On Bilateral Multiple Sums and Rogers–Ramanujan Type Identities

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Abstract. We establish some new bilateral double-sum Rogers–Ramanujan identities involving parameters. As applications, these identities yield several new multi-sum Rogers–Ramanujan type identities. Our proofs utilize the theory of basic hypergeometric series in conjunction with the integral method.

Key words: Rogers–Ramanujan type identities; bilateral summations; multiple sums; q-series; integral method

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1 Introduction

We begin by recalling the basic notation and definitions used throughout this paper. Assume that $|q| < 1$ for convergence. The q-shifted factorials are defined as follows [8]:

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n-1} \big(1 - aq^k\big), \qquad (a;q)_\infty := \prod_{k=0}^{\infty} \big(1 - aq^k\big).$$

For negative subscripts, we define

$$(a;q)_{-n} := \frac{1}{(aq^{-n};q)_n}.$$

We also adopt the compact product notation

$$(a_1,\dots,a_m;q)_k := (a_1;q)_k \cdots (a_m;q)_k, \qquad k \in \mathbb{Z} \cup \{\infty\}.$$

In 1894, Rogers [15] discovered and proved the following fundamental identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{\big(q,q^4;q^5\big)_\infty}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{\big(q^2,q^3;q^5\big)_\infty}. \tag{1.2}$$

These are now known as the Rogers–Ramanujan identities, since they were later rediscovered by Ramanujan before 1913 [9]. These results have inspired extensive research into similar q-series identities, among which Slater's famous list of 130 such identities stands out [17].

A natural direction of generalization is to consider multi-sum Rogers–Ramanujan type identities. As a multi-sum generalization of (1.1) and (1.2), the Andrews–Gordon identity [1] stated that for integers $k > 1$ and $1 \le i \le k$,

$$\sum_{n_1,\dots,n_{k-1}\ge 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}(q;q)_{n_{k-1}}} = \frac{\left(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1}\right)_\infty}{(q;q)_\infty}, \tag{1.3}$$

where $N_j = n_j + \cdots + n_{k-1}$ for $1 \le j \le k-1$ and $N_k = 0$. Bressoud [4] later provided an even-modulus analogue: for integers $k > 1$ and $1 \le i \le k$,

$$\sum_{n_1,\dots,n_{k-1}\ge 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}\left(q^2;q^2\right)_{n_{k-1}}} = \frac{\left(q^i, q^{2k-i}, q^{2k}; q^{2k}\right)_\infty}{(q;q)_\infty}, \tag{1.4}$$

with N_j defined as above.

Rogers–Ramanujan type identities play a significant role in several fields, including combinatorics, mathematical physics and number theory. In particular, they reveal profound connections between the theory of q-series and modular forms. For instance, after multiplying with suitable powers of q, the right-hand sides of identities (1.1) and (1.2) become modular forms which is not easy to observe from their sum-side representations. A central question in this area is to characterize which basic hypergeometric series can be expressed as modular forms.

In this aspect, Nahm [10, 11, 12] considered a specific class of q-hypergeometric series known as Nahm sum or Nahm series

$$f_{A,B,C}(q) := \sum_{n=(n_1,\dots,n_r)^{\mathrm{T}}\in\mathbb{N}^r} \frac{q^{\frac{1}{2}n^{\mathrm{T}}An+n^{\mathrm{T}}B+C}}{(q;q)_{n_1}\cdots(q;q)_{n_r}},$$

where r is a positive integer, A is a real positive definite symmetric $r \times r$ matrix, B is an r-dimensional column vector, and C is a rational scalar. Motivated by considerations from physics, Nahm [12] posed the problem of classifying all rational triples (A, B, C) for which $f_{A,B,C}$ is modular; such a triple (A, B, C) is called as a modular triple.

Nahm further conjectured a necessary and sufficient condition on A for it to be the matrix part of a modular triple. In a systematic study, Zagier [22] showed that there are exactly seven modular triples in the rank $r = 1$ case:

$$\begin{aligned} &(1/2, 0, -1/40), \quad (1/2, 1/2, 1/40), \quad (1, 0, -1/48), \quad (1, 1/2, 1/24),\\ &(1, -1/2, 1/24), \quad (2, 0, -1/60), \quad (2, 1, 11/60). \end{aligned}$$

Notably, the last two triples correspond to the Rogers–Ramanujan identities (1.1) and (1.2).

The following identity is first given by Ramanujan, which is commonly known as Ramanujan's ${}_1\psi_1$ summation [8, Appendix II.29]

$$\sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(b;q)_k} z^k = \frac{(q, az, q/az, b/a; q)_\infty}{(b, z, b/az, q/a; q)_\infty}, \qquad |b/a| < |z| < 1. \tag{1.5}$$

The identity is a bilateral extension of the q-binomial theorem [8, Appendix II.3]

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z| < 1,$$

which is a fundamental identity in the theory of basic hypergeometric series. A natural question is whether similar bilateral extensions exist for Rogers–Ramanujan type identities. In 2023, Schlosser [16] employed bilateral summation formulas to study Nahm sum and derived new bilateral Rogers–Ramanujan type identities by analytic methods.

Following the notation of [18], we define a Rogers–Ramanujan type identity of the index $(n_1, n_2, \ldots, n_k)$ as a finite sum of the form:

$$\sum_{(i_1,\ldots,i_k)\in S} \frac{(-1)^{t(i_1,\ldots,i_k)} q^{Q(i_1,\ldots,i_k)}}{(q^{n_1};q^{n_1})_{i_1}\cdots(q^{n_k};q^{n_k})_{i_k}} = \prod_{(a,n)\in P} (q^a;q^n)_\infty^{r(a,n)},$$

where $t(i_1,\ldots,i_k)$ is an integer-valued function, $Q(i_1,\ldots,i_k)$ is a rational polynomial in $i_1,\ldots,i_k$, the $n_1, n_2, \ldots, n_k$ are positive integers with $\gcd(n_1, n_2, \ldots, n_k) = 1$, $S \subset \mathbb{Z}^k$, $P \subset \mathbb{Q}^2$ is finite, and $r(a, n)$ is integer-valued. For instance, the Andrews–Gordon identity (1.3) and Bressoud's identity (1.4) are of indices $(1, 1, \ldots, 1)$ and $(1, 1, \ldots, 1, 2)$, respectively.

Recently, Cao and Wang [5] used the integral method to derive several new Rogers–Ramanujan type identities of indices

$$(1,1), \quad (1,2), \quad (1,1,1), \quad (1,1,2), \quad (1,1,3), \quad (1,2,2), \quad (1,2,3), \quad (1,2,4).$$

Some of these include additional parameters, thereby giving infinite families of such identities. For example, they proved [5, Theorem 3.4]

$$\sum_{i,j\geq 0} \frac{u^{i-j} q^{\binom{i}{2}+\binom{j+1}{2}+a\binom{j-i}{2}}}{(q;q)_i(q;q)_j} = \frac{\left(-uq^a, -q/u, q^{a+1}; q^{a+1}\right)_\infty}{(q;q)_\infty}. \tag{1.6}$$

Motivated by the integral method and recent work on bilateral Rogers–Ramanujan identities, we present the following main results.

Theorem 1.1. *For $a \in \mathbb{N}^+$, we have*

$$\sum_{i,j\in\mathbb{Z}} \frac{x^i y^j q^{a\binom{i}{2}+a\binom{j+1}{2}+\binom{j-i}{2}}}{(xq^a;q^a)_i(yq^a;q^a)_j} = \frac{(q;q)_\infty}{(xq^a, yq^a; q^a)_\infty (q^a;q^a)_\infty^{a-1}} \Theta_a\left(qy, q^2y, \ldots, q^{a-1}y, xy; q^a\right), \tag{1.7}$$

where

$$\Theta_a(x_1, x_2, \ldots, x_a; q) := \sum_{n_1,n_2,\ldots,n_a\in\mathbb{Z}} q^{n_1^2+\cdots+n_a^2+n_1n_2+n_1n_3+\cdots+n_{a-1}n_a} x_1^{n_1} x_2^{n_2} \cdots x_a^{n_a}.$$

In particular, when $a = 1$, we have

$$\sum_{i,j\in\mathbb{Z}} \frac{x^i y^j q^{i^2-ij+j^2}}{(xq;q)_i(yq;q)_j} = \frac{(q;q)_\infty\left(-xyq, -q/xy, q^2; q^2\right)_\infty}{(xq, yq; q)_\infty}. \tag{1.8}$$

When $a = 2$, we have

$$\begin{aligned}\sum_{i,j\in\mathbb{Z}} & \frac{x^i y^j q^{\frac{3}{2}i^2+\frac{3}{2}j^2-ij-\frac{1}{2}i+\frac{1}{2}j}}{\left(xq^2;q^2\right)_i\left(yq^2;q^2\right)_j} \\ &= \frac{(q;q)_\infty}{\left(xq^2, yq^2, q^2; q^2\right)_\infty}\left(\left(q^4, -yq^3, \frac{-q}{y}; q^4\right)_\infty \left(q^{12}, -x^2yq^5, \frac{-q^7}{x^2y}; q^{12}\right)_\infty\right. \\ &\quad \left. + xq\left(q^4, -yq, \frac{-q^3}{y}; q^4\right)_\infty \left(q^{12}, -x^2yq^{11}, \frac{-q}{x^2y}; q^{12}\right)_\infty\right). \end{aligned} \tag{1.9}$$

Moreover, when $x = y = 1$, we have

$$\Theta_a\left(q, q^2, \ldots, q^{a-1}, 1; q^a\right) = \frac{(q^a;q^a)_\infty^a\left(-q, -q^a, q^{a+1}; q^{a+1}\right)_\infty}{(q;q)_\infty}. \tag{1.10}$$

As applications, these identities yield the following.

Corollary 1.2. *We have*

$$\sum_{i,j\geq 0}\frac{q^{i^2-ij+j^2}}{(q;q)_i(q;q)_j}=\frac{(-q,-q,q^2;q^2)_\infty}{(q;q)_\infty}, \tag{1.11}$$

$$\sum_{i,j,k\geq 0}\frac{q^{i^2+j^2+k^2+ik+jk}}{(q;q)_i(q;q)_j(q;q)_k}=\frac{(-q,-q,q^2;q^2)_\infty}{(q;q)_\infty}, \tag{1.12}$$

$$\sum_{n_1,n_2,\ldots,n_\ell\geq 0}\frac{q^{Q(n_1,n_2,\ldots,n_\ell)}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_\ell}}=\frac{(-q,-q,q^2;q^2)_\infty}{(q;q)_\infty}, \tag{1.13}$$

$$\sum_{i,j\geq 0}\frac{q^{\frac{3}{2}i^2+\frac{3}{2}j^2-ij-\frac{1}{2}i+\frac{1}{2}j}}{(q^2;q^2)_i(q^2;q^2)_j}=\frac{(q^3,q^3,q^6;q^6)_\infty}{(q;q)_\infty}, \tag{1.14}$$

$$\sum_{i,j\geq 0}\frac{q^{2i^2+2j^2-ij-i+j}}{(q^3,q^3)_i(q^3,q^3)_j}=\frac{(q^2,q^2)_\infty^2}{(q,q)_\infty(q^3,q^3)_\infty}, \tag{1.15}$$

$$\sum_{i,j\geq 0}\frac{q^{3i^2+3j^2-ij-2i+2j}}{(q^5,q^5)_i(q^5,q^5)_j}=\frac{(q^2,q^2)_\infty^2(q^3,q^3)_\infty(q^{12},q^{12})_\infty}{(q,q)_\infty(q^4,q^4)_\infty(q^5,q^5)_\infty(q^6,q^6)_\infty}, \tag{1.16}$$

where $Q(n_1,n_2,\ldots,n_\ell)=(n_1+n_3+\cdots+n_\ell)^2-(n_1+n_3+\cdots+n_\ell)(n_2+n_3+\cdots+n_\ell)+(n_2+n_3+\cdots+n_\ell)^2+\sum_{i\in\{4,\ldots,\ell\}}(n_1+n_i+\cdots+n_\ell)(n_2+n_i+\cdots+n_\ell)+n_1n_2$ *with integers* $\ell\geq 4$.

Remark 1.3. Setting $u=a=1$ in (1.6), we can also obtain (1.11). This identity is a special case of a more general family of identities for Nahm sums associated with the tensor product of Cartan matrices [21, equation (1.6)]. Setting $\nu=0$ in [19, equation (4.37)], we can also obtain (1.12). Setting $\alpha=\frac{3}{2}$, $\nu=-\frac{1}{6}$ and substituting $q\to q^2$ in [20, equation (3.1)], we can also obtain (1.14). Setting $\alpha=\frac{4}{3}$, $\nu=-\frac{1}{4}$ and substituting $q\to q^3$ in [20, equation (3.1)], we can also obtain (1.15). Setting $\alpha=\frac{6}{5}$, $\nu=-\frac{1}{3}$ and substituting $q\to q^5$ in [20, equation (3.1)], we can also obtain (1.16).

The rest of this paper is organized as follows. In Section 2, we review some fundamental identities, the integral method and modular functions. In Section 3, we prove Theorem 1.1 using the integral method and derive several multi-sum Rogers–Ramanujan identities as consequences.

2 Preliminaries

2.1 Some fundamental identities and the integral method

In this section, we recall several fundamental identities and techniques from the theory of q-series that will be used throughout this paper.

We have that

$$\lim_{a\to\infty}(a;q)_n a^{-n}=(-1)^n q^{\binom{n}{2}}.$$

We also require the Jacobi triple product identity [8, p. 15]

$$(q,z,q/z;q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}}z^n.$$

Starting from Ramanujan's ${}_1\psi_1$ summation formula (1.5), we substitute $z \to z/a$ and take $a \to \infty$, yielding

$$\sum_{k\in\mathbb{Z}} \frac{(-z)^k q^{\binom{k}{2}}}{(b;q)_k} = \frac{(q,z,q/z;q)_\infty}{(b,b/z;q)_\infty}. \tag{2.1}$$

Two special cases of (2.1) will be particularly useful. First, setting $b = xq$ and replacing z with xz, we obtain

$$\sum_{i\in\mathbb{Z}} \frac{(-xz)^i q^{\binom{i}{2}}}{(xq;q)_i} = \frac{(q,xz,q/xz;q)_\infty}{(xq,q/z;q)_\infty}. \tag{2.2}$$

Second, setting $b = yq$ and replacing z with yq/z, we obtain

$$\sum_{j\in\mathbb{Z}} \frac{(-yq/z)^j q^{\binom{j}{2}}}{(yq;q)_j} = \frac{(q,yq/z,z/y;q)_\infty}{(yq,z;q)_\infty}. \tag{2.3}$$

For a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a(n)z^n$, we denote by $[z^n]f(z)$ the coefficient of z^n, i.e., $[z^n]f(z) = a(n)$. A key observation is that this coefficient can be extracted via contour integration

$$\oint_K f(z)\frac{\mathrm{d}z}{2\pi \mathrm{i} z} = [z^0]f(z),$$

where K is any positively oriented simple closed contour encircling the origin. This principle underlies the integral method, which we employ to prove Rogers–Ramanujan type identities. The general approach is to express the sum side of an identity as a finite linear combination of integrals involving infinite products, and then evaluate each integral explicitly.

2.2 Modular functions

According to [3] and [13], let $\mathcal{H} = \{\tau \mid \mathrm{Im}(\tau) > 0\}$ denote the complex upper half-plane. For each

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2^+(\mathbb{Z}),$$

where $M_2^+(\mathbb{Z})$ is the set of integer 2×2 matrices with positive determinant, the bilinear transformation $M(\tau)$ is defined by

$$M\tau = M(\tau) = \frac{a\tau + b}{c\tau + d}.$$

The slash operator is defined by

$$(f|M)(\tau) = f(M\tau),$$

and satisfies

$$f|MS = f|M|S,$$

for matrices M and S. The modular group $\Gamma(1)$ is defined by

$$\Gamma(1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2^+(\mathbb{Z}) \mid ad - bc = 1 \right\}.$$

We consider the following subgroup Γ of the modular group with finite index

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

The Dedekind eta function is defined to be

$$\eta_m(\tau) = \eta(m\tau) = q^{\frac{m}{24}} \prod_{n=1}^{\infty} (1 - q^{mn}).$$

The generalized Dedekind eta function is defined to be

$$\eta_{\delta,g}(\tau) = q^{\frac{\delta}{2}P_2(g/\delta)} \prod_{m \equiv \pm g \, (\mathrm{mod}\, \delta)} (1 - q^m), \tag{2.4}$$

where $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second periodic Bernoulli polynomial, $\{t\} = t - [t]$ is the fractional part of $t, g, \delta, m \in \mathbb{Z}^+$ and $0 < g < \delta$. The function $\eta_{\delta,g}(\tau)$ is a modular function on $\mathrm{SL}_2(\mathbb{Z})$ with a multiplier system. Let N be a fixed positive integer. A generalized Dedekind eta-product of level N has the form

$$f(\tau) = \prod_{\delta | N, \ 0 < g < \delta} \eta_{\delta,g}^{r_{\delta,g}}(\tau), \tag{2.5}$$

where

$$r_{\delta,g} \in \begin{cases} \frac{1}{2}\mathbb{Z}, & g = \frac{\delta}{2}, \\ \mathbb{Z}, & \text{otherwise.} \end{cases}$$

Robins [14] has found sufficient conditions under which a generalized eta-product is a modular function on $\Gamma_1(N)$.

Theorem 2.1 ([14, Theorem 3])**.** *The function $f(\tau)$, defined in (2.5), is a modular function on $\Gamma_1(N)$ if*

$$(i) \quad \sum_{\delta | N, \ g} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} \equiv 0 \pmod 2, \quad \textit{and}$$

$$(ii) \quad \sum_{\delta | N, \ g} \frac{N}{\delta} P_2(0) r_{\delta,g} \equiv 0 \pmod 2.$$

Cho, Koo and Park [6] have found a set of inequivalent cusps for $\Gamma_1(N) \cap \Gamma_0(mN)$. The group $\Gamma_1(N)$ corresponds to the case $m = 1$.

Theorem 2.2 ([6, Corollary 4])**.** *Let $a, c, a', c' \in \mathbb{Z}$ with $(a, c) = (a', c') = 1$.*

(i) *The cusps $\frac{a}{c}$ and $\frac{a'}{c'}$ are equivalent mod $\Gamma_1(N)$ if and only if*

$$\begin{pmatrix} a' \\ c' \end{pmatrix} \equiv \pm \begin{pmatrix} a + nc \\ c \end{pmatrix} \pmod N$$

for some integers n.

(ii) *The following is a complete set of inequivalent cusps mod Γ_1:*

$$\mathcal{S} = \left\{ \frac{y_{c,j}}{x_{c,j}} \,\middle|\, 0 < c | N, \, 0 < s_{c,i}, a_{c,j} \leq N, \, (s_{c,i}, N) = (a_{c,j}, N) = 1, \right.$$

$$s_{c,i} = s_{c,i'} \Leftrightarrow s_{c,1} \equiv \pm s_{c',i'} \left(\bmod \frac{N}{c}\right),$$
$$a_{c,j} = a_{c,j'} \Leftrightarrow \begin{cases} a_{c,j} \equiv \pm a_{c,j'} \pmod{c}, & \textit{if } c = \frac{N}{2} \textit{ or } N, \\ a_{c,j} \equiv \pm a_{c,j'} \pmod{c}, & \textit{otherwise.} \end{cases}$$
$$x_{c,i}, y_{c,j} \in \mathbb{Z} \textit{ chosen so that } x_{c,i} \equiv cs_{c,i},\ y_{c,j} \equiv a_{c,j} \pmod{N},\ (x_{c,i}, y_{c,j}) = 1 \Big\}.$$

(iii) The fan width of the cusp $\frac{a}{c}$ *is given by*

$$\kappa\Big(\frac{a}{c}, \Gamma_1(N)\Big) = \begin{cases} 1, & \textit{if } N = 4 \textit{ and } (c, 4) = 2, \\ \dfrac{N}{(c, N)}, & \textit{otherwise.} \end{cases}$$

In this theorem, it is understood, as usual that the fraction $\frac{\pm 1}{0}$ corresponds to ∞. Robins [14] has calculated the invariant order of $\eta_{\delta,g}(\tau)$ at any cusp. This gives a method for calculating the invariant order at any cusp of a generalized eta-product.

Theorem 2.3 ([14])**.** *The order at the cusp* $\zeta = \frac{a}{c}$ *(assuming* $(a, c) = 1$*) of the generalized eta-function* $\eta_{\delta,g}(\tau)$ *(defined in (2.4) and assuming* $0 < g < \delta$*) is*

$$ord(\eta_{\delta,g}(\tau); \zeta) = \frac{\varepsilon^2}{2\delta} P_2\Big(\frac{ag}{\varepsilon}\Big),$$

where $\varepsilon = (\delta, c)$*.*

Theorem 2.4 ([7, Corollary 2.5])**.** *Let* $f_1(\tau), f_2(\tau), \ldots, f_n(\tau)$ *be generalized eta-products that are modular functions on* $\Gamma_1(N)$*. Let* $\mathcal{S}_N$ *be a set of inequivalent cusps for* $\Gamma_1(N)$*. Define the constant*

$$B = \sum_{s \in \mathcal{S}_N,\ s \neq \infty} \min(\{\mathrm{Ord}(f_j, s, \Gamma_1(N)) \mid 1 \leq j \leq n\} \cup \{0\}), \tag{2.6}$$

and consider

$$g(\tau) := \alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_n f_n(\tau) + 1,$$

where each $\alpha_j \in \mathbb{C}$*. Then* $g(\tau) = 0$ *if and only if* $\mathrm{Ord}(g(\tau), \infty, \Gamma_1(N)) > -B$*.*

A more comprehensive description can be found in [7]. We have utilized a MAPLE package known as `thetaids`, which facilitates the implementation of the aforementioned algorithm.[1]

3 The proof of Theorem 1.1 and Corollary 1.2

In this section, we prove Theorem 1.1 using the integral method. We also show how special choices of parameters in Theorem 1.1 lead to several Rogers–Ramanujan type identities.

The proof of Theorem 1.1. First of all, we notice a fact about the q-shifted factorials:

$$\frac{(z, q/z; q)_\infty}{(z, q^a/z; q^a)_\infty} = \prod_{l=1}^{a-1} \big(zq^l, q^{a-l}/z; q^a\big)_\infty. \tag{3.1}$$

[1] See `http://qseries/org/fgarvan/qmaple/thetaids/`.

Then we consider the following contour integral:

$$\begin{aligned}
I &:= \oint \frac{(q^a, xz, q^a/xz; q^a)_\infty (q^a, yq^a/z, z/y; q^a)_\infty}{(xq^a, q^a/z; q^a)_\infty (yq^a, z; q^a)_\infty} (q, z, q/z; q)_\infty \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \\
&= \frac{(q;q)_\infty}{(xq^a, yq^a; q^a)_\infty (q^a;q^a)_\infty^{a-1}} \\
&\quad \times \oint (q^a, xz, q^a/xz; q^a)_\infty (q^a, yq^a/z, z/y; q^a)_\infty \prod_{l=1}^{a-1} \left(q^a, zq^l, q^{a-l}/z; q^a\right)_\infty \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \\
&= \frac{(q;q)_\infty}{(xq^a, yq^a; q^a)_\infty (q^a;q^a)_\infty^{a-1}} \\
&\quad \times \oint \sum_{i\in\mathbb{Z}} (-1)^i q^{a\binom{i}{2}} (xz)^i \sum_{j\in\mathbb{Z}} (-1)^j q^{a\binom{j}{2}} \left(\frac{yq^a}{z}\right)^j \prod_{l=1}^{a-1} \sum_{n_l\in\mathbb{Z}} (-1)^{n_l} q^{a\binom{n_l}{2}} \left(zq^l\right)^{n_l} \frac{\mathrm{d}z}{2\pi \mathrm{i} z}.
\end{aligned}$$

Here we use (3.1) for the second equality and the third equality follows the Jacobi triple product identity.

Multiplying the series and extracting the coefficient of z^0, we require $i - j + n_1 + \cdots + n_{a-1} = 0$. We let $i = n_a$ and then $j = \sum_{l=1}^{a} n_l$. Thus,

$$\begin{aligned}
I &= \frac{(q;q)_\infty}{(xq^a, yq^a; q^a)_\infty (q^a;q^a)_\infty^{a-1}} \sum_{n_1,n_2,\ldots,n_a\in\mathbb{Z}} x^{n_a} y^{\sum_{l=1}^a n_l} q^{a\binom{\sum_{l=1}^a n_l}{2} + a\sum_{l=1}^a n_l + a\sum_{l=1}^a \binom{n_l}{2} + \sum_{l=1}^{a-1} l n_l} \\
&= \frac{(q;q)_\infty}{(xq^a, yq^a; q^a)_\infty (q^a;q^a)_\infty^{a-1}} \Theta_a\left(qy, q^2y, \ldots, q^{a-1}y, xy; q^a\right).
\end{aligned}$$

On the other hand, starting from (2.2) and (2.3), we substitute $q \to q^a$. Then the same integral can be written as:

$$\begin{aligned}
I &= \oint \sum_{i\in\mathbb{Z}} \frac{(-xz)^i q^{a\binom{i}{2}}}{(xq^a;q^a)_i} \sum_{j\in\mathbb{Z}} \frac{(-yq^a/z)^j q^{a\binom{j}{2}}}{(xq^a;q^a)_j} \sum_{k\in\mathbb{Z}} (-1)^k q^{\binom{k}{2}} z^k \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \\
&= \sum_{i,j\in\mathbb{Z}} \frac{x^i y^j q^{a\binom{i}{2} + a\binom{j+1}{2} + \binom{j-i}{2}}}{(xq^a;q^a)_i (yq^a;q^a)_j}.
\end{aligned}$$

Similarly, the second equality holds because we require $i - j + k = 0$, i.e., $k = j - i$ to extract the coefficient of z^0. Equating the two expressions for I completes the proof of (1.7).

Setting $a = 1$ in (1.7), we can obtain (1.8) by the Jacobi triple product identity:

$$\begin{aligned}
\sum_{i,j\in\mathbb{Z}} \frac{x^i y^j q^{i^2 - ij + j^2}}{(xq;q)_i (yq;q)_j} &= \frac{(q;q)_\infty}{(xq, yq; q)_\infty} \sum_{i\in\mathbb{Z}} (xy)^i q^{i^2} \\
&= \frac{(q;q)_\infty}{(xq, yq; q)_\infty} \left(-xyq, -q/xy, q^2; q^2\right)_\infty.
\end{aligned}$$

Setting $a = 2$ in (1.7), we can obtain

$$\sum_{i,j\in\mathbb{Z}} \frac{x^i y^j q^{\frac{3}{2}i^2 + \frac{3}{2}j^2 - ij - \frac{1}{2}i + \frac{1}{2}j}}{\left(xq^2;q^2\right)_i \left(yq^2;q^2\right)_j} = \frac{(q;q)_\infty}{\left(xq^2, yq^2, q^2; q^2\right)_\infty} \sum_{i,j\in\mathbb{Z}} x^j y^{i+j} q^{2i^2 + 2j^2 + 2ij + i}. \tag{3.2}$$

Then by the Jacobi triple product identity, we have

$$\sum_{i,j\in\mathbb{Z}} x^j y^{i+j} q^{2i^2 + 2j^2 + 2ij + i}$$

$$
\begin{aligned}
&= \sum_{i,j\in\mathbb{Z}} x^j y^{i+j} q^{2[(i+\frac{1}{2}j)^2+\frac{3}{4}j^2]+i} \\
&= \sum_{i,j\in\mathbb{Z}} x^{2j} y^{i+2j} q^{2[(i+j)^2+3j^2]+i} + \sum_{i,j\in\mathbb{Z}} x^{2j-1} y^{i+2j-1} q^{2[(i+j-\frac{1}{2})^2+3j^2-3j+\frac{3}{4}]+i} \\
&= \sum_{k,j\in\mathbb{Z}} x^{2j} y^{k+j} q^{2(k^2+3j^2)+k-j} + \sum_{k,j\in\mathbb{Z}} x^{2j-1} y^{k+j-1} q^{2(k^2-k+3j^2-3j+1)+k-j} \\
&= \sum_{k\in\mathbb{Z}} y^k q^{2k^2+k} \sum_{j\in\mathbb{Z}} (x^2y)^j q^{6j^2-j} + xq \sum_{k\in\mathbb{Z}} y^k q^{2k^2-k} \sum_{j\in\mathbb{Z}} (x^2y)^j q^{6j^2+5j} \\
&= \Big(q^4, -yq^3, \frac{-q}{y}; q^4\Big)_\infty \Big(q^{12}, -x^2yq^5, \frac{-q^7}{x^2y}; q^{12}\Big)_\infty \\
&\quad + xq\Big(q^4, -yq, \frac{-q^3}{y}; q^4\Big)_\infty \Big(q^{12}, -x^2yq^{11}, \frac{-q}{x^2y}; q^{12}\Big)_\infty.
\end{aligned}
$$

Combining the above identity and (3.2), we complete the proof of (1.9).

From Wang [20, equation (3.1)], we have

$$
\sum_{i,j\geq 0} \frac{q^{\frac{\alpha}{2}i^2+(1-\alpha)ij+\frac{\alpha}{2}j^2+\alpha\nu i-\alpha\nu j}}{(q;q)_i(q;q)_j} = \frac{\big(-q^{\frac{\alpha}{2}+\alpha\nu}, -q^{\frac{\alpha}{2}-\alpha\nu}, q^\alpha; q^\alpha\big)_\infty}{(q;q)_\infty}. \tag{3.3}
$$

Setting $\alpha = \frac{a+1}{a}$, $\nu = -\frac{a-1}{2(a+1)}$ and substituting $q \to q^a$ in (3.3), we obtain

$$
\sum_{i,j\geq 0} \frac{q^{\frac{a+1}{2}i^2-ij+\frac{a+1}{2}j^2-\frac{a-1}{2}i+\frac{a-1}{2}j}}{(q^a;q^a)_i(q^a;q^a)_j} = \frac{\big(-q, -q^a, q^{a+1}; q^{a+1}\big)_\infty}{(q^a;q^a)_\infty}.
$$

Setting $x = y = 1$ in (1.7), we obtain

$$
\sum_{i,j\geq 0} \frac{q^{\frac{a+1}{2}i^2-ij+\frac{a+1}{2}j^2-\frac{a-1}{2}i+\frac{a-1}{2}j}}{(q^a;q^a)_i(q^a;q^a)_j} = \frac{(q;q)_\infty}{(q^a;q^a)_\infty^{a+1}} \Theta_a\big(q, q^2, \ldots, q^{a-1}, 1; q^a\big).
$$

Combining the above two identities, we can prove (1.10). ■

The proof of Corollary 1.2. Setting $x = y = 1$ in (1.8) and noting that $1/(q;q)_{-n} = 0$ for $n > 0$, the bilateral sum reduces to a sum over non-negative indices, yielding (1.11).

To prove (1.13), we use the identity [2, p. 20]

$$
\frac{1}{(q;q)_i(q;q)_j} = \sum_{k\geq 0} \frac{q^{(i-k)(j-k)}}{(q;q)_k(q;q)_{i-k}(q;q)_{j-k}}.
$$

Substituting this into (1.11) and shifting the summation indices by setting $i \to i+k$ and $j \to j+k$, we obtain identity (1.12). Finally, applying the same identity from [2, p. 20] iteratively yields identity (1.13).

Setting $x = y = 1$ in (1.9), we obtain

$$
\sum_{i,j\geq 0} \frac{q^{\frac{3}{2}i^2+\frac{3}{2}j^2-ij-\frac{1}{2}i+\frac{1}{2}j}}{(q^2;q^2)_i(q^2;q^2)_j} = \frac{1}{(q^2;q^2)_\infty}\big((-q^5, -q^7, q^{12}; q^{12})_\infty + q(-q, -q^{11}, q^{12}; q^{12})_\infty\big).
$$

Therefore, (1.14) holds if we can prove

$$
(-q^5, -q^7, q^{12}; q^{12})_\infty + q(-q, -q^{11}, q^{12}; q^{12})_\infty = \frac{(q^3;q^3)_\infty^2 (q^2;q^2)_\infty}{(q^6;q^6)_\infty (q;q)_\infty}. \tag{3.4}
$$

We first rewrite (3.4) as the following modular function identity for generalized eta-products on $\Gamma_1(24)$:

$$0 = 1 - \frac{\eta_{24,1}\eta_{24,10}\eta_{24,11}}{\eta_{24,3}\eta_{24,6}\eta_{24,9}} - \frac{\eta_{24,2}\eta_{24,5}\eta_{24,7}}{\eta_{24,3}\eta_{24,6}\eta_{24,9}}. \tag{3.5}$$

We use Theorem 2.1 to check that each generalized eta-product is a modular function on $\Gamma_1(24)$. We then use Theorems 2.2 and 2.3 to calculate the order of each generalized eta-product at each cusp of $\Gamma_1(24)$. We calculate the constant in equation (2.6) to find that $B = -4$. We let $g(\tau)$ be the right-hand side of (3.5) and easily show that $\mathrm{Ord}(g(\tau), \infty, \Gamma_1(24)) > 4$. Thus (3.5) follows by Theorem 2.4.

Setting $x = y = 1$ and $a = 3$ in (1.7), we obtain

$$\sum_{i,j\geq 0} \frac{q^{2i^2+2j^2-ij-i+j}}{\left(q^3,q^3\right)_i \left(q^3,q^3\right)_j} = \frac{(q,q)_\infty}{\left(q^3,q^3\right)_\infty^4} \sum_{i,j,k\in\mathbb{Z}} q^{3(i^2+j^2+k^2+ij+ik+jk)+i+2j}.$$

Therefore, (1.15) holds if we can prove

$$\sum_{i,j,k\in\mathbb{Z}} q^{3(i^2+j^2+k^2+ij+ik+jk)+i+2j} = \frac{\left(q^3,q^3\right)_\infty^3 \left(q^2,q^2\right)_\infty^2}{(q,q)_\infty^2}. \tag{3.6}$$

Firstly, we have

$$\begin{aligned}
\sum_{i,j,k\in\mathbb{Z}} q^{3(i^2+j^2+k^2+ij+ik+jk)+i+2j} &= \sum_{i,j,k\in\mathbb{Z}} q^{3[(i+\frac{j+k}{2})^2+(\frac{j+k}{2})^2+\frac{j^2+k^2}{2}]+i+2j} \\
&= \sum_{i,j,k\in\mathbb{Z}} q^{3[(i+j+\frac{k+1}{2})^2+(j+\frac{k+1}{2})^2+2j^2+2j+\frac{k^2+1}{2}]+i+4j+2} \\
&\quad + \sum_{i,j,k\in\mathbb{Z}} q^{3[(i+j+\frac{k}{2})^2+(j+\frac{k}{2})^2+2j^2+\frac{k^2}{2}]+i+4j} \\
&= \sum_{i,j,k\in\mathbb{Z}} q^{3[(i+j+k)^2+(j+k)^2+2j^2+2k^2]+i+4j} \\
&\quad\quad + \sum_{i,j,k\in\mathbb{Z}} q^{3[(i+j+k+\frac{1}{2})^2+(j+k+\frac{1}{2})^2+2j^2+2k^2+2k+\frac{1}{2}]+i+4j} \\
&\quad\quad + \sum_{i,j,k\in\mathbb{Z}} q^{3[(i+j+k+\frac{1}{2})^2+(j+k+\frac{1}{2})^2+2j^2+2j+2k^2\,\frac{1}{2}]+i+4j+2} \\
&\quad\quad + \sum_{i,j,k\in\mathbb{Z}} q^{3[(i+j+k)^2+(j+k)^2+2j^2+2j+2k^2-2k+1]+i+4j+2}.
\end{aligned}$$

The method to simplify these four series is similar, so we take the first series as an example. Setting $j + k = m$ and $i + m = l$, we obtain

$$\begin{aligned}
&\sum_{i,j,k\in\mathbb{Z}} q^{3[(i+j+k)^2+(j+k)^2+2j^2+2k^2]+i+4j} \\
&\quad = \sum_{l,m,k\in\mathbb{Z}} q^{3(m^2+l^2+2(m-k)^2+2k^2)+l+3m-4k} \\
&\quad = \sum_{l,m,k\in\mathbb{Z}} q^{3(2m^2+l^2+(m-2k)^2)+l+3m-4k} \\
&\quad = \sum_{l,m,k\in\mathbb{Z}} q^{3(8m^2+l^2+(2m-2k)^2)+l+6m-4k} + \sum_{l,m,k\in\mathbb{Z}} q^{3(2(2m+1)^2+l^2+(2m-2k+1)^2)+l+6m+3-4k}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l,m,n\in\mathbb{Z}} q^{3(8m^2+l^2+4n^2)+l+2m+4n} + \sum_{l,m,k\in\mathbb{Z}} q^{3(2(2m+1)^2+l^2+(2n+1)^2)+l+2m+3+4n} \\
&= \sum_{l\in\mathbb{Z}} q^{3l^2+l} \sum_{m\in\mathbb{Z}} q^{24m^2+2m} \sum_{n\in\mathbb{Z}} q^{12n^2+4n} + q^{12} \sum_{l\in\mathbb{Z}} q^{3l^2+l} \sum_{m\in\mathbb{Z}} q^{24m^2+26m} \sum_{n\in\mathbb{Z}} q^{12n^2+16n} \\
&= \left(q^6,-q^2,-q^4;q^6\right)_\infty \left(q^{48},-q^{26},-q^{22};q^{48}\right)_\infty \left(q^{24},-q^{16},-q^{8};q^{24}\right)_\infty \\
&\quad + q^6\left(q^6,-q^2,-q^4;q^6\right)_\infty \left(q^{48},-q^{2},-q^{46};q^{48}\right)_\infty \left(q^{24},-q^{4},-q^{20};q^{24}\right)_\infty .
\end{aligned}
$$

Here we let $m-k=n$ for the fourth equality and the last equality follows the Jacobi triple product identity. Therefore, (3.6) holds if we can prove

$$
\begin{aligned}
&\frac{\eta_4\eta_6^2\eta_{16}\eta_{24}^2\eta_{96,44}}{\eta_2\eta_8\eta_{12}\eta_{96,22}\eta_{96,26}} + \frac{\eta_6^2\eta_8^2\eta_{48}^2\eta_{96,4}}{\eta_2\eta_{16}\eta_{24}\eta_{96,2}\eta_{96,46}} + \frac{\eta_2^2\eta_3\eta_{12}\eta_{24}\eta_{48}\eta_{48,20}\eta_{96,20}}{\eta_1\eta_4\eta_6\eta_{48,10}\eta_{48,14}\eta_{96,10}\eta_{96,38}} \\
&\quad + \frac{\eta_2^2\eta_3\eta_{12}\eta_{24}\eta_{48}\eta_{48,4}\eta_{96,28}}{\eta_1\eta_4\eta_6\eta_{48,2}\eta_{48,22}\eta_{96,14}\eta_{96,34}} + \frac{\eta_6^2\eta_8^2\eta_{48}^2\eta_{96,44}}{\eta_2\eta_{16}\eta_{24}\eta_{96,22}\eta_{96,26}} + \frac{\eta_4\eta_6^2\eta_{16}\eta_{24}^2\eta_{96,4}}{\eta_2\eta_8\eta_{12}\eta_{96,2}\eta_{96,46}} \\
&\quad + \frac{\eta_2^2\eta_3\eta_{12}\eta_{24}\eta_{48}\eta_{48,4}\eta_{96,20}}{\eta_1\eta_4\eta_6\eta_{48,2}\eta_{48,22}\eta_{96,10}\eta_{96,38}} + \frac{\eta_2^2\eta_3\eta_{12}\eta_{24}\eta_{48}\eta_{48,20}\eta_{96,28}}{\eta_1\eta_4\eta_6\eta_{48,10}\eta_{48,14}\eta_{96,14}\eta_{96,34}} = \frac{\eta_3^3\eta_2^2}{\eta_1^2}.
\end{aligned}
\tag{3.7}
$$

We first rewrite (3.7) as the following modular function identity for generalized eta-products on $\Gamma_1(96)$:

$$
0 = 1 - x(L_1+L_2+L_3+L_4+L_5+L_6+L_7+L_8), \tag{3.8}
$$

where

$$
\begin{aligned}
x &= \frac{\eta_{96,1}\eta_{96,5}\eta_{96,7}\eta_{96,11}^2\eta_{96,13}^2\eta_{96,17}^2\eta_{96,19}^2\eta_{96,23}^2\eta_{96,25}^2\eta_{96,29}^2\eta_{96,31}^2\eta_{96,35}^2\eta_{96,37}^2\eta_{96,41}^2\eta_{96,43}^2\eta_{96,47}^2}{\eta_{96,3}\eta_{96,6}^3\eta_{96,9}\eta_{96,12}^3\eta_{96,15}\eta_{96,18}^3\eta_{96,21}\eta_{96,24}^3\eta_{96,27}\eta_{96,30}^3\eta_{96,33}\eta_{96,36}^3\eta_{96,39}\eta_{96,42}^3\eta_{96,45}}, \\
L_1 &= \frac{\eta_{96,1}\eta_{96,5}\eta_{96,6}\eta_{96,7}\eta_{96,12}\eta_{96,18}\eta_{96,24}^2\eta_{96,30}\eta_{96,36}\eta_{96,42}\eta_{96,44}}{\eta_{96,2}\eta_{96,8}\eta_{96,10}\eta_{96,14}\eta_{96,22}^2\eta_{96,26}^2\eta_{96,34}\eta_{96,38}\eta_{96,40}\eta_{96,46}}, \\
L_2 &= \frac{\eta_{96,1}\eta_{96,5}\eta_{96,6}\eta_{96,7}\eta_{96,8}\eta_{96,12}\eta_{96,18}\eta_{96,24}^2\eta_{96,30}\eta_{96,36}\eta_{96,40}\eta_{96,42}}{\eta_{96,2}^2\eta_{96,10}\eta_{96,14}\eta_{96,20}\eta_{96,22}\eta_{96,26}\eta_{96,28}\eta_{96,34}\eta_{96,38}\eta_{96,44}\eta_{96,46}^2}, \\
L_3 &= \frac{\eta_{96,2}\eta_{96,6}\eta_{96,12}\eta_{96,18}\eta_{96,20}^2\eta_{96,22}\eta_{96,24}^2\eta_{96,26}\eta_{96,28}\eta_{96,30}\eta_{96,36}\eta_{96,42}\eta_{96,46}}{\eta_{96,10}\eta_{96,11}\eta_{96,13}\eta_{96,17}\eta_{96,19}\eta_{96,23}\eta_{96,25}\eta_{96,29}\eta_{96,31}\eta_{96,35}\eta_{96,37}\eta_{96,38}\eta_{96,41}\eta_{96,43}\eta_{96,47}}, \\
L_4 &= \frac{\eta_{96,4}\eta_{96,6}\eta_{96,10}\eta_{96,12}\eta_{96,18}\eta_{96,24}^2\eta_{96,28}\eta_{96,30}\eta_{96,36}\eta_{96,38}\eta_{96,42}\eta_{96,44}}{\eta_{96,11}\eta_{96,13}\eta_{96,17}\eta_{96,19}\eta_{96,23}\eta_{96,25}\eta_{96,29}\eta_{96,31}\eta_{96,35}\eta_{96,37}\eta_{96,41}\eta_{96,43}\eta_{96,47}}, \\
L_5 &= \frac{\eta_{96,4}\eta_{96,6}\eta_{96,12}\eta_{96,14}\eta_{96,18}\eta_{96,20}\eta_{96,24}^2\eta_{96,30}\eta_{96,34}\eta_{96,36}\eta_{96,42}\eta_{96,44}}{\eta_{96,11}\eta_{96,13}\eta_{96,17}\eta_{96,19}\eta_{96,23}\eta_{96,25}\eta_{96,29}\eta_{96,31}\eta_{96,35}\eta_{96,37}\eta_{96,41}\eta_{96,43}\eta_{96,47}}, \\
L_6 &= \frac{\eta_{96,2}\eta_{96,6}\eta_{96,12}\eta_{96,18}\eta_{96,20}\eta_{96,22}\eta_{96,24}^2\eta_{96,26}\eta_{96,28}^2\eta_{96,30}\eta_{96,36}\eta_{96,42}\eta_{96,46}}{\eta_{96,11}\eta_{96,13}\eta_{96,14}\eta_{96,17}\eta_{96,19}\eta_{96,23}\eta_{96,25}\eta_{96,29}\eta_{96,31}\eta_{96,34}\eta_{96,35}\eta_{96,37}\eta_{96,41}\eta_{96,43}\eta_{96,47}}, \\
L_7 &= \frac{\eta_{96,1}\eta_{96,5}\eta_{96,6}\eta_{96,7}\eta_{96,8}\eta_{96,12}\eta_{96,18}\eta_{96,24}^2\eta_{96,30}\eta_{96,36}\eta_{96,40}\eta_{96,42}}{\eta_{96,2}\eta_{96,4}\eta_{96,10}\eta_{96,14}\eta_{96,20}\eta_{96,22}^2\eta_{96,26}^2\eta_{96,28}\eta_{96,34}\eta_{96,38}\eta_{96,46}}, \\
L_8 &= \frac{\eta_{96,1}\eta_{96,4}\eta_{96,5}\eta_{96,6}\eta_{96,7}\eta_{96,12}\eta_{96,18}\eta_{96,24}^2\eta_{96,30}\eta_{96,36}\eta_{96,42}}{\eta_{96,2}^2\eta_{96,8}\eta_{96,10}\eta_{96,14}\eta_{96,22}\eta_{96,26}\eta_{96,34}\eta_{96,38}\eta_{96,40}\eta_{96,46}^2}.
\end{aligned}
$$

We use Theorem 2.1 to check that each generalized eta-product is a modular function on $\Gamma_1(96)$. We then use Theorems 2.2 and 2.3 to calculate the order of each generalized eta-product at each cusp of $\Gamma_1(96)$. We calculate the constant in equation (2.6) to find that $B=-192$. We let $g(\tau)$ be the right-hand side of (3.8) and easily show that $\mathrm{Ord}(g(\tau),\infty,\Gamma_1(96)) > 192$. Thus, (3.8) follows by Theorem 2.4.

Setting $x=y=1$ and $a=5$ in (1.7), together with (1.10), we can prove (1.16). ■

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References

[1] Andrews G.E., On the general Rogers–Ramanujan theorem, *Mem. Amer. Math. Soc.*, Vol. 152, American Mathematical Society, Providence, RI, 1974.

[2] Andrews G.E., Multiple q-series identities, *Houston J. Math.* **7** (1981), 11–22.

[3] Berndt B.C., Ramanujan's notebooks. Part III, Springer, New York, 1991.

[4] Bressoud D.M., Analytic and combinatorial generalizations of the Rogers–Ramanujan identities, *Mem. Amer. Math. Soc.* **24** (1980), 54 pages.

[5] Cao Z., Wang L., Multi-sum Rogers–Ramanujan type identities, *J. Math. Anal. Appl.* **522** (2023), 126960, 24 pages, arXiv:2205.12786.

[6] Cho B., Koo J.K., Park Y.K., Arithmetic of the Ramanujan–Göllnitz–Gordon continued fraction, *J. Number Theory* **129** (2009), 922–947.

[7] Frye J., Garvan F., Automatic proof of theta-function identities, in Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory, *Texts Monogr. Symbol. Comput.*, Springer, Cham, 2019, 195–258, arXiv:1807.08051.

[8] Gasper G., Rahman M., Basic hypergeometric series, 2nd ed., *Encyclopedia Math. Appl.*, Vol. 96, Cambridge University Press, Cambridge, 2004.

[9] Hardy G.H., The Indian Mathematician Ramanujan, *Amer. Math. Monthly* **44** (1937), 137–155.

[10] Nahm W., Conformal field theory, dilogarithms, and three-dimensional manifolds, *Adv. Appl. Clifford Algebras* **4** (1994), suppl. 1, 179–191.

[11] Nahm W., Conformal field theory and the dilogarithm, in XIth International Congress of Mathematical Physics (Paris, 1994), International Press, Cambridge, MA, 1995, 662–667.

[12] Nahm W., Conformal field theory and torsion elements of the Bloch group, in Frontiers in Number Theory, Physics, and Geometry. II, Springer, Berlin, 2007, 67–132, arXiv:hep-th/0404120.

[13] Rankin R.A., Modular forms and functions, Cambridge University Press, Cambridge, 1977.

[14] Robins S., Generalized Dedekind η-products, in The Rademacher Legacy to Mathematics (University Park, PA, 1992), *Contemp. Math.*, Vol. 166, American Mathematical Society, Providence, RI, 1994, 119–128.

[15] Rogers L.J., Second memoir on the expansion of certain infinite products, *Proc. Lond. Math. Soc.* **25** (1893/94), 318–343.

[16] Schlosser M.J., Bilateral identities of the Rogers–Ramanujan type, *Trans. Amer. Math. Soc. Ser. B* **10** (2023), 1119–1140, arXiv:1806.01153.

[17] Slater L.J., Further identities of the Rogers–Ramanujan type, *Proc. London Math. Soc.* **54** (1952), 147–167.

[18] Wang L., New proofs of some double sum Rogers–Ramanujan type identities, *Ramanujan J.* **62** (2023), 251–272, arXiv:2203.15572.

[19] Wang L., Explicit forms and proofs of Zagier's rank three examples for Nahm's problem, *Adv. Math.* **450** (2024), 109743, 46 pages, arXiv:2211.04375.

[20] Wang L., Identities on Zagier's rank two examples for Nahm's problem, *Res. Math. Sci.* **11** (2024), 49, 34 pages, arXiv:2210.10748.

[21] Warnaar S.O., Zudilin W., Dedekind's η-function and Rogers–Ramanujan identities, *Bull. Lond. Math. Soc.* **44** (2012), 1–11, arXiv:1001.1571.

[22] Zagier D., The dilogarithm function, in Frontiers in Number Theory, Physics, and Geometry. II, Springer, Berlin, 2007, 3–65.