n_p]e_\alpha$, where e_α is f_α or g_α according as p is even or odd respectively, we have*

$$\begin{aligned} Z_{(V_L^T)^+}(u,\tau) = {}& \left(\sum_{\sigma} \prod_{(rs)(t)} \delta_{i_r,i_s} \tilde{\overline{F}}_{n_t}(\tau) \overline{F}_{n_r+n_s}(\tau) \right) \left\{ Z_{(V_L^T)^+}(f_\alpha,\tau) - \frac{1}{2} Z_{V_L^T}(f_\alpha,\tau) \right\} \\ & + \frac{1}{2} Z_{V_L^T}(u,\tau), \end{aligned}$$

where σ ranges over all involutions of the index set $S_p = \{1,2,\ldots,p\}$ and (rs), (t) range over the 2-cycles and 1-cycles respectively in the decomposition of σ in S_p as a product of disjoint 2-cycles and 1-cycles.

Proof. For $u, v \in V_L^-$, from Theorem 3.8, we have

$$\begin{aligned}
Z_{(V_L^T)^+}(u[-n]v,\tau) &= \overline{F}_{0+n}(\tau) Z_{(V_L^T)^+}(u[0]v,\tau) + \sum_{m\geq 1} \frac{1}{m} \overline{F}_{m+n}(\tau) Z_{(V_L^T)^+}(u[m]v,\tau) \\
&\quad + \frac{1}{2} \sum_{m\geq 1} \frac{1}{m} \left(\overline{E}_{m+n}(\tau) - \overline{F}_{m+n}(\tau)\right) Z_{V_L^T}(u[m]v,\tau).
\end{aligned}$$

Let $v := h_{i_2}[-n_2] \cdots h_{i_p}[-n_p] e_\alpha$, $v' := h_{i_2}[-n_2] \cdots h_{i_p}[-n_p] e'_\alpha$, where we use same notation as earlier for e'_α. Then similarly as earlier, we have

$$\begin{aligned}
Z_{(V_L^T)^+}(u,\tau) &= \sum_{j=2}^{p} \delta_{i_1,i_j} \overline{F}_{n_1+n_j}(\tau) \left\{ Z_{(V_L^T)^+}(v\backslash j,\tau) - \frac{1}{2} Z_{V_L^T}(v\backslash j,\tau) \right\} \\
&\quad + \frac{1}{2} \sum_{j=2}^{p} \delta_{i_1,i_j} \overline{E}_{n_1+n_j}(\tau) Z_{V_L^T}(v\backslash j,\tau) + \tilde{\overline{F}}_{n_1}(\tau) Z_{(V_L^T)^+}(v',\tau),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
Z_{(V_L^T)^+}(u,\tau) - \frac{1}{2} Z_{V_L^T}(u,\tau) &= \sum_{j=2}^{p} \delta_{i_1,i_j} \overline{F}_{n_1+n_j}(\tau) \left\{ Z_{(V_L^T)^+}(v\backslash j,\tau) - \frac{1}{2} Z_{V_L^T}(v\backslash j,\tau) \right\} \\
&\quad + \tilde{\overline{F}}_{n_1}(\tau) Z_{(V_L^T)^+}(v',\tau).
\end{aligned}$$

Due to the structure of T, we have $Z_{V_L^T}(v',\tau) = 0$. Thus, we can rewrite the above equation as

$$\begin{aligned}
Z_{(V_L^T)^+}(u,\tau) - \frac{1}{2} Z_{V_L^T}(u,\tau) &= \sum_{j=2}^{p} \delta_{i_1,i_j} \overline{F}_{n_1+n_j}(\tau) \left\{ Z_{(V_L^T)^+}(v\backslash j,\tau) - \frac{1}{2} Z_{V_L^T}(v\backslash j,\tau) \right\} \\
&\quad + \tilde{\overline{F}}_{n_1}(\tau) \left\{ Z_{(V_L^T)^+}(v',\tau) - \frac{1}{2} Z_{V_L^T}(v',\tau) \right\}.
\end{aligned}$$

We thus have

$$\begin{aligned}
&Z_{(V_L^T)^+}(u,\tau) - \frac{1}{2} Z_{V_L^T}(u,\tau) \\
&\quad = \left(\sum_{\sigma} \prod_{(rs)(t)} \delta_{i_r,i_s} \tilde{\overline{F}}_{n_t}(\tau) \overline{F}_{n_r+n_s}(\tau) \right) \left\{ Z_{(V_L^T)^+}(f_\alpha,\tau) - \frac{1}{2} Z_{V_L^T}(f_\alpha,\tau) \right\}. \qquad \blacksquare
\end{aligned}$$

5 1-point functions

In this section, we write down the explicit formulas for the 1-point functions of all states in the $\mathbb{Z}_2$-orbifold of the lattice vertex operator algebra. Recall that $l = k/8$ where k is the rank of the lattice L.

Lemma 5.1. *We have*

$$Z_{V_L}(\mathbf{1},\tau) = \frac{\theta_L(\tau)}{\eta(\tau)^k}, \tag{5.1}$$

$$Z_{V_L^T}(\mathbf{1},\tau) = \eta(\tau)^{k/2} \left(\frac{\Theta_2(\tau)}{2} \right)^{-k/2}, \tag{5.2}$$

$$Z_{V_L^+}(\mathbf{1},\tau) = \frac{1}{2} \frac{\theta_L(\tau)}{\eta(\tau)^k} + \frac{1}{2} \eta(\tau)^{k/2} \left(\frac{\Theta_1(\tau)}{2} \right)^{-k/2}, \tag{5.3}$$

$$Z_{(V_L^T)^+}(\mathbf{1},\tau) = \frac{1}{2} \eta(\tau)^{k/2} \left\{ \left(\frac{\Theta_2(\tau)}{2} \right)^{-k/2} + (-1)^l \left(\frac{\Theta_3(\tau)}{2} \right)^{-k/2} \right\}. \tag{5.4}$$

Proof. We know equations (5.1), (5.3) from [7, Lemma 7] and equations (5.2) and (5.4) can be obtained using similar counting. ∎

Lemma 5.2. *For $\alpha \in L\backslash 2L$, we have $Z_{V_L^+}(f_\alpha, \tau) = 0$, $Z_{(V_L^T)^+}(f_\alpha, \tau) = 0$. For $\alpha \in 2L$,*

$$Z_{V_L^+}(f_\alpha, \tau) = \eta(\tau)^{k/2} \left(\frac{\Theta_1(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle - k/2},$$

$$\frac{1}{2} Z_{V_L^T}(f_\alpha, \tau) = \eta(\tau)^{k/2} \left\{ \left(\frac{\Theta_2(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle - k/2} \right\},$$

$$Z_{(V_L^T)^+}(f_\alpha, \tau) = \eta(\tau)^{k/2} \left\{ \left(\frac{\Theta_2(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle - k/2} + (-1)^l \left(\frac{\Theta_3(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle - k/2} \right\}.$$

Proof. Similarly as in [3]. ∎

We have the following formulas for 1-point functions in the $\mathbb{Z}_2$-orbifold V of V_L given by

$$V = V_L^+ \oplus \left(V_L^T\right)^+.$$

Lemma 5.3. *Suppose V is the $\mathbb{Z}_2$-orbifold of the lattice VOA V_L corresponding to the (-1)-involution as defined in Section 2.3, then we have*

$$Z_V(\mathbf{1}, \tau) = \frac{1}{2} \frac{\theta_L(\tau)}{\eta(\tau)^k} + \frac{1}{2} \eta(\tau)^{k/2} \left\{ \left(\frac{\Theta_1(\tau)}{2}\right)^{-k/2} + \left(\frac{\Theta_2(\tau)}{2}\right)^{-k/2} + (-1)^l \left(\frac{\Theta_3(\tau)}{2}\right)^{-k/2} \right\}$$

and for $\alpha \in 2L$,

$$Z_V(f_\alpha, \tau) = \eta(\tau)^{k/2} \left\{ \left(\frac{\Theta_1(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle - k/2} + \left(\frac{\Theta_2(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle - k/2} + (-1)^l \left(\frac{\Theta_3(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle - k/2} \right\}$$

for $\alpha \in L\backslash 2L$, $Z_V(f_\alpha, \tau) = 0$.

Proof. Using Lemmas 5.1 and 5.2, this follows. ∎

Theorem 5.4. *Suppose V is the $\mathbb{Z}_2$-orbifold of the lattice VOA V_L corresponding to the (-1)-involution as defined in Section 2.3, then for positive integers $n_i \geq 1$, corresponding to the Heisenberg state $u := h_{i_1}[-n_1] \cdots h_{i_p}[-n_p]\mathbf{1} \in V$, we have*

$$\begin{aligned}
Z_V(u, \tau) = \frac{1}{2} \sum_{\Delta \subseteq \Lambda} \frac{\theta_L(\tau, P_\Delta)}{\eta(\tau)^k} &\Bigg(\sum_{\sigma \in \mathrm{Inv}_0(\underline{p}\backslash\Delta)} \prod_{(rs)} \delta_{i_r, i_s} \hat{E}_{n_r + n_s}(\tau) \Bigg) \\
&+ \frac{1}{2} \eta(\tau)^{k/2} \left(\frac{\Theta_1(\tau)}{2}\right)^{-k/2} \Bigg(\sum_{\sigma \in \mathrm{Inv}_0(\underline{p})} \prod_{(rs)} \delta_{i_r, i_s} \hat{F}_{n_r + n_s}(\tau) \Bigg) \\
&+ \frac{1}{2} \eta(\tau)^{k/2} \left(\frac{\Theta_2(\tau)}{2}\right)^{-k/2} \Bigg(\sum_{\sigma \in \mathrm{Inv}_0(\underline{p})} \prod_{(rs)} \delta_{i_r, i_s} \overline{E}_{n_r + n_s}(\tau) \Bigg) \\
&+ \frac{(-1)^l}{2} \eta(\tau)^{k/2} \left(\frac{\Theta_3(\tau)}{2}\right)^{-k/2} \Bigg(\sum_{\sigma \in \mathrm{Inv}_0(\underline{p})} \prod_{(rs)} \delta_{i_r, i_s} \overline{F}_{n_r + n_s}(\tau) \Bigg),
\end{aligned}$$

where $\Lambda = \{j \in \underline{p} \mid n_j = 1\}$, Δ is a subset of Λ of even cardinality and

$$P_\Delta(\alpha) = \prod_{j \in \Delta} \langle h_{i_j}, \alpha \rangle.$$

Proof. Using Lemma 5.1 and [7, Theorem 22], we have

$$\begin{aligned}Z_{V_L^+}(u,\tau) &= \frac{1}{2}\sum_{\Delta\subseteq\Lambda}\frac{\theta_L(\tau,P_\Delta)}{\eta(\tau)^k}\Bigg(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p}\backslash\Delta)}\prod_{(rs)}\delta_{i_r,i_s}\hat{E}_{n_r+n_s}(\tau)\Bigg)\\ &\quad+\frac{1}{2}\eta(\tau)^{k/2}\left(\frac{\Theta_1(\tau)}{2}\right)^{-k/2}\Bigg(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p})}\prod_{(rs)}\delta_{i_r,i_s}\hat{F}_{n_r+n_s}(\tau)\Bigg).\end{aligned}$$

Further, using Theorem 4.4, Lemma 5.1, Corollary 4.2 and the above equation, we have the required result. ■

Theorem 5.5. *Suppose V is the $\mathbb{Z}_2$-orbifold of the lattice VOA V_L corresponding to the (-1)-involution as defined in Section 2.3, then for positive integers $n_i\geq 1$, $\alpha\in 2L$, corresponding to the state $u:=h_{i_1}[-n_1]\cdots h_{i_p}[-n_p]e_\alpha\in V$, where e_α is f_α or g_α accordingly as p is even or odd, respectively,*

$$\begin{aligned}Z_V(u,\tau) &= \eta(\tau)^{k/2}\left(\frac{\Theta_1(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle-k/2}\Bigg(\sum_{\sigma}\prod_{(rs)(t)}\delta_{i_r,i_s}\tilde{F}_{n_t}(\tau)\hat{F}_{n_r+n_s}(\tau)\Bigg)\\ &\quad+\eta(\tau)^{k/2}\left(\frac{\Theta_2(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle-k/2}\Bigg(\sum_{\sigma}\prod_{(rs)(t)}\delta_{i_r,i_s}\tilde{\overline{E}}_{n_t}\overline{E}_{n_r+n_s}(\tau)\Bigg)\\ &\quad+(-1)^l\eta(\tau)^{k/2}\left(\frac{\Theta_3(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle-k/2}\Bigg(\sum_{\sigma}\prod_{(rs)(t)}\delta_{i_r,i_s}\tilde{\overline{F}}_{n_t}\overline{F}_{n_r+n_s}(\tau)\Bigg),\end{aligned}$$

where σ ranges over all involutions of the index set $S_p=\{1,2,\ldots,p\}$ and (rs), (t) range over the 2-cycles and 1-cycles respectively in the decomposition of σ in S_p as a product of disjoint 2-cycles and 1-cycles. For $\alpha\in L\backslash 2L$, $u:=h_{i_1}[-n_1]\cdots h_{i_p}[-n_p]e_\alpha$, we have $Z_V(u,\tau)=0$.

Proof. Using Lemma 5.2 and [7, Theorem 19], we have

$$Z_{V_L^+}(u,\tau)=\eta(\tau)^{k/2}\left(\frac{\Theta_1(\tau)}{2}\right)^{\langle\alpha,\alpha\rangle-k/2}\Bigg(\sum_{\sigma}\prod_{(rs)(t)}\delta_{i_r,i_s}\tilde{F}_{n_t}(\tau)\hat{F}_{n_r+n_s}(\tau)\Bigg).$$

Further, using Theorems 4.3 and 4.5, Lemma 5.2, we can compute $Z_{(V_L^T)^+}(u,\tau)$. Combining the two traces, we have the required result. ■

6 Modular Invariance in the $\mathbb{Z}_2$-orbifold

Before we prove modular invariance in the orbifold, we shall state a lemma which is an immediate consequence of [7, Proposition 23].

Lemma 6.1. *For a state $u=h_{i_1}[-n_1]\cdots h_{i_p}[-n_p]\mathbf{1}$, let*

$$G(u,\tau)=\sum_{\Delta\subseteq\Lambda}\frac{\theta_L(\tau,P_\Delta)}{\eta(\tau)^k}\Bigg(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p}\backslash\Delta)}\prod_{(rs)}\delta_{i_r,i_s}\hat{E}_{n_r+n_s}(\tau)\Bigg),$$

where $\Lambda=\{j\in\underline{p}\mid n_j=1\}$ and Δ is a subset of Λ of even cardinality. When L is unimodular of rank k, $G(u,\tau)$ is modular of weight equal to $L[0]$-weight of u.

Proposition 6.2. *Suppose V is the $\mathbb{Z}_2$-orbifold of the lattice VOA V_L corresponding to the (-1)-involution as defined in Section 2.3, then for every homogeneous state u (with respect to $L[0]$-grading) in V, we have*

$$Z_V(u, S\tau) = \tau^{wt(u)} Z_V(u,\tau), \qquad Z_V(u, T\tau) = \mathrm{e}^{-2k\pi\mathrm{i}/24} Z_V(u,\tau),$$

where $S, T \in \mathrm{SL}_2(\mathbb{Z})$ represent as usual the elements $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ respectively and hence $Z_V(u,\tau)$ is modular of weight $wt(u)$ (with respect to $L[0]$) over the full modular group $\mathrm{SL}_2(\mathbb{Z})$ up to a character.

Proof. Observe that the proposition is true for any homogeneous state u in $\left(V_L^T\right)^+$ since for such states $Z_V(u,\tau) = 0$. Thus, we prove the proposition for states in V_L^+ below. The S-invariance follows from the following:

$$\begin{aligned}
&\hat{F}_k(S\tau) = \tau^k \overline{E}_k(\tau), \qquad \overline{E}_k(S\tau) = \tau^k \hat{F}_k(\tau), \qquad \overline{F}_k(S\tau) = \tau^k \overline{F}_k(\tau),\\
&\Theta_1(S\tau) = (-\mathrm{i}\tau)^{1/2}\Theta_2(\tau), \qquad \Theta_2(S\tau) = (-\mathrm{i}\tau)^{1/2}\Theta_1(\tau), \qquad \Theta_3(S\tau) = (-\mathrm{i}\tau)^{1/2}\Theta_3(\tau),\\
&\eta(S\tau) = (-\mathrm{i}\tau)^{1/2}\eta(\tau).
\end{aligned}$$

Using Lemma 6.1, we also have $G(u, S\tau) = \tau^{wt(u)} G(u,\tau)$. The T-invariance follows from the following:

$$\begin{aligned}
&\hat{F}_k(T\tau) = \hat{F}_k(\tau), \qquad \overline{F}_k(T\tau) = \overline{E}_k(\tau), \qquad \hat{E}_k(T\tau) = \hat{E}_k(\tau), \qquad \overline{E}_k(T\tau) = \overline{F}_k(\tau),\\
&\Theta_1(T\tau) = \mathrm{e}^{\frac{3\pi\mathrm{i}}{12}}\Theta_1(\tau), \qquad \Theta_2(T\tau) = \Theta_3(\tau), \qquad \Theta_3(T\tau) = \Theta_2(\tau),\\
&\eta(T\tau) = \mathrm{e}^{\frac{\pi\mathrm{i}}{12}}\eta(\tau).
\end{aligned}$$

Using Lemma 6.1, we also have $G(u, T\tau) = \mathrm{e}^{-2k\pi\mathrm{i}/24} G(u,\tau)$. Thus, for a Heisenberg state $u = h_{i_1}[-n_1]\cdots h_{i_p}[-n_p]\mathbf{1}$,

$$\begin{aligned}
Z_V(u, T\tau) &= \frac{1}{2}G(u, T\tau) + \frac{1}{2}\eta(T\tau)^{k/2}\left(\frac{\Theta_1(T\tau)}{2}\right)^{-k/2}\left(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p})}\prod_{(rs)}\delta_{i_r,i_s}\hat{F}_{n_r+n_s}(T\tau)\right)\\
&\quad + \frac{1}{2}\eta(T\tau)^{k/2}\left(\frac{\Theta_2(T\tau)}{2}\right)^{-k/2}\left(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p})}\prod_{(rs)}\delta_{i_r,i_s}\overline{E}_{n_r+n_s}(T\tau)\right)\\
&\quad + \frac{(-1)^l}{2}\eta(T\tau)^{k/2}\left(\frac{\Theta_3(T\tau)}{2}\right)^{-k/2}\left(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p})}\prod_{(rs)}\delta_{i_r,i_s}\overline{F}_{n_r+n_s}(T\tau)\right)\\
&= \frac{\mathrm{e}^{-2k\pi\mathrm{i}/24}}{2}G(u,\tau) + \frac{\mathrm{e}^{-2k\pi\mathrm{i}/24}}{2}\eta(\tau)^{k/2}\left(\frac{\Theta_1(\tau)}{2}\right)^{-k/2}\\
&\qquad \times\left(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p})}\prod_{(rs)}\delta_{i_r,i_s}\hat{F}_{n_r+n_s}(\tau)\right)\\
&\quad + \frac{(-1)^l.(-1)^l.\mathrm{e}^{k\pi\mathrm{i}/24}}{2}\eta(\tau)^{k/2}\left(\frac{\Theta_3(\tau)}{2}\right)^{-k/2}\left(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p})}\prod_{(rs)}\delta_{i_r,i_s}\overline{F}_{n_r+n_s}(\tau)\right)\\
&\quad + \frac{(-1)^l\mathrm{e}^{k\pi\mathrm{i}/24}}{2}\eta(\tau)^{k/2}\left(\frac{\Theta_2(\tau)}{2}\right)^{-k/2}\left(\sum_{\sigma\in\mathrm{Inv}_0(\underline{p})}\prod_{(rs)}\delta_{i_r,i_s}\overline{E}_{n_r+n_s}(\tau)\right).
\end{aligned}$$

Rewriting $(-1)^l$ as $\mathrm{e}^{k\pi\mathrm{i}/8}$, we have $Z_V(u, T\tau) = \mathrm{e}^{-2k\pi\mathrm{i}/24} Z_V(u,\tau)$. Similarly, for a lattice state $u = h_{i_1}[-n_1]\cdots h_{i_p}[-n_p]e_\alpha$, we have the T-invariance given by

$$Z_V(u, T\tau) = \mathrm{e}^{-2k\pi\mathrm{i}/24} Z_V(u,\tau). \qquad \blacksquare$$

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