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The Linearized Floer Equation in a Chart

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Abstract. In this article, we are considering the Hessian of the area functional in a non-Darboux chart. This does not seem to have been considered before and leads to an interesting new mathematical structure which we introduce in this article and refer to as *almost extendable weak Hessian field*. Our main result is a Fredholm theorem for Robbin–Salamon operators associated to *non-continuous* Hessians which we prove by taking advantage of this new structure.

Key words: Fredholm; Robbin–Salamon operators; Floer theory; para-Darboux

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1 Introduction

In a Darboux chart, the Hessian of the area functional is the constant operator $A_0 = \mathrm{i}\partial_t$. In this note, we look at the Hessian of the area functional in a *non-Darboux* chart. To the best of our knowledge, this was never done before. Here, in particular, an additional summand shows up whose discontinuity challenges the previous methods to obtain the Fredholm property for the associated Robbin–Salamon operators.

We discover an interesting structure from the point of view of scale geometry. We think that the structure is of crucial importance in order to understand the structure of Floer theory in general. Therefore, we give this structure a name and refer to it as an *almost extendable weak Hessian field*. We then study the Robbin–Salamon operator $\mathbb{D} = \partial_s + A$ associated to a connecting path in an almost extendable weak Hessian field A.

Our main result is that this Robbin–Salamon operator is Fredholm. The difficulty in proving this result is that the Hessian is not necessarily continuous. Therefore, the improvement, due to Rabier [14], of the classical Robbin–Salamon theorem [15] is not necessarily applicable to the situation at hand. However, the new notion of almost extendability discovered in this article allows us to decompose the Hessian in the sum of two operators where one is still continuous while the other one is of lower order, in symbols $A = F + C$. In contrast to the Hessian itself, the two summands are not necessarily symmetric any more. Luckily the theorem of Rabier can as well deal with non-symmetric situations as long as continuity is guaranteed. We check that the conditions of Rabier apply to the operator associated to the first summand in our situation $\mathbb{F} = \partial_s + F$. Hence $\mathbb{F}$ is a Fredholm operator. Since the second summand C is of lower order, we show that it gives rise to a compact multiplication operator. This then proves our main result, because the Fredholm property is invariant under compact perturbation.

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This article is part of our endeavor to understand the basic structure behind Floer theory in order to make Floer theory applicable to a broader class of problems involving Hamiltonian delay equation as explained in [3, 4].

2 The area functional in a Darboux chart

On $\mathbb{C}^n$, the standard symplectic form $\omega = \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j$ has the primitive

$$\lambda = \frac{1}{2}\sum_{j=1}^{n}(x_j \mathrm{d}y_j - y_j \mathrm{d}x_j).$$

We are considering the area functional

$$\mathcal{A}_0\colon\ C^\infty\big(\mathbb{S}^1,\mathbb{C}^n\big) \to \mathbb{R}, \qquad u \mapsto \int_{\mathbb{S}^1} u^*\lambda,$$

where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. We write $u \in C^\infty\big(\mathbb{S}^1,\mathbb{C}^n\big)$ as a Fourier series

$$u(t) = \sum_{k\in\mathbb{Z}} u_k e^{2\pi \mathrm{i} k t}, \qquad \dot u(t) = \sum_{\ell\in\mathbb{Z}} 2\pi\mathrm{i}\ell u_\ell e^{2\pi\mathrm{i}\ell t},$$

where $u_k \in \mathbb{C}^n$ for $k \in \mathbb{Z}$. Expressing the area functional via the Fourier decomposition,[1]

$$\begin{aligned}
\mathcal{A}_0(u) &= \int_0^1 \lambda_u \dot u \,\mathrm{d}t\\
&= -\int_0^1 \frac{1}{2}\mathrm{Re}\,\langle u, \mathrm{i}\dot u\rangle_{\mathbb{C}^n}\,\mathrm{d}t\\
&= -\int_0^1 \frac{1}{2}\mathrm{Re}\sum_{k,\ell\in\mathbb{Z}} 2\pi\ell\big\langle u_k e^{2\pi\mathrm{i}kt}, \mathrm{i}^2 u_\ell e^{2\pi\mathrm{i}\ell t}\big\rangle_{\mathbb{C}^n}\mathrm{d}t\\
&= \mathrm{Re}\sum_{k,\ell\in\mathbb{Z}} \pi\ell\,\langle u_k, u_\ell\rangle_{\mathbb{C}^n} \underbrace{\int_0^1 e^{2\pi\mathrm{i}(k-\ell)t}\,\mathrm{d}t}_{=\delta_{k\ell}}\\
&= \sum_{k\in\mathbb{Z}} \pi k |u_k|^2_{C^n}.
\end{aligned}$$

In particular, we see that the area functional in a Darboux chart is just a quadratic function in the Fourier coefficients. Therefore, the Hessian is constant independent of u with eigenvalues $2\pi k$ where $k \in \mathbb{Z}$. Abbreviating $H_k = W^{k,2}\big(\mathbb{S}^1,\mathbb{C}^n\big)$, we see that the Hessian of $\mathcal{A}_0$ at every point $u \in H_1$ is a Fredholm operator of index zero from H_k to H_{k-1} for every $k \in \mathbb{N}$.

3 Euclidean inner product and symplectic forms

3.1 Associated anti-symmetric matrix

Definition 3.1. Let $\langle\cdot,\cdot\rangle$ be the Euclidean inner product on $\mathbb{R}^{2n}$. Let $\mathfrak{U} \subset \mathbb{R}^{2n}$ be an open subset carrying an exact symplectic form $\omega = \mathrm{d}\lambda$. Then the identity

$$\langle\cdot,\cdot\rangle = \omega_x(\cdot, B_x\cdot) \tag{3.1}$$

for $x \in \mathfrak{U}$ determines a map $\mathfrak{U} \to \mathcal{L}\big(\mathbb{R}^{2n}\big)$, $x \mapsto B_x$. Each linear map $B_x = B_x^\omega$ is unique by non-degeneracy of ω_x.

[1] Let $z = x + \mathrm{i}y$ and $\xi = \hat x + \mathrm{i}\hat y$ where $x, y, \hat x, \hat y \in \mathbb{R}^n$. Hence $z = (x_1 + \mathrm{i}y_1, \dots, x_n + \mathrm{i}y_n)$ and $\xi = (\hat x_1 + \mathrm{i}\hat y_1, \dots, \hat x_n + \mathrm{i}\hat y_n)$ and we calculate $\mathrm{Re}\,\langle z, \mathrm{i}\xi\rangle_{\mathbb{C}^n} = \mathrm{Re}\sum_{k=1}^n (x_k + \mathrm{i}y_k)\overline{(-\hat y_k + \mathrm{i}\hat x_k)} = -\lambda_z\xi$.

Lemma 3.2. *At any point $x \in \mathfrak{U}$ the linear map $B_x\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is invertible and satisfies*

(i) $\omega(\xi, B_x\eta) = -\omega(B_x\xi, \eta)$ *(ω-anti-symmetry of B_x),*

(ii) $\langle \xi, B_x\eta\rangle = -\langle B_x\xi, \eta\rangle$ *(anti-symmetry of B_x),*

(iii) $\langle \xi, -B_x^2\eta\rangle = \langle -B_x^2\xi, \eta\rangle$ *(symmetry of $-B_x^2$),*

(iv) $\langle -B_x^2\xi, \xi\rangle = |B\xi|^2 > 0$, $\xi \neq 0$ *(positive definiteness of $-B_x^2$),*

(v) $\sqrt{-B_x^2}$ *and its inverse are both symmetric positive definite*

for all $\xi, \eta \in \mathbb{R}^{2n}$.

As we shall see in Remark 3.5 further below, the invertible linear maps $B_x\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ are orientation preserving, in symbols $B_x \in \mathrm{GL}^+(2n, \mathbb{R})$.

Proof. Suppose $B_x\eta = 0$, then it follows from (3.1) and non-degeneracy of $\langle\cdot,\cdot\rangle$ that $\eta = 0$. Hence $B_x\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is injective and therefore surjective.

(i) By (3.1) and symmetry of $\langle\cdot,\cdot\rangle$ and anti-symmetry of ω, we obtain

$$\omega_x(\xi, B_x\eta) = \langle \xi, \eta\rangle = \langle \eta, \xi\rangle = \omega_x(\eta, B_x\xi) = -\omega_x(B_x\xi, \eta).$$

(ii) By (3.1) and anti-symmetry of ω and symmetry of $\langle\cdot,\cdot\rangle$, we obtain

$$\langle \xi, B_x\eta\rangle = \omega_x\bigl(\xi, B_x^2\eta\bigr) \overset{\text{(i)}}{=} \omega_x\bigl((-B_x)^2\xi, \eta\bigr) = -\omega_x\bigl(\eta, B_x^2\xi\bigr) = -\langle \eta, B_x\xi\rangle = \langle -B_x\xi, \eta\rangle .$$

(iii) holds by applying twice (ii).

(iv) By (ii) and injectivity, we get

$$\langle -B_x^2\xi, \xi\rangle = \langle B_x\xi, B_x\xi\rangle = |B_x\xi|^2 > 0.$$

(v) By Heron's construction of the square root of a symmetric positive definite matrix Q, Lemma D.2, the square root is symmetric positive definite and depends smoothly on Q. Taking the inverse preserves symmetry and positive definiteness. ∎

For the previous and the following lemma, see also [10, Proposition 2.5.6].

3.2 Compatible almost complex structure

Lemma 3.3. *The linear map $J_B := \sqrt{-B^2}^{-1}B\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is an almost complex structure on $\mathfrak{U}$ compatible with ω, in symbols*

(a) $J_BJ_B = -\mathbb{1}$ *(almost complex structure),*

(b) $g_{J_B}(\xi, \eta) := \omega(\xi, J_B\eta) = g_{J_B}(\eta, \xi)$ *(g_{J_B} symmetric),*

(c) $g_{J_B}(\xi, \xi) = \langle \xi, \sqrt{-B^2}^{-1}\xi\rangle > 0$, $\xi \neq 0$ *(g_{J_B} positive definite)*

for all vector fields ξ, η along $\mathfrak{U} \subset \mathbb{R}^{2n}$.

Proof. (a) Since B commutes with $-B^2$, then by Corollary D.4 it also commutes with $\sqrt{-B^2}$, hence with the inverse of $\sqrt{-B^2}$. This is used in equality 2 below

$$\begin{aligned} J_BJ_B &= \sqrt{-B^2}^{-1}B\sqrt{-B^2}^{-1}B \overset{2}{=} \sqrt{-B^2}^{-1}\sqrt{-B^2}^{-1}BB \\ &= \bigl(\sqrt{-B^2}\sqrt{-B^2}\bigr)^{-1}BB = (-BB)^{-1}BB = -\mathbb{1}. \end{aligned}$$

(b) Equality 3 below holds analogously to equality 2 in (a), equality 4 uses (3.1), equality 5 is by Lemma 3.2 part (v), equality 6 is by symmetry of the Euclidean inner product:

$$\begin{aligned} \underline{g_{J_B}(\xi,\eta)} &:= \omega(\xi, J_B\eta) = \omega\big(\xi, \sqrt{-B^2}^{\,-1} B\eta\big) \stackrel{3}{=} \omega\big(\xi, B\sqrt{-B^2}^{\,-1}\eta\big) \stackrel{4}{=} \underline{\big\langle \xi, \sqrt{-B^2}^{\,-1}\eta\big\rangle} \\ &\stackrel{5}{=} \big\langle \sqrt{-B^2}^{\,-1}\xi, \eta\big\rangle = \big\langle \eta, \sqrt{-B^2}^{\,-1}\xi\big\rangle = g_{J_B}(\eta,\xi). \end{aligned}$$

The last step uses equality of the two underlined terms, just commute ξ and η.

(c) Step 2 below holds analogously to equality 2 in (a), Step 3 by (3.1), namely

$$g_{J_B}(\xi,\xi) := \omega\big(\xi, \sqrt{-B^2}^{\,-1}B\xi\big) \stackrel{2}{=} \omega\big(\xi, B\sqrt{-B^2}^{\,-1}\xi\big) \stackrel{3}{=} \big\langle \xi, \sqrt{-B^2}^{\,-1}\xi\big\rangle > 0$$

pointwise at each $x \in \mathfrak{U}$ whenever $\xi(x) \neq 0$. This proves Lemma 3.3. ∎

Lemma 3.4. *Any complex structure*[2] *J on a real vector space V of finite dimension $2n$ has determinant* 1.

Proof. For a complex basis on V the matrix representation of J is the complex $n\times n$ matrix $J_{\mathbb{C}} = \mathrm{i}\mathbb{1}_n$. Its determinant is $\det(J_{\mathbb{C}}) = \mathrm{i}^n$. In general, writing a complex $n\times n$ matrix $Z = X + \mathrm{i}Y$ as sum of real and imaginary parts, define the corresponding real $2n\times 2n$ matrix as follows and observe the identity

$$Z_{\mathbb{R}} := \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathrm{i}\mathbb{1} & -\mathrm{i}\mathbb{1} \end{pmatrix}^{-1} \begin{pmatrix} X - \mathrm{i}Y & 0 \\ 0 & X + \mathrm{i}Y \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathrm{i}\mathbb{1} & -\mathrm{i}\mathbb{1} \end{pmatrix}.$$

Take the determinant to obtain the formula

$$\det Z_{\mathbb{R}} = \det \begin{pmatrix} \bar{Z} & 0 \\ 0 & Z \end{pmatrix} = \det \bar{Z} \cdot \det Z = \overline{\det Z} \cdot \det Z = |\det Z|^2.$$

Hence $\det J = |\det J_{\mathbb{C}}|^2 = |\mathrm{i}^n|^2 = 1$. ∎

Remark 3.5. The linear map B_x determined by (3.1) has positive determinant

$$\det B = \det\big(\sqrt{-B^2} J_B\big) = \det \sqrt{-B^2} \cdot \det J_B = \det \sqrt{-B^2} > 0. \tag{3.2}$$

Here identity one is by definition of J_B and identity three by Lemma 3.4. Strictly positive holds true by Lemma 3.2 (v).

4 Action functional

We denote by $\mathbb{S}^1$ the circle $\mathbb{R}/\mathbb{Z}$. Consider the Hilbert space triple

$$H_0 := L^2\big(\mathbb{S}^1, \mathbb{R}^{2n}\big), \qquad H_1 := W^{1,2}\big(\mathbb{S}^1, \mathbb{R}^{2n}\big), \qquad H_2 := W^{2,2}\big(\mathbb{S}^1, \mathbb{R}^{2n}\big). \tag{4.1}$$

Given an open subset $\mathfrak{U} \subset \mathbb{R}^{2n}$, let $\omega = \mathrm{d}\lambda$ be an exact symplectic form on $\mathfrak{U}$. Define open subsets of H_1 and H_2 by

$$U_\ell := \big\{u \in H_\ell \mid u(t) \in \mathfrak{U}\ \forall t \in \mathbb{S}^1\big\} \subset C^0\big(\mathbb{S}^1, \mathfrak{U}\big), \qquad \ell = 1, 2. \tag{4.2}$$

Convention. We write $\dot u(t) := \frac{\mathrm{d}}{\mathrm{d}t} u(t)$. All sums in Section 4 run from 1 to $2n$.

Definition 4.1. The *symplectic action* (functional) is defined by

$$\mathcal{A}_0\colon\ H_1 \supset U_1 \to \mathbb{R}, \qquad u \mapsto \int_{\mathbb{S}^1} u^*\lambda. \tag{4.3}$$

[2] A linear map $J\colon V \to V$ such that $J^2 = -\mathbb{1}$.

4.1 First derivative and gradient

Lemma 4.2 (gradient)**.** *The derivative at $u \in U_1$ in direction $\xi \in H_1$ is*

$$d\mathcal{A}_0|_u \xi = \int_0^1 \omega_{u_t}(\xi_t, \dot u_t)\, dt = \int_0^1 \langle \xi_t, B_{u_t}^{-1} \dot u_t \rangle dt =: \langle \xi, B_u^{-1} \dot u \rangle_{H_0},$$

where $B_{u_t} \in \mathcal{L}(\mathbb{R}^{2n})$ is determined by (3.1). So the (L^2)-gradient is given by

$$(\operatorname{grad} \mathcal{A}_0|_u)(t) = B_{u(t)}^{-1} \dot u(t)$$

pointwise for every $t \in \mathbb{S}^1$.

Proof (global version). Given $u \in U_1$ and $\xi \in H_1$, let u_r be a smooth family with $u_0 = u$ and $\frac{d}{dr}\big|_{r=0} u_r = \xi$. We compute

$$\begin{aligned} d\mathcal{A}_0|_u \xi &= \frac{d}{dr}\Big|_{r=0} \mathcal{A}_0(u_r) = \int_{\mathbb{S}^1} \frac{d}{dr}\Big|_{r=0} u_r^* \lambda \stackrel{3}{=} \int_{\mathbb{S}^1} u^* L_\xi \lambda \\ &= \int_{\mathbb{S}^1} u^*(d i_\xi + i_\xi d)\lambda \stackrel{5}{=} \int_{\mathbb{S}^1} d u^* i_\xi \lambda + \int_{\mathbb{S}^1} u^* i_\xi \omega \stackrel{6}{=} 0 + \int_0^1 \omega_{u_t}(\xi_t, \dot u_t)\, dt. \end{aligned}$$

Here Step 3 is by definition of the Lie derivative and Step 4 is Cartan's formula. Step 5 uses that the exterior derivative d commutes with pull-back. Step 6 is by Stokes' theorem and the fact that the integral is over the empty set $\partial\mathbb{S}^1 = \emptyset$, so the integral is 0. This proves Lemma 4.2. ∎

Proof (local version). Let $u\colon \mathbb{S}^1 \to \mathfrak{U}$ be in U_1 and $\xi\colon \mathbb{S}^1 \to \mathbb{R}^{2n}$ in H_1. Take the derivative of $t \mapsto (\lambda|_u \xi)(t)$ to get

$$\begin{aligned} \frac{d}{dt}((u^*\lambda)\xi)(t) &= \frac{d}{dt} \sum_i \lambda_i(u(t)) \xi_i(t) \\ &= \sum_{i,j} \partial_j \lambda_i(u(t)) \dot u_j(t) \xi_i(t)\, dt + \sum_i \lambda_i(u(t)) \dot\xi_i(t)\, dt. \\ &\stackrel{3}{=} \sum_{i,j} \partial_i \lambda_j(u(t)) \dot u_i(t) \xi_j(t)\, dt + \underline{\sum_i \lambda_i(u(t)) \dot\xi_i(t)\, dt}, \end{aligned}$$

where in Step 3 we renamed the summation indices (i,j) by (j,i). Let u_r be a variation associated to u and ξ. Substitute the underlined term in what follows

$$\begin{aligned} \frac{d}{dr}\Big|_{r=0} (u_r^*\lambda)(t) &\stackrel{1}{=} \sum_{i,j} \partial_j \lambda_i(u(t)) \xi_j(t) \dot u_i(t)\, dt + \underline{\sum_i \lambda_i(u(t)) \dot\xi_i(t)\, dt} \\ &= \sum_{i,j} \Big(\underbrace{\partial_j \lambda_i(u(t)) \xi_j(t) \dot u_i(t) - \partial_i \lambda_j(u(t)) \dot u_i(t) \xi_j(t)}_{= \omega_{ji}(u(t)) \xi_j(t) \dot u_i(t)} \Big) dt + \tfrac{d}{dt}((u^*\lambda)\xi)(t). \end{aligned} \tag{4.4}$$

We integrate and use 1-periodicity in time t to obtain the formula

$$d\mathcal{A}_0|_u \xi = \int_{\mathbb{S}^1} \frac{d}{dr}\Big|_{r=0} u_r^* \lambda = \int_0^1 \sum_{i,j} \omega_{ji}(u_t) \xi_j(t) \dot u_i(t)\, dt = \int_0^1 \omega_{u_t}(\xi_t, \dot u_t)\, dt.$$

This proves Lemma 4.2. ∎

4.2 Second derivative

Consider an open subset $\mathfrak{U}$ of $\mathbb{R}^{2n}$ with coordinates $x = (x_1, \dots, x_{2n})$. A primitive λ of ω is of the form $\lambda = \sum_{i=1}^{2n} \lambda_i(x) \mathrm{d}x_i$. We denote the second derivatives of the coefficient functions $x \mapsto \lambda_i(x)$ and their differences by

$$\Lambda_{kji}(x) := \partial_k \partial_j \lambda_i(x), \qquad L_{kji}(x) := \Lambda_{kji}(x) - \Lambda_{ikj}(x), \qquad i, j, k = 1, \dots, 2n,$$

at any point $x \in \mathfrak{U} \subset \mathbb{R}^{2n}$.

To compute the Hessian of the function $\mathcal{A}_0 \colon H_1 \supset U_1 \to \mathbb{R}$ we introduce the following covariant tensor field of type $(0,3)$ in the notation of [12, Section 2]. The set of vector fields, respectively, functions, along $\mathfrak{U} \subset \mathbb{R}^{2n}$ are denoted by

$$\mathcal{X}(\mathfrak{U}) := C^\infty\big(\mathfrak{U}, \mathbb{R}^{2n}\big), \qquad \mathcal{F}(\mathfrak{U}) := C^\infty(\mathfrak{U}, \mathbb{R}).$$

Definition 4.3. A tensor field of type $(0,3)$ is defined by the sum

$$L\colon\ \mathcal{X}(\mathfrak{U}) \times \mathcal{X}(\mathfrak{U}) \times \mathcal{X}(\mathfrak{U}) \to \mathcal{F}(\mathfrak{U}), \qquad (\eta, \xi, \zeta) \mapsto \sum_{i,j,k} L_{kji} \eta_k \xi_j \zeta_i.$$

Remark 4.4 (Darboux charts)**.** Locally, in Darboux coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$, the symplectic form ω has as a primitive the Liouville form $\lambda = \sum_{i=1}^n p_i \mathrm{d}q_i$. Observe that all coefficients, namely the p_i, are linear functions in the coordinates. Therefore, the tensor field vanishes $L \equiv 0$.

We discuss some symmetries of L. By the theorem of Schwarz, $\Lambda_{kji} = \Lambda_{jki}$ and therefore

$$L_{kji} - L_{jki} - L_{ijk} = 0. \tag{4.5}$$

Furthermore, by definition of the L_{kji}, the sum of cyclic permutations vanishes

$$L_{kji} + L_{ikj} + L_{jik} = 0.$$

Consequently, the sum of cyclic permutations in L vanishes as well

$$L(\eta, \xi, \zeta) + L(\zeta, \eta, \xi) + L(\xi, \zeta, \eta) = 0.$$

By the consequence (4.5) of Schwarz, the following sum vanishes as well:

$$L(\eta, \xi, \zeta) - L(\xi, \eta, \zeta) - L(\zeta, \xi, \eta) = 0.$$

Lemma 4.5. *The second derivative of $\mathcal{A}_0 \colon U_1 \to \mathbb{R}$ at u evaluated on two vector fields $\xi, \eta \in H_1$ is the symmetric bilinear form given by*

$$\mathrm{d}^2 \mathcal{A}_0|_u (\xi, \eta) = \int_0^1 \omega_u(\xi, \dot{\eta}) \, \mathrm{d}t + \int_0^1 L_u(\eta, \xi, \dot{u}) \, \mathrm{d}t.$$

Proof. Given $u \in U_1$ and $\xi, \eta \in H_1$, let $u_{r,\rho}$ be an associated two-parameter variation, that is, $u_{0,0} = u$ and

$$\frac{\mathrm{d}}{\mathrm{d}r} u_{r,\rho}(t)\Big|_{(r,\rho)=(0,0)} = \xi(t), \qquad \frac{\mathrm{d}}{\mathrm{d}\rho} u_{r,\rho}(t)\Big|_{(r,\rho)=(0,0)} = \eta(t).$$

We use equality 1 in (4.4) in Step 2 to compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{(r,\rho)=(0,0)}\left(u_{r,\rho}^*\lambda\right) &= \frac{\mathrm{d}}{\mathrm{d}\rho}\Big|_{\rho=0}\Big(\sum_{i,j}\partial_j\lambda_i\big(u_{(0,\rho)}\big)\xi_j\big(\dot u_{(0,\rho)}\big)_i\,\mathrm{d}t+\sum_i\lambda_i\big(u_{(0,\rho)}\big)\dot\xi_i\,\mathrm{d}t\Big)\\ &=\sum_{i,j,k}\partial_k\partial_j\lambda_i(u)\eta_k\xi_j\dot u_i\,\mathrm{d}t+\sum_{i,j}\partial_j\lambda_i(u)\big(\xi_j\dot\eta_i+\eta_j\dot\xi_i\big)\mathrm{d}t\end{aligned} \tag{4.6}$$

evaluated pointwise at t. In the following calculation, we use the definition (4.3) of the action functional in Step 2, and (4.6) in Step 3, to obtain

$$\begin{aligned}\mathrm{d}^2\mathcal{A}_0(u)(\xi,\eta) &= \frac{\mathrm{d}}{\mathrm{d}\rho}\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{(r,\rho)=(0,0)}\mathcal{A}_0(u_{r,\rho})\\ &\overset{2}{=}\int_{\mathbb{S}^1}\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{(r,\rho)=(0,0)}{u_{r,\rho}}^*\lambda\\ &\overset{3}{=}\int_0^1\Big(\sum_{i,j,k}\partial_k\partial_j\lambda_i(u)\eta_k\xi_j\dot u_i\,\mathrm{d}t+\sum_{i,j}\partial_j\lambda_i(u)\big(\xi_j\dot\eta_i+\eta_j\dot\xi_i\big)\Big)\mathrm{d}t\\ &\overset{4}{=}\int_0^1\Big(\sum_{i,j,k}\underbrace{\partial_k\partial_j\lambda_i(u)}_{=:\Lambda_{kji}(u)}\eta_k\xi_j\dot u_i\,\mathrm{d}t+\sum_{i,j}\underbrace{\partial_j\lambda_i(u)\big(\xi_j\dot\eta_i-\dot\eta_j\xi_i\big)}_{=\omega_{ji}(u)\xi_j\dot\eta_i}\Big)\mathrm{d}t\\ &\quad-\int_0^1\sum_{i,j,k}\partial_k\partial_j\lambda_i(u)\dot u_k\eta_j\xi_i\,\mathrm{d}t\\ &=\int_0^1\omega_u(\xi,\dot\eta)\,\mathrm{d}t+\int_0^1\sum_{i,j,k}\underbrace{\big(\Lambda_{kji}(u)-\Lambda_{ikj}(u)\big)}_{=:L_{kji}(u)}\eta_k\xi_j\dot u_i\,\mathrm{d}t,\end{aligned}$$

where Step 4 is by integration by parts (boundary terms vanish by periodicity). Symmetry holds by the theorem of Schwarz. This proves Lemma 4.5. ∎

4.3 Para-Darboux Hessian

Using the metric isomorphism, we turn the $(0,3)$ tensor L into the $(1,2)$ tensor

$$\bar L\colon\ \mathcal{X}(\mathfrak{U})\times\mathcal{X}(\mathfrak{U})\to\mathcal{X}(\mathfrak{U}),\qquad(\eta,\zeta)\mapsto\bar L(\eta,\zeta) \tag{4.7}$$

determined by the identity

$$\big\langle\bar L(\eta,\zeta),\xi\big\rangle=L(\eta,\xi,\zeta).$$

Now we can write the Hessian of $\mathcal{A}_0$ at $u\in U_1$ and for vector fields $\xi,\eta\in H_1$ in terms of the L^2 inner product. Namely, by Lemma 4.5 we get

$$\mathrm{d}^2\mathcal{A}_0|_u(\xi,\eta)=\big\langle\xi,B_u^{-1}\dot\eta+\bar L_u(\eta,\dot u)\big\rangle_{H_0}=\langle\xi,A_0^u\eta\rangle_{H_0}, \tag{4.8}$$

where $H_0=L^2\big(\mathbb{S}^1,\mathbb{R}^{2n}\big)$ and the loop $t\mapsto B_{u(t)}\in\mathrm{GL}^+\big(2n,\mathbb{R}^{2n}\big)$ is determined pointwise at t by (3.1) and the last identity determines the linear operator A_0^u.

Definition 4.6. The *para-Darboux Hessian operator* of the action $\mathcal{A}_0$ at $u\in U_1$ is, due to (4.8), the bounded linear map

$$A_0^u\colon\ H_1\to H_0,\qquad\eta\mapsto B_u^{-1}\dot\eta+\bar L_u(\eta,\dot u). \tag{4.9}$$

For any $u\in U_2$, see (4.2), this is a bounded linear map $A_0^u|_{H_2}\colon H_2\to H_1$.

Lemma 4.7 (H_0-symmetry)**.** $\forall u \in U_1\colon\ \langle \xi, A_0^u \eta\rangle_{H_0} = \langle A_0^u \xi, \eta\rangle_{H_0}$ $\forall \xi, \eta \in H_1$.

Proof. By Lemma 4.5, the Hessian is symmetric, now use the identity (4.8). ∎

Remark 4.8 (Morse–Bott)**.** The critical points of $\mathcal{A}_0$ are precisely the constant loops, in symbols

$$\operatorname{Crit} \mathcal{A}_0 = \mathfrak{U}.$$

Let u be a constant loop in $\mathfrak{U}$. Then the second term of the para-Darboux Hessian operator vanishes. Hence the para-Darboux Hessian operator is just given by

$$A_u^0 \eta = B_u^{-1} \dot{\eta} \qquad \text{and} \qquad \ker A_0^u = \{\eta \in H_1 \mid \dot{\eta} = 0\} \simeq \mathbb{R}^{2n} = T_u \mathfrak{U}.$$

5 Perturbed action functional

As discussed in Remark 4.8, the functional $\mathcal{A}_0$ is not Morse, but only Morse–Bott. In order to get a Morse functional, we look at perturbations of the area functional $\mathcal{A}_0$. To this end, we introduce time-dependent Hamiltonians which are 1-periodic in time. The Hilbert spaces (H_0, H_1, H_2) are defined by (4.1) and open subsets $U_1 \subset H_1$ and $U_2 \subset H_2$ by (4.2).

Definition 5.1 (perturbed action)**.** For C^3 functions $h\colon \mathbb{S}^1 \times \mathfrak{U} \to \mathbb{R}$, notation $h_t(x) := h(t, x)$, the *perturbed action functional* is defined by

$$\mathcal{A}_h\colon\ U_1 \to \mathbb{R}, \qquad u \mapsto \int_{\mathbb{S}^1} u^*\lambda - \int_0^1 h_t(u(t))\, \mathrm{d}t.$$

To define the *Hamiltonian vector field* X_{h_t}, we choose the convention that

$$\mathrm{d}h_t(\cdot) = \omega(\cdot, X_{h_t})$$

whenever $t \in \mathbb{S}^1$. The derivative of $\mathcal{A}_h$ at $u \in U_1$ in direction $\xi \in H_1$ is given by

$$\mathrm{d}\mathcal{A}_h|_u \xi = \int_0^1 \left(\omega_{u_t}(\xi_t, \dot{u}_t) - \mathrm{d}h_t|_{u_t}\xi_t\right) \mathrm{d}t = \int_0^1 \omega_{u_t}(\xi_t, \dot{u}_t - X_{h_t}(u_t))\, \mathrm{d}t.$$

In particular, critical points of $\mathcal{A}_h$ are solutions of the ODE

$$\dot{u}(t) = X_{h_t}(u(t))$$

for $t \in \mathbb{S}^1$, i.e., 1-periodic orbits of the Hamiltonian vector field of h.

5.1 Perturbed para-Darboux Hessian

Given $t \in \mathbb{S}^1$, the *Hessian operator of the function* $h_t\colon \mathfrak{U} \to \mathbb{R}$ at $x \in \mathfrak{U}$ is the linear map whose matrix with respect to the canonical basis is given by

$$a_t|_x = (\partial_i \partial_j h_t|_x)_{i,j=1}^{2n}\colon\ \mathbb{R}^{2n} \to \mathbb{R}^{2n}. \tag{5.1}$$

It is determined by the identity

$$\mathrm{d}^2 h_t|_x(v, w) = \langle v, a_t|_x w\rangle, \qquad \mathrm{d}^2 h_t|_x(v, w) := \frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_0 \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_0 h_t(x + \varepsilon v + \tau w),$$

for all $v, w \in \mathbb{R}^{2n}$. By Schwarz's theorem, $\mathrm{d}^2 h_t|_x$ is symmetric, so $a_t|_x = {a_t|_x}^T$.

Definition 5.2. The *Hessian operator of the Hamiltonian perturbation* h at $u \in U_1$ is the bounded linear map $a^u \colon H_0 \to H_0$ defined pointwise at t by

$$(a^u\eta)(t) = a_t|_{u_t}\eta_t. \tag{5.2}$$

Still for $u \in U_1$, this map is also bounded as a map $H_1 \to H_1$. Since the matrix is symmetric, it holds that $\langle \xi, a_0^u \eta\rangle_{H_0} = \langle a_0^u\xi, \eta\rangle_{H_0}$ for all vector fields $\xi, \eta \in H_0$.

The second derivative of the perturbed action at $u \in U_1$ is, as a consequence of Lemma 4.5, for $\xi, \eta \in H_1$ given by

$$\mathrm{d}^2\mathcal{A}_h(u)(\xi,\eta) = \int_0^1 \left(\omega_{u_t}(\xi_t,\dot\eta_t) + L_{u_t}(\eta_t,\xi_t,\dot u_t) - \mathrm{d}^2 h|_{u_t}(\eta_t,\xi_t)\right)\mathrm{d}t.$$

Definition 5.3. The *perturbed para-Darboux Hessian operator* at $u \in U_1$ is the bounded linear map given by the difference

$$A^u = A_0^u - a^u \colon\; H_1 \to H_0, \qquad \eta \mapsto B_u^{-1}\dot\eta + \bar L_u(\eta,\dot u) - a^u\eta \tag{5.3}$$

of the para-Darboux Hessian operator A_0^u in (4.9) and the perturbation term a^u in (5.2). For any $u \in U_2$, see (4.2), this operator is also bounded as a linear map $A_2^u := A^u|_{H_2} \colon H_2 \to H_1$.

Lemma 5.4 (H_0-symmetry)**.** *At any $u \in U_1$, it holds $\langle \xi, A^u\eta\rangle_{H_0} = \langle A^u\xi,\eta\rangle_{H_0}$ for all $\xi,\eta \in H_1$.*

Proof. Lemma 4.7 and symmetry of the matrix (5.1). ∎

5.2 Fredholm operators

Hypothesis 5.5 (on u)**.** Let u_- and u_+ be non-degenerate critical points of the perturbed action $\mathcal{A}_h$. Pick a *basic path* $\hat u$ from u_- to u_+ (see [5]), i.e., $\hat u \in C^2(\mathbb{R}, U_2)$ with the property that there exists $T > 0$ such that $\hat u(s) = u_-$ whenever $s \le -T$ and $\hat u(s) = u_+$ whenever $s \ge T$. Let (H_0, H_1, H_2) be given by (4.1). Abbreviate

$$W^{1,2}_{H_j} := W^{1,2}(\mathbb{R}, H_j), \qquad L^2_{H_k} := L^2(\mathbb{R}, H_k). \tag{5.4}$$

Let

$$u \in C^0(\mathbb{R}, U_1), \qquad u - \hat u \in W^{1,2}_{H_1} \cap L^2_{H_2}.$$

It can be shown that the non-degeneracy condition can be rephrased with the help of the Hamiltonian flow φ_t^h, i.e., the flow of the Hamiltonian vector field, characterized by the requirement $\varphi_0^h = \mathrm{id}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^h = X_{h_t} \circ \varphi_t^h$. Namely, $\mathcal{A}_h$ is Morse if and only if for every critical point u of $\mathcal{A}_h$ we have $\ker\left(\mathrm{d}\varphi_1^h u(0) - \mathbb{1}\right) = \{0\}$.

Theorem 5.6. *For u as in Hypothesis 5.5 and A^u as in (5.3), the operators*

$$\begin{aligned}
\mathbb{D}^u &= \partial_s + A^u \colon\; W^{1,2}_{H_0} \cap L^2_{H_1} \to L^2_{H_0},\\
\mathbb{D}^u_2 &= \partial_s + A^u_2 \colon\; W^{1,2}_{H_1} \cap L^2_{H_2} \to L^2_{H_1}
\end{aligned}$$

are both Fredholm operators of the same Fredholm index

$$\operatorname{index}\mathbb{D}^u = \operatorname{index}\mathbb{D}^u_2.$$

Theorem 5.6 holds true by the abstract Theorem 6.11 which applies since the perturbed para-Darboux Hessian field A is almost extendable by Theorem 7.6.

Remark 5.7 (Conley–Zehnder index)**.** In the case where one of the asymptotic loops u_- or u_+ is contractible, one can choose a filling disk for this loop. This filling disk, together with the cylinder u, induces a filling disk for the other asymptotic loop. To each of these filling disks, one can associate a Conley–Zehnder index by conjugating the linearized flow with a symplectic trivialization over the filling disk. In this case, one should be able to show that the Fredholm index corresponds to the difference of the Conley–Zehnder indices.

Remark 5.8 (linearized downward L^2 gradient flow)**.** Since A^u is the L^2 Hessian of the functional $\mathcal{A}_h$ at $u \in U_1$ with respect to the standard inner product on $\mathbb{R}^{2n}$, the kernel of the operator $\mathbb{D}^u$ corresponds to the linearized downward L^2 gradient flow of $\mathcal{A}_h$.

Main difficulties and how to overcome them

Before introducing the abstract setup in Section 6 below which we use to prove Theorem 5.6, we first explain the main difficulties and the main ideas how to overcome these difficulties.

The fact that $\mathbb{D}^u$ is Fredholm almost directly follows from Rabier's theorem which itself generalizes previous work by Robbin and Salamon. In particular, Rabier does not need any differentiability assumption on $s \mapsto A(s)$, but requires continuity. However, since u as a map $\mathbb{R} \to U_2$, $s \mapsto u_s$, is only of class L^2_{loc}, it is not necessarily continuous. In particular, the map $\mathbb{R} \to \mathcal{L}(H_2, H_1)$, $s \mapsto A_2^{u_s}$, is not necessarily of class C^0. Thus the improvement of the Robbin–Salamon Fredholm theorem [15] from C^1 to C^0 by Rabier [14] cannot be applied to $\mathbb{D}_2^u$.

Decomposition. In order to deal with this difficulty, we decompose the operators $\mathbb{D}^u$ and $\mathbb{D}_2^u$ into two parts. To this end, we introduce the notation

$$A^{u_s} = \underbrace{B_{u_s}^{-1}\partial_t - a^{u_s}}_{=:F^{u_s}} + \underbrace{\bar{L}_{u_s}(\cdot, \dot{u}_s)}_{=:C^{\dot{u}_s}} = F^{u_s} + C^{\dot{u}_s}$$

and

$$\mathbb{D}^u\xi = \underbrace{(\partial_s + F^u)}_{=:\mathbb{F}^u}\xi + \underbrace{\left[s \mapsto C^{\dot{u}_s}\xi_s\right]}_{=:M_{C^{\dot{u}}}\xi} = \mathbb{F}^u\xi + M_{C^{\dot{u}}}\xi.$$

The advantage of this decomposition is the following. The map

$$\mathbb{R} \to \mathcal{L}(H_2, H_1), \qquad s \mapsto F_2^{u_s} = B_{u_s}^{-1}\partial_t|_{H_2} - a_2^{u_s}$$

is continuous since it avoids $\dot{u}_s$, while the non-continuous map

$$\mathbb{R} \to \mathcal{L}(H_2, H_1), \qquad s \mapsto C_2^{\dot{u}_s} = \bar{L}_{u_s}(\cdot, \dot{u}_s)|_{H_2}$$

is of lower order since it avoids ∂_t.

Since $C^{\dot{u}}$ is of lower order, we show, maybe after an asymptotic correction, with the help of Theorem B.3 that the multiplication operator $M_{C^{\dot{u}}}$ as a map $W^{1,2}_{H_1} \cap L^2_{H_2} \to L^2_{H_1}$ is compact. But Fredholm property and index are stable under compact perturbation. Thus to show that

$$\mathbb{D}_2^u\colon\; W^{1,2}_{H_1} \cap L^2_{H_2} \to L^2_{H_1}, \qquad \xi \mapsto \partial_s\xi + A_2^u\xi$$

is Fredholm of the same Fredholm index as $\mathbb{D}^u$ is equivalent to showing this for

$$\mathbb{F}_2^u = \partial_s + F^u\colon\; W^{1,2}_{H_1} \cap L^2_{H_2} \to L^2_{H_1}, \qquad \xi \mapsto \partial_s\xi + \underbrace{B_u^{-1}\partial_t\xi - a^u\xi}_{=:F^u\xi}.$$

The fact that $\mathbb{R} \to \mathcal{L}(H_2, H_1)$, $s \mapsto F^{u_s}$, is continuous, while $s \mapsto A^{u_s}$ is not, makes it easier to prove the Fredholm property for $\mathbb{F}_2^u$ (Rabier's theorem applies) than for $\mathbb{D}_2^u$ $\big($it does not apply by non-continuity of $s \mapsto C^{\dot{u}_s}\big)$. The disadvantage of F^{u_s} when compared to A^{u_s} is that, while A^{u_s} is H_0-symmetric for every s, necessarily not so is F^{u_s}. However, Rabier's theorem can as well deal with non-symmetric operators.

6 Almost extendability

Consider a *Hilbert space pair* (H_0, H_1), that is H_0 and H_1 are both infinite-dimensional Hilbert spaces such that as sets $H_1 \subset H_0$ and inclusion $\iota\colon H_1 \hookrightarrow H_0$ is compact and dense.

Then, as we explain in [3, Section 2], there exists an unbounded monotone function

$$h = h(H_0, H_1)\colon\ \mathbb{N} \to (0, \infty),$$

called *pair growth function*, such that the pair (H_0, H_1) is isometric to the pair $\left(\ell^2, \ell^2_h\right)$, see [3, Appendix A], and from now on we identify the pairs

$$(H_0, H_1) = \left(\ell^2, \ell^2_h\right), \qquad h = h(H_0, H_1).$$

Here ℓ^2_h is defined as follows. In general, for $f\colon \mathbb{N} \to (0, \infty)$ unbounded monotone ℓ^2_f is the space of all sequences $x = (x_\nu)_{\nu\in\mathbb{N}}$ with $\sum_{\nu=1}^\infty f(\nu) x_\nu^2 < \infty$. The space ℓ^2_f becomes a Hilbert space if we endow it with the inner product

$$\langle x, y\rangle_f = \sum_{\nu=1}^\infty f(\nu) x_\nu y_\nu, \qquad \|x\|_f := \sqrt{\langle x, x\rangle_f}.$$

Therefore, we can define for every real number r a Hilbert space $H_r := \ell^2_{f^r}$. For each $s < r$, the inclusion $H_r \subset H_s$ is compact and dense.

The resulting triple of Hilbert spaces H_0, H_1, and $H_{r=2}$ is called a *Hilbert space triple*, notation (H_0, H_1, H_2). Pick a Hilbert space triple (H_0, H_1, H_2).

6.1 Weak Hessian fields and decompositions

Fix a Hilbert space triple (H_0, H_1, H_2). Let $U_1 \subset H_1$ be open. Set $U_2 := U_1 \cap H_2$.

Definition 6.1 (weak Hessian [3]). An element A of the Banach space[3] $\mathcal{L}(H_1, H_0) \cap \mathcal{L}(H_2, H_1)$ is said a *weak Hessian* if it satisfies these two axioms:

`(symmetry)` $\forall x, y \in H_1$: $\langle Ax, y\rangle_0 = \langle x, Ay\rangle_0$ called H_0-*symmetry.*

`(fredholm)` $A\colon H_1 \to H_0$, $A_2 := A|_{H_2}\colon H_2 \to H_1$ are Fredholm of index zero.

We are interested not in a single weak Hessian, but in fields of weak Hessians.

Definition 6.2 (weak Hessian field). A *weak Hessian field* on U_1 is a continuous map $A \in C^0(U_1, \mathcal{L}(H_1, H_0)) \cap C^0(U_2, \mathcal{L}(H_2, H_1))$, notation $u \mapsto A^u$, satisfying the two conditions:

`(Symmetry)` At any point $u \in U_1$ there is H_0-symmetry in the sense that

$$\forall x, y \in H_1\colon\ \ \langle A^u x, y\rangle_0 = \langle x, A^u y\rangle_0\,. \tag{6.1}$$

`(Fredholm)` $\forall u \in U_1$: $A^u\colon H_1 \to H_0$ is Fredholm of index zero.

$\forall u \in U_2$: $A^u_2\colon H_2 \to H_1$ is Fredholm of index zero.

Remark 6.3. Given a weak Hessian field A, then along level two each operator A^u is a weak Hessian, in symbols for each $u \in U_2$, since $U_2 \subset U_1$, the operator A^u lies in $\mathcal{L}(H_1, H_0) \cap \mathcal{L}(H_2, H_1)$ and satisfies `(symmetry)` and `(fredholm)`.

[3] $\mathcal{L}(H_1, H_0) \cap \mathcal{L}(H_2, H_1)$ is a Banach space under the norm $\max\left\{\|\cdot\|_{\mathcal{L}(H_1,H_0)}, \|\cdot\|_{\mathcal{L}(H_2,H_1)}\right\}$.

Definition 6.4 (extendability)**.** We say that a weak Hessian field A on U_1 *extends* if A extends to a continuous map $U_1 \to \mathcal{L}(H_1, H_0) \cap \underline{\mathcal{L}(H_2, H_1)}$, still denoted by $u \mapsto A^u$, such that the restriction $A_2^u := A^u|_{H_2} \colon H_2 \to H_1$ is Fredholm of index zero at every point u of $\underline{U_1}$, and not only of U_2.

Remark 6.5 (equivalent formulation)**.** A weak Hessian field A on U_1 *extends* if and only if A extends to a continuous map $U_1 \to \mathcal{L}(H_1, H_0) \cap \underline{\mathcal{L}(H_2, H_1)}$, still denoted by $u \mapsto A^u$, such that A^u is a weak Hessian at every point $u \in U_1$.

In general, the extendability condition is too strong as the example of the area functional in a non-Darboux chart shows, see (4.9).[4]

Definition 6.6 (almost extendability)**.**

(i) We say that a weak Hessian field A on U_1 *almost extends* if there exists a decomposition

$$A = F + C$$

with

$$F \in C^0(U_1, \mathcal{L}(H_1, H_0) \cap \mathcal{L}(H_2, H_1)) \tag{6.2}$$

and

$$\exists r \in [0,1)\colon \;\; C \in C^0\big(U_1, \mathcal{L}\big(\underline{H_r}, H_0\big)\big) \cap C^0\big(U_2, \mathcal{L}\big(\underline{H_1}, H_1\big)\big) \tag{6.3}$$

such that the following two axioms hold:

(F) $\forall u \in \underline{U_1}\colon\;\; F_2^u := F^u|_{H_2} \colon H_2 \to H_1$ is Fredholm of index zero.

(C) $\forall u \in U_1$ there exists an H_1-open neighborhood V_u of u and a constant κ such that for all $v, w \in V_u \cap H_2$ it holds the *scale Lipschitz estimate*[5]

$$\|C^v - C^w\|_{\mathcal{L}(H_1)} \le \kappa\big(|v-w|_{H_2} + \min\{|v|_{H_2}, |w|_{H_2}\} \cdot |v-w|_{H_1}\big). \tag{6.4}$$

(ii) If A almost extends we call the pair (F, C) a *decomposition of* $\boldsymbol{A}$.

Remark 6.7 (extendable $\Rightarrow$ almost extendable)**.** If a weak Hessian field A is extendable, then it is almost extendable (choose $F = A$ and $C = 0$).

Lemma 6.8. *Assume a weak Hessian field A along U_1 is almost extendable with decomposition (F, C). Then, for any $u \in U_1$, the level one map $F^u \colon H_1 \to H_0$ is Fredholm of index zero, too.*

Proof. We write $F^u = A^u - C^u|_{H_1}$, where $A^u \colon H_1 \to H_0$ is Fredholm of index zero by axiom (Fredholm). Since $C^u \in \mathcal{L}(H_r, H_0)$, we can write $C^u|_{H_1}$ as a composition $C^u \circ \iota_r \colon H_1 \to H_r \to H_0$ of a bounded and a compact operator. Hence $C^u|_{H_1}$ is compact. By the stability of the Fredholm property and the index under compact perturbation, we have that F^u is a Fredholm operator and $\operatorname{index} F^u = \operatorname{index} A^u = 0$. This proves Lemma 6.8. ■

[4] For $u \in U_1 \subset W^{1,2}$ $\big(\dot u \in L^2\big)$ and $\eta \in W^{2,2}$, the product $\eta \dot u$ does not necessarily lie in $W^{1,2}$, however it does if $\dot u \in W^{1,2}$ $(u \in U_2)$; see (7.1).

[5] Axiom (6.4) is motivated by calculation (7.5) for the para-Darboux Hessian.

6.2 Main theorem

Let (H_0, H_1, H_2) be a Hilbert space triple and $U_1 \subset H_1$ an open subset. Write

$$W^{1,2}_{H_j} := W^{1,2}(\mathbb{R}, H_j), \qquad L^2_{H_j} := L^2(\mathbb{R}, H_j).$$

To avoid constants, maybe after replacing the norms by equivalent norms, we assume in the following that

$$|\cdot|_{H_0} \le |\cdot|_{H_1} \le |\cdot|_{H_2}.$$

Definition 6.9 (connecting paths)**.** Fix two points $u_-, u_+ \in U_2 := U_1 \cap H_2$. Fix a *basic path* $\hat{u}$ from u_- to u_+ (see [5]), i.e., $\hat{u} \in C^2(\mathbb{R}, U_2)$ with the property that there exists $T > 0$ such that $\hat{u}(s) = u_-$ whenever $s \le -T$ and $\hat{u}(s) = u_+$ whenever $s \ge T$.

A *connecting path from* $\boldsymbol{u_-}$ *to* $\boldsymbol{u_+}$ is a continuous map $u \colon \mathbb{R} \to U_1$ such that the difference $u - \hat{u}$ lies in the intersection Hilbert space $W^{1,2}_{H_1} \cap L^2_{H_2}$, i.e.,

$$u \in C^0(\mathbb{R}, U_1), \qquad u - \hat{u} \in W^{1,2}_{H_1} \cap L^2_{H_2}. \tag{6.5}$$

Remark 6.10 (independence of choice of basic path)**.** The notion of connecting path does not depend on the choice of the basic path $\hat{u}$. Indeed, suppose $\hat{v}$ is another basic path, then $u - \hat{v} = u - \hat{u} + \hat{u} - \hat{v}$ where $u - \hat{u} \in W^{1,2}_{H_1} \cap L^2_{H_2}$ by assumption and the other difference is C^2 and of compact support, namely $\hat{u} - \hat{v} \in C^2_{\mathrm{c}}(\mathbb{R}, H_2) \subset W^{1,2}_{H_1} \cap L^2_{H_2}$.

Theorem 6.11. *Let A be an almost extendable weak Hessian field on U_1. Consider two points $u_-, u_+ \in U_2$ and a connecting path u. Assume that both asymptotic operators $A^{u_\mp}$ are isomorphisms as maps $H_1 \to H_0$. Then the operators*

$$\mathbb{D}^u = \partial_s + A^u \colon\; W^{1,2}_{H_0} \cap L^2_{H_1} \to L^2_{H_0}$$

and

$$\mathbb{D}^u_2 = \partial_s + A^u_2 \colon\; W^{1,2}_{H_1} \cap L^2_{H_2} \to L^2_{H_1}$$

are both Fredholm operators of the same Fredholm index, $\operatorname{index} \mathbb{D}^u = \operatorname{index} \mathbb{D}^u_2$.

Remark 6.12. That A is almost extendable is only used to show that $\mathbb{D}^u_2$ is a Fredholm operator. That $\mathbb{D}^u$ is a Fredholm operator follows for every weak Hessian field on U_1 satisfying the asymptotic non-degeneracy condition at $u_\mp$.

Lemma 6.13. *Suppose that A is an almost extendable weak Hessian field. Let $u_* \in U_2$. Then there exists a decomposition $A = F + C$ satisfying $C^{u_*} = 0$.*

Proof. Since A is extendable, there exists a decomposition $A = \tilde{F} + \tilde{C}$. In particular, there is $r \in [0, 1)$ such that $\tilde{C} \in C^0(U_1, \mathcal{L}(H_r, H_0)) \cap C^0(U_2, \mathcal{L}(H_1))$. Since $u_* \in U_2 \subset U_1$, it holds that $\tilde{C}^{u_*} \in \mathcal{L}(H_r, H_0) \cap \mathcal{L}(H_1)$.

As U_1 is open, there exists $\varepsilon > 0$ such that the ball $B^{H_1}_\varepsilon(u_*) = \{v \in H_1 \mid |v - u_*|_{H_1} < \varepsilon\}$ is contained in U_1. Choose a smooth cut-off function $\beta \colon \mathbb{R} \to [0, 1]$ with the property that $\beta \equiv 1$ on $(-\infty, 0]$ and $\beta \equiv 0$ on $\left[\varepsilon^2, \infty\right)$ and define the map

$$v \mapsto P^v := \begin{cases} \beta\left(|u_* - v|^2_{H_1}\right) \tilde{C}^{u_*}, & v \in B^{H_1}_\varepsilon(u_*), \\ 0, & \text{else}, \end{cases}$$

which is element of $C^\infty(U_1, \mathcal{L}(H_r, H_0) \cap \mathcal{L}(H_1))$. The map defined by

$$C := \tilde{C} - P$$

lies in $C^0(U_1,\mathcal{L}(H_r,H_0))\cap C^0(U_2,\mathcal{L}(H_1))$ since $\tilde{C}$ does and so does P, because inclusion $U_2\hookrightarrow U_1$ is continuous. Furthermore, the map defined by

$$F:=\tilde{F}+P$$

lies in $C^0(U_1,\mathcal{L}(H_1,H_0)\cap\mathcal{L}(H_2,H_1))$ since $\tilde{F}$ does and so does P, because the inclusions $H_1\hookrightarrow H_r$ and $H_2\hookrightarrow H_1$ are continuous. Note that

$$C^{u_*}=\tilde{C}^{u_*}-P^{u_*}=\tilde{C}^{u_*}-\beta(0)\tilde{C}^{u_*}=0,\qquad F+C=\tilde{F}+\tilde{C}=A.$$

Claim. *The pair (F,C) is a decomposition of A.*

To see this, we need to verify the two axioms `(F)` and `(C)`.

`(F)` Let $u\in U_1$, then

$$F_2^u=F^u|_{H_2}=\tilde{F}^u|_{H_2}+P^u|_{H_2}=\tilde{F}_2^u+\beta\bigl(|u_*-u|_{H_1}^2\bigr)\tilde{C}^{u_*}|_{H_2}.$$

Since $\bigl(\tilde{F},\tilde{C}\bigr)$ is a decomposition by axiom `(F)` for $\tilde{F}$, we have that $\tilde{F}_2^u\colon H_2\to H_1$ is Fredholm of index zero. Since $\tilde{C}^{u_*}\in\mathcal{L}(H_1)$ and the inclusion $H_2\hookrightarrow H_1$ is compact, it follows that $\tilde{C}^{u_*}|_{H_2}\colon H_1\to H_1$ is compact. This proves `(F)` for F since Fredholm property and index are preserved under compact perturbation.

`(C)` To show the local Lipschitz condition for C, we show that P is globally Lipschitz, then the local Lipschitz condition for $C=\tilde{C}-P$ follows from the local Lipschitz condition for $\tilde{C}$. To this end, by the triangle inequality, first we see that the function

$$U_1\to\mathbb{R},\qquad v\mapsto|u_*-v|_{H_1}$$

is Lipschitz.[6] Since the squaring function $\mathbb{R}\ni x\mapsto x^2$ on the compact set $[0,\varepsilon]$ is Lipschitz and β is Lipschitz, the function

$$U_1\to\mathbb{R},\qquad v\mapsto\beta\bigl(|u_*-v|_{H_1}^2\bigr)$$

is Lipschitz (Lipschitz is preserved under composition). As $\tilde{C}^{u_*}$ does not depend on v, the map

$$U_1\to\mathcal{L}(H_r,H_0)\cap\mathcal{L}(H_1),\qquad v\mapsto P^v$$

is Lipschitz, say with Lipschitz constant L. That is, for all $v,w\in U_1$, it holds

$$\|P^v-P^w\|_{\mathcal{L}(H_r,H_1)\cap\mathcal{L}(H_1)}\le L|v-w|_{H_1}.$$

If $v,w\in U_2$, this implies the estimate

$$\|P^v-P^w\|_{\mathcal{L}(H_1)}\le\|P^v-P^w\|_{\mathcal{L}(H_r,H_1)\cap\mathcal{L}(H_1)}\le L|v-w|_{H_1}\le L|v-w|_{H_2}.$$

Hence $P\colon U_2\to\mathcal{L}(H_1)$ is Lipschitz continuous even globally on U_2. This proves axiom `(C)` for C, hence the claim.

The proof of Lemma 6.13 is complete. ■

Remark 6.14 (nullifying C on finitely many points)**.** The proof of Lemma 6.13 shows that one can improve the statement of the lemma to a stronger statement: Assume A is an almost extendable weak Hessian field on U_1 and $\Delta\subset U_2$ is a finite subset. Then there is a decomposition $A=F+C$ such that C vanishes along the points of Δ. The case $\Delta=\{u_-,u_+\}$ matters.

[6] Add $0=-w+w$ to v, then $|u_*-v|_{H_1}-|u_*-w|_{H_1}\le|u_*-w|_{H_1}+|w-v|_{H_1}-|u_*-w|_{H_1}$ by the triangle inequality, interchange v and w to get $|u_*-w|_{H_1}-|u_*-v|_{H_1}\le|v-w|_{H_1}$.

6.3 Proof of main theorem

The proof of the main theorem, Theorem 6.11, has three parts.

Part I. *The operator* $\mathbb{D}^u = \partial_s + A^u \colon W^{1,2}_{H_0} \cap L^2_{H_1} \to L^2_{H_0}$ *is Fredholm.*

Proof. Verifying (H1)–(H3) and applying Corollary A.4 proves Part I.

(H1) True since (H_0, H_1) is a Hilbert space pair.

(H2) Since $W^{1,2}(\mathbb{R}, H_1)$ embeds in $C^0(\mathbb{R}, H_1)$, see, e.g., [5, Appendix A.6], the map $\mathbb{R} \ni s \mapsto u_s \in U_1 \subset H_1$ is continuous. Since moreover $A \colon U_1 \to \mathcal{L}(H_1, H_0)$ is continuous as well, the map $\mathbb{R} \to \mathcal{L}(H_1, H_0)$, $s \mapsto A^{u_s}$ is continuous. Hence (H2) is satisfied.

(H3) By (6.5), the path u converges to $u_\mp$, as $s \to \mp\infty$, hence A^{u_s} converges to operators $A^{u_\mp}$, as $s \to \mp\infty$, and as maps $H_1 \to H_0$ these are invertible by assumption. Hence (H3) is satisfied. This proves Part I. ∎

Part II. *The operator* $\mathbb{D}^u_2 = \partial_s + A^u_2 \colon W^{1,2}_{H_1} \cap L^2_{H_2} \to L^2_{H_1}$ *is Fredholm.*

Proof. By assumption, A^u is almost extendable, so there exists a decomposition $A^u = F^u + C^u$. By Remark 6.14, we can assume in addition that $C^{u_-} = 0 = C^{u_+}$.

By invariance of the Fredholm property under compact perturbation and since $\mathbb{D}^u_2 = \partial_s + F^u_2 + C^u_2$, to prove Part II it suffices to show the following:

II-a. *The operator* $\mathbb{F}^u_2 = \partial_s + F^u_2 \colon W^{1,2}_{H_1} \cap L^2_{H_2} \to L^2_{H_1}$ *is Fredholm.*

II-b. *The multiplication operator* $M^{C^u}_2 \colon W^{1,2}_{H_1} \cap L^2_{H_2} \to L^2_{H_1}$ *is compact.*

Proof of II-a. It suffices to verify hypotheses (H1)–(H5) in Rabier's Theorem A.1.

(H1) The pair $(H, W) := (H_1, H_2)$ is a Hilbert space pair.

(H2) The map $\mathbb{R} \to U_1 \to \mathcal{L}(H_2, H_1)$, $s \mapsto u_s \mapsto F^{u_s}_2$ is a composition of continuous maps, by (6.5) and (6.2), hence continuous.

(H3) Since u is a connecting path from u_- to u_+, we have that $\lim_{s\to\mp\infty} u_s = u_\mp$. Since $F \in C^0(U_1, \mathcal{L}(H_2, H_0))$, by (6.2), it holds that $\lim_{s\to\mp\infty} \|F^{u_s} - F^{u_\mp}\|_{\mathcal{L}(H_2,H_1)} = 0$. It remains to show that the asymptotic limit operators $F^{u_\mp}$ are isomorphisms as linear maps $H_2 \to H_1$. To see this, observe that $F^{u_\mp} = A^{u_\mp}$, since $C^{u_\mp} = 0$ as mentioned above, and that by hypothesis $A^{u_\mp} \colon H_1 \to H_0$ are isomorphisms. In particular, the kernel vanishes, hence so does the kernel of the restriction $\ker A^{u_\mp}_2 \colon H_2 \to H_1$. Since $u_\mp$ are elements of U_2, by axiom `(Fredholm)` of weak Hessian field, the operator $A^{u_\mp}_2 \colon H_2 \to H_1$ is Fredholm of index zero. Hence injective is equivalent to surjective and therefore by the open mapping theorem the two maps $A^{u_\mp}_2$ are isomorphisms. But $F^{u_\mp}_2 = A^{u_\mp}_2$.

(H4) For $F^{u_s}_2$ whenever $s \in \mathbb{R}$: By Lemma A.3, based on Definition 6.2 for A, we know that $A^{u_s} \colon H_1 \to H_0$ satisfies (H4). Our first goal is to show that $F^{u_s} = A^{u_s} - C^{u_s}|_{H_1} \colon H_1 \to H_0$ satisfies (H4). To see this, observe that $C^{u_s}|_{H_1} \colon H_1 \to H_0$ is a compact operator: indeed, $C^{u_s} \in \mathcal{L}(H_r, H_0)$ by (6.3) and inclusion $H_r \hookrightarrow H_1$ is compact since $r < 1$. But (H4) is stable under compact perturbation by Proposition A.7. Thus $F^{u_s} \colon H_1 \to H_0$ satisfies (H4) whenever $s \in \mathbb{R}$ proving the goal. Now by Lemma A.8, based on axiom `(F)`, the restriction $F^{u_s}_2 \colon H_2 \to H_1$ satisfies (H4) as well.

(H5) Since $F^{u_\mp} = A^{u_\mp}$, as we saw during the verification of (H3), and the spectrum of $A^{u_\mp}$ is real but does not contain zero, by invertibility, it holds that $i\mathbb{R} \cap \operatorname{spec} A^{u_\mp} = \varnothing$.

Rabier's Theorem A.1 concludes the proof of Step II-a. ∎

Proof of II-b. By Theorem B.3, it suffices to show that $C^u \in L^2(\mathbb{R}, \mathcal{L}(H_1))$.

Basic path. Let $\hat{u}$ be a basic path from u_- to u_+ with interval, say $[-T,T]$.

Positive end. By axiom (C), for C there is an H_1-open neighborhood V_{u_+} of u_+, and a constant $\kappa_+ > 0$ such that all $v, w \in V_{u_+} \cap H_2$ satisfy the scale Lipschitz estimate (6.4). In particular, since $w := u_+ \in U_2$ and $C^{u_+} = 0$, for any $v \in V_{u_+} \cap H_2$ there is the local Lipschitz estimate

$$\|C^v\|_{\mathcal{L}(H_1)} \le \kappa\Big(|v-u_+|_{H_2} + \underbrace{\min\{|v|_{H_2}, |u_+|_{H_2}\}}_{\le |u_+|_{H_2}} \cdot \underbrace{|v-u_+|_{H_1}}_{\le |v-u_+|_{H_2}}\Big) \le \kappa_+ |v-u_+|_{H_2}, \tag{6.6}$$

where $\kappa_+ = \kappa_+(u_+) = \kappa + |u_+|_{H_2}$. Since the connecting path $s \mapsto u_s$ converges to u_+ in H_1, there exists a time $T_+ \ge T$ such that $u_s \in V_{u_+}$ whenever $s \ge T_+$.

Negative end. Similarly, there exist V_{u_-}, $\kappa_- > 0$, $T_- \ge T$ such that

$$\|C^v\|_{\mathcal{L}(H_1)} \le \kappa_- |v-u_-|_{H_2} \tag{6.7}$$

whenever $v \in V_{u_-} \cap H_2$ and $u_s \in V_{u_-}$ whenever $s \le -T_-$.

Compact part. For every $s \in [-T_-, T_+]$, by axiom (C), we choose an H_1-open neighborhood V_s of u_s in U_1 such that there exists a constant $\kappa_s > 0$ with the property that for all $v, w \in V_s \cap H_2$ it holds the scale Lipschitz estimate

$$\|C^v - C^w\|_{\mathcal{L}(H_1)} \le \kappa_s(|v-w|_{H_2} + \min\{|v|_{H_2}, |w|_{H_2}\} \cdot |v-w|_{H_1}). \tag{6.8}$$

Since $u\colon \mathbb{R} \to U_1$ is continuous and the interval $[-T_-, T_+]$ is compact, the image $u_{[-T_-,T_+]}$ is a compact subset of U_1. Therefore, there exists a positive integer N and times $-T_- < s_1 < s_2 < \dots < s_N < T_+$ such that the V_{s_j} cover the image

$$u_{[-T_-,T_+]} \subset \bigcup_{j=1}^N V_{s_j}.$$

As H_2 is dense in H_1, for any $j = 1, \dots, N$, we can choose $p_j \in V_{s_j} \cap H_2$ as illustrated by Figure 1.

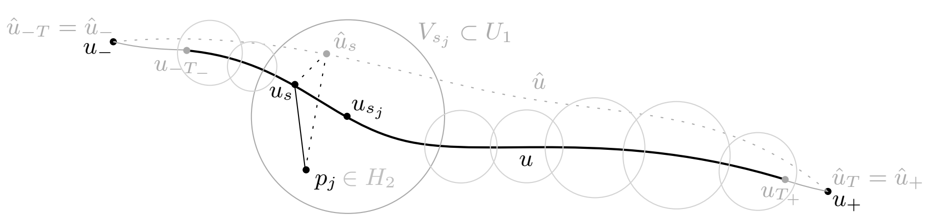

Figure 1. Basic path $\hat{u}$ and open cover of connecting path u along time $[-T_-, T_+]$.

Estimates. Firstly, we assume that $s \in [-T_-, T_+]$. Then u_s lies in one of the finitely many open sets, i.e., there exists $j = j(s)$ such that $u_s \in V_{s_j}$. We estimate

$$\begin{aligned}
\|C^{u_s}\|_{\mathcal{L}(H_1)} &\le \|C^{p_j}\|_{\mathcal{L}(H_1)} + \|C^{u_s} - C^{p_j}\|_{\mathcal{L}(H_1)} \\
&\le \|C^{p_j}\|_{\mathcal{L}(H_1)} + \kappa_{s_j}\Big(|u_s - p_j|_{H_2} + \underbrace{\min\{|u_s|_{H_2}, |p_j|_{H_2}\}}_{\le |p_j|_{H_2}} \cdot \underbrace{|u_s - p_j|_{H_1}}_{\le |u_s - p_j|_{H_2}}\Big) \\
&\le \underbrace{\max_{\ell=1,\dots,N} \|C^{p_\ell}\|_{\mathcal{L}(H_1)}}_{=:a_u} + \underbrace{\max_{\ell=1,\dots,N} (\kappa_{s_\ell} + |p_\ell|_{H_2})}_{=:b_u}(|u_s - \hat{u}_s|_{H_2} + |\hat{u}_s - p_j|_{H_2}) \\
&\le \underbrace{a_u + b_u \max_{\substack{\sigma \in [-T,T] \\ \ell=1,\dots,N}} |\hat{u}_\sigma - p_\ell|_{H_2}}_{=:c_u} + b_u |u_s - \hat{u}_s|_{H_2}.
\end{aligned}$$

Inequality 1 is by adding zero and the triangle inequality. Inequality 2 uses (6.8) for $s = s_j$. Inequality 3 is by taking maxima and adding again zero $\hat{u}_s - \hat{u}_s$, as illustrated by Figure 1. Inequality 4 uses that a basic path $\hat{u}$ is continuous as a map to U_2 and constant outside $[-T, T]$, so the maximum exists.

Secondly, for $s \geq T_+$, by (6.6) it holds $\|C^{u_s}\|_{\mathcal{L}(H_1)} \leq \kappa_+ |u_s - u_+|_{H_2}$.

Thirdly, for $s \leq -T_-$, by (6.7) it holds $\|C^{u_s}\|_{\mathcal{L}(H_1)} \leq \kappa_- |u_s - u_-|_{H_2}$.

To prove that $C^u \in L^2(\mathbb{R}, \mathcal{L}(H_1))$, we estimate

$$\begin{aligned} \|C^u\|^2_{L^2(\mathbb{R},\mathcal{L}(H_1))} &= \left(\int_{-\infty}^{-T_-} + \int_{-T_-}^{T_+} + \int_{T_+}^{\infty} \right) \|C^{u_s}\|^2_{\mathcal{L}(H_1)} \, \mathrm{d}s \\ &\leq \left(\kappa_-^2 + \kappa_+^2 \right) \|u - u_-\|^2_{L^2_{H_2}} + (T_- + T_+) 2c_u^2 + b_u \|u - \hat{u}\|^2_{L^2([-T,T],H_2)} \end{aligned}$$

which is finite. This proves Step II-b. ■

Since $\mathbb{D}_2^u = \mathbb{F}_2^u + M_2^{C^u}$ and since Fredholm property and index are invariant under compact perturbation, we get from Steps II-a and II-b that $\mathbb{D}_2^u$ is a Fredholm operator and that

$$\operatorname{index} \mathbb{D}_2^u = \operatorname{index} \mathbb{F}_2^u.$$

This concludes the proof of Part II. ■

Part III. *Equal Fredholm indices* $\operatorname{index} \mathbb{D}^u = \operatorname{index} \mathbb{D}_2^u$.

Proof of Part III – using a homotopy argument. Because the space $C^0_{\mathrm{c}}(\mathbb{R}, H_2)$ is dense in $L^2(\mathbb{R}, H_2)$, see, e.g., [5, Theorem A.5.4], the connecting path $u \in C^0(\mathbb{R}, U_1)$ from u_+ to u_- can be approximated by a connecting path which additionally is continuous as a map to H_2 and satisfies that the limit as $s \to \mp\infty$ exists in H_2 and is $u_\mp$. In particular, after a homotopy inside the space of connecting paths from u_- to u_+, we can assume that our connecting path u is additionally continuous on level H_2.

Since the Fredholm index is homotopy invariant, it suffices to show the equality of indices only for connecting paths which are additionally continuous on level H_2. For such a path, the map $s \mapsto A_2^{u_s}$ is continuous as a map $\mathbb{R} \to \mathcal{L}(H_2, H_1)$. Since both operators $A^{u_s} \colon H_1 \to H_0$ and $A_2^{u_s} \colon H_2 \to H_1$ are Fredholm of index zero, [3, Corollary 3.4] shows that the two operators have the same spectrum, in particular they have the same spectral flow. By [4, Theorem A], the Fredholm index of $\mathbb{D}^u = \partial_s + A^u$ is the spectral flow of $s \mapsto A^{u_s}$ and the Fredholm index of $\mathbb{D}_2^u = \partial_s + A_2^u$ is the spectral flow of $s \mapsto A_2^{u_s}$. Since the spectral flows agree, the two indices agree. This proves Part III. ■

The proof of Theorem 6.11 is complete.

7 Para-Darboux Hessian is almost extendable

Recall the setup from Section 4, where an open subset $\mathfrak{U} \subset \mathbb{R}^{2n}$ carries an exact symplectic form $\omega = \mathrm{d}\lambda$. The Hilbert space triple (H_0, H_1, H_2) is given by the $W^{k,2}(\mathbb{S}^1, \mathbb{R}^{2n})$-Sobolev spaces (4.1). Open subsets $U_1 \subset H_1$ and $U_2 \subset H_2$ are defined by (4.2). Recall further that the identity (3.1) determines a smooth map

$$B \colon \mathfrak{U} \to \mathrm{GL}(2n, \mathbb{R}), \qquad x \mapsto B_x.$$

In (4.7), we defined a (1,2) tensor $\bar{L} \colon \mathcal{X}(\mathfrak{U}) \times \mathcal{X}(\mathfrak{U}) \to \mathcal{X}(\mathfrak{U})$ where $\mathcal{X}(\mathfrak{U})$ is the set of vector fields along $\mathfrak{U}$.

In order to prove the lemma and the theorem below, we need the following result from fractional Sobolev theory.

Theorem 7.1 ([1, Theorem 7.4])**.** *Assume ρ_1, ρ_2, ρ are real numbers satisfying*

(1) $\rho_1 \geq \rho \geq 0$ *and* $\rho_2 \geq \rho \geq 0$*;*

(2) $\rho_1 + \rho_2 > \frac{1}{2} + \rho$*.*

Then the following is true. If $v \in W^{\rho_1,2}(\mathbb{S}^1)$ and $w \in W^{\rho_2,2}(\mathbb{S}^1)$, then $vw \in W^{\rho,2}(\mathbb{S}^1)$ and pointwise multiplication of functions is a continuous bi-linear map

$$W^{\rho_1,2}(\mathbb{S}^1) \times W^{\rho_2,2}(\mathbb{S}^1) \to W^{\rho,2}(\mathbb{S}^1).$$

Most important for us is the theorem in the forms

$$W^{1,2} \times L^2 \to L^2, \qquad W^{2,2} \times W^{1,2} \to W^{1,2}, \tag{7.1}$$

and

$$W^{r,2} \times L^2 \stackrel{r>\frac{1}{2}}{\longrightarrow} L^2, \qquad W^{1,2} \times W^{1,2} \to W^{1,2}. \tag{7.2}$$

7.1 Unperturbed case

Lemma 7.2. *The para-Darboux Hessians, one for each $u \in U_1$, defined by*

$$A_0^u \colon\; H_1 \to H_0, \qquad \eta \mapsto B_u^{-1}\dot\eta + \bar L_u(\eta, \dot u)$$

determine a weak Hessian field A_0 on U_1.

Proof. By (7.1), the map $u \mapsto A_0^u$ is element of the space $C^0(U_1, \mathcal{L}(H_1, H_0))$ and of the space $C^0(U_2, \mathcal{L}(H_2, H_1))$. The (`Symmetry`) axiom (6.1) holds true by Lemma 4.7. It remains to check the (`Fredholm`) axiom. By Theorem C.1 for the loop $t \mapsto \omega_{u_t}$, the operator as a map

$$B_u^{-1}\partial_t \colon\; H_1 \to H_0, \qquad H_2 \to H_1, \qquad \forall u \in U_1 \tag{7.3}$$

is Fredholm of index zero. Using the Sobolev estimate $|\eta|_{L^\infty} \leq |\eta|_{H_1}$, the term

$$C^u \colon\; \eta \mapsto \left[t \mapsto \bar L_{u_t}(\eta_t, \dot u_t)\right]$$

is bounded as a map $H_0 \to H_0$ if $u \in U_1$ and as a map $H_1 \to H_1$ if $u \in U_2$. So, by compactness of the embeddings $H_1 \hookrightarrow H_0$ and $H_2 \hookrightarrow H_1$, both operators C^u are compact. But Fredholm property and index are stable under compact perturbation. This proves Lemma 7.2. ∎

Theorem 7.3. *The para-Darboux weak Hessian field A_0 is almost extendable. Moreover, the pair (F, C) defined for $u \in U_1$ by*

$$F^u \colon\; \eta \mapsto B_u^{-1}\dot\eta, \qquad C^u \colon\; \eta \mapsto \bar L_u(\eta, \dot u),$$

is a decomposition.

Proof of Theorem 7.3. The proof has four Steps 1, 2, (`C`), and (`F`).

Step 1. By (7.2), the map $u \mapsto C^u$ is element of the space $C^0(U_1, \mathcal{L}(\underline{H_r}, H_0))$ and of the space $C^0(U_2, \mathcal{L}(\underline{H_1}, H_1))$, where $H_r = W^{r,2}(\mathbb{S}^1, \mathbb{R}^{2n})$.

Step 2. We need to show that the map $F \colon u \mapsto F^u = B_u^{-1}\partial_t$ is element of the space $C^0(U_1, \mathcal{L}(H_1, H_0) \cap \mathcal{L}(H_2, H_1))$; see (6.2).

Proof. We prove that $F \in C^0(U_1, \mathcal{L}(H_2, H_1))$; similarly $F \in C^0(U_1, \mathcal{L}(H_1, H_0))$.

Fix $u \in U_1$ and let $K = K(u)$ be a compact neighborhood of the (compact) image of u in $\mathfrak{U} \subset \mathbb{R}^{2n}$. Pick $v \in U_1$ near u taking values in K as well. The smooth map $x \mapsto B_x^{-1}$ is Lipschitz continuous since $K = K(u)$ is compact. Let λ_u be the Lipschitz constant. Now, given any $\xi \in H_2$, we estimate

$$\begin{aligned}
|(F^u - F^v)\xi|_{W^{1,2}}^2 &= \left|\left(B_u^{-1} - B_v^{-1}\right)\dot\xi\right|_{L^2}^2 + \left|\left(B_u^{-1} - B_v^{-1}\right)\ddot\xi + \left(\partial_t\left(B_u^{-1} - B_v^{-1}\right)\right)\dot\xi\right|_{L^2}^2 \\
&\le \lambda_u^2 |u-v|_{L^\infty}^2 |\xi|_{W^{1,2}}^2 + 2\lambda_u^2 |u-v|_{L^\infty}^2 |\xi|_{W^{2,2}}^2 \\
&\quad + 4\left|dB_u^{-1}\left(\dot u \underline{- \dot v}, \dot\xi\right)\right|_{L^2}^2 + 4\left|\left(\underline{dB_u^{-1}} - dB_v^{-1}\right)\left(\dot v, \dot\xi\right)\right|_{L^2}^2 \\
&\le 3\lambda_u^2 |u-v|_{W^{1,2}}^2 |\xi|_{W^{2,2}}^2 + 4\left|dB_u^{-1}\right|_{L^\infty(K)}^2 |u-v|_{W^{1,2}}^2 \left|\dot\xi\right|_{L^\infty}^2 \\
&\quad + 4\lambda_u^2 |u-v|_{L^\infty}^2 |v|_{W^{1,2}}^2 \left|\dot\xi\right|_{L^\infty}^2 \\
&\le \left(3\lambda_u^2 + 4\left|dB_u^{-1}\right|_{L^\infty(K)}^2 + 4\lambda_u^2 |v|_{W^{1,2}}^2\right) |u-v|_{W^{1,2}}^2 |\xi|_{W^{2,2}}^2.
\end{aligned}$$

Here inequality one uses Lipschitz continuity of $K \ni x \mapsto B_x^{-1}$ with Lipschitz constant λ_u. We also added *zero* and used the triangle inequality. Inequality two uses the Sobolev embedding $W^{1,2} \hookrightarrow C^0$ with constant 1, we also used Lipschitz continuity again. This proves Step 2. ■

Step (C). The map $u \mapsto C^u$ satisfies the scale Lipschitz estimate (6.4).

Proof. It remains to check the local scale Lipschitz axiom (C), see (6.4), for the map C. To this end suppose $u \in U_1 = W^{1,2}(\mathbb{S}^1, \mathfrak{U})$. Since $W^{1,2} \subset C^0$, the image of u is a compact subset of $\mathfrak{U}$. Therefore, there exist, firstly, an open subset $\mathfrak{V}$ which contains the image of u and, secondly, a constant c such that

$$\begin{aligned}
&\left\|\bar L_x\right\|_{\mathcal{L}(\mathbb{R}^{2n}\times\mathbb{R}^{2n};\mathbb{R}^{2n})} \le c, \\
&\left\|\bar L_x - \bar L_x\right\|_{\mathcal{L}(\mathbb{R}^{2n}\times\mathbb{R}^{2n};\mathbb{R}^{2n})} \le c|x-y|_{\mathbb{R}^{2n}}, \\
&\left\|d\bar L_x\right\|_{\mathcal{L}(\mathbb{R}^{2n}\times\mathbb{R}^{2n}\times\mathbb{R}^{2n};\mathbb{R}^{2n})} \le c, \\
&\left\|d\bar L_x - d\bar L_x\right\|_{\mathcal{L}(\mathbb{R}^{2n}\times\mathbb{R}^{2n}\times\mathbb{R}^{2n};\mathbb{R}^{2n})} \le c|x-y|_{\mathbb{R}^{2n}},
\end{aligned} \tag{7.4}$$

whenever $x, y \in \mathfrak{V}$. We define an open neighborhood of u in U_1 by

$$V_u := W^{1,2}\left(\mathbb{S}^1, \mathfrak{V}\right) \cap B_1^{H_1}(u),$$

where $B_1^{H_1}(u)$ is the open radius 1 ball in H_1 centered at u. Pick elements $v, w \in V_u \cap H_2 = V_u \cap W^{2,2}$. Note that

$$|v|_{W^{1,2}} \le |u|_{W^{1,2}} + 1, \qquad |w|_{W^{1,2}} \le |u|_{W^{1,2}} + 1.$$

Now for $\xi \in H_1$, we estimate

$$\begin{aligned}
|(C^v - C^w)\xi|_{W^{1,2}}^2 &\le \left|\bar L_v(\xi, \dot v) - \bar L_w(\xi, \dot w)\right|_{L^2}^2 + \left|\partial_t\left(\bar L_v(\xi, \dot v) - \bar L_w(\xi, \dot w)\right)\right|_{L^2}^2 \\
&\le 2\left|\bar L_v(\xi, \dot v - \dot w)\right|_{L^2}^2 + 2\left|\left(\bar L_v - \bar L_w\right)(\xi, \dot w)\right|_{L^2}^2 \\
&\quad + 7\left|d\bar L_v(\dot v - \dot w, \xi, \dot v)\right|_{L^2}^2 + 7\left|d\bar L_v(\dot w, \xi, \dot v - \dot w)\right|_{L^2}^2 \\
&\quad + 7\left|\left(d\bar L_v - d\bar L_w\right)(\dot w, \xi, \dot w)\right|_{L^2}^2 \\
&\quad + 7\left|\bar L_v\left(\dot\xi, \dot v - \dot w\right)\right|_{L^2}^2 + 7\left|\left(\bar L_v - \bar L_w\right)\left(\dot\xi, \dot w\right)\right|_{L^2}^2 \\
&\quad + 7\left|\bar L_v(\xi, \ddot v - \ddot w)\right|_{L^2}^2 + 7\left|\left(\bar L_v - \bar L_w\right)(\xi, \ddot w)\right|_{L^2}^2,
\end{aligned}$$

where in inequality two we added four times zero and used Lemma 7.4 for $k = 7$ summands. Now we estimate summand by summand starting below

$$\begin{aligned}\left|\left(\bar{L}_v - \bar{L}_w\right)(\xi, \ddot{w})\right|_{L^2}^2 &= \int_0^1 \left|\left(\bar{L}_{v_t} - \bar{L}_{w_t}\right)(\xi_t, \ddot{w}_t)\right|^2 \mathrm{d}t \\ &\le \int_0^1 c^2 |v_t - w_t|^2 |\xi_t|^2 |\ddot{w}_t|^2 \,\mathrm{d}t \\ &\le c^2 |v-w|_{L^\infty}^2 |\xi|_{L^\infty}^2 |\ddot{w}|_{L^2}^2 \\ &\le c^2 |v-w|_{W^{1,2}}^2 |\xi|_{W^{1,2}}^2 |w|_{W^{2,2}}^2 .\end{aligned}$$

Here inequality one uses (7.4) and the Sobolev embedding $L^\infty \hookrightarrow W^{1,2}$ with constant 1. Similarly, we estimate

$$\begin{aligned}&\left|\bar{L}_v(\xi, \ddot{v} - \ddot{w})\right|_{L^2}^2 \le c^2 |\xi|_{W^{1,2}}^2 |v-w|_{W^{2,2}}^2, \\ &\left|\left(\bar{L}_v - \bar{L}_w\right)\left(\dot{\xi}, \dot{w}\right)\right|_{L^2}^2 \le c^2 |v-w|_{W^{1,2}}^2 |\xi|_{W^{1,2}}^2 |w|_{W^{2,2}}^2, \\ &\left|\bar{L}_v\left(\dot{\xi}, \dot{v} - \dot{w}\right)\right|_{L^2}^2 \le c^2 |\xi|_{W^{1,2}}^2 |v-w|_{W^{2,2}}^2, \\ &\left|\left(\mathrm{d}\bar{L}_v - \mathrm{d}\bar{L}_w\right)(\dot{w}, \xi, \dot{w})\right|_{L^2}^2 \le c^2 \left(1 + |u|_{W^{1,2}}\right)^2 |v-w|_{W^{1,2}}^2 |\xi|_{W^{1,2}}^2 |w|_{W^{2,2}}^2, \\ &\left|\mathrm{d}\bar{L}_v(\dot{w}, \xi, \dot{v} - \dot{w})\right|_{L^2}^2 \le c^2 \left(1 + |u|_{W^{1,2}}\right)^2 |\xi|_{W^{1,2}}^2 |v-w|_{W^{2,2}}^2, \\ &\left|\mathrm{d}\bar{L}_v(\dot{v} - \dot{w}, \xi, \dot{v})\right|_{L^2}^2 \le c^2 \left(1 + |u|_{W^{1,2}}\right)^2 |v-w|_{W^{2,2}}^2 |\xi|_{W^{1,2}}^2, \\ &\left|\left(\bar{L}_v - \bar{L}_w\right)(\xi, \dot{w})\right|_{L^2}^2 \le c^2 |v-w|_{W^{1,2}}^2 |\xi|_{W^{1,2}}^2 |w|_{W^{2,2}}^2, \\ &\left|\bar{L}_v(\xi, \dot{v} - \dot{w})\right|_{L^2}^2 \le c^2 |\xi|_{W^{1,2}}^2 |v-w|_{W^{2,2}}^2 .\end{aligned}$$

Continuing the above estimate and returning to the notation $H_k = W^{k,2}$, we get

$$|(C^v - C^w)\xi|_{H_1}^2 \le \underbrace{7c^2\left(7 + 4|u|_{H_1}^2\right)}_{=:\kappa^2}\left(|v-w|_{H_2}^2 + |w|_{H_2}^2 |v-w|_{H_1}^2\right)|\xi|_{H_1}^2 .$$

This implies that the operator norm is bounded by

$$\|C^v - C^w\|_{\mathcal{L}(H_1)} \le \kappa(|v-w|_{H_2} + |w|_{H_2}|v-w|_{H_1}).$$

Interchanging the roles of v and w, we get

$$\|C^v - C^w\|_{\mathcal{L}(H_1)} \le \kappa(|v-w|_{H_2} + |v|_{H_2}|v-w|_{H_1}).$$

The above two estimates imply the scale Lipschitz estimate

$$\|C^v - C^w\|_{\mathcal{L}(H_1)} \le \kappa(|v-w|_{H_2} + \min\{|v|_{H_2}, |w|_{H_2}\}|v-w|_{H_1}), \tag{7.5}$$

which is precisely (6.4) in axiom (C). This proves Step (C). ■

Step (F). $\forall u \in U_1\colon\ F_2^u := B_u^{-1}\partial_t|_{H_2}\colon H_2 \to H_1$ is Fredholm of index zero.

Proof. The proof was given in (7.3) as a consequence of Theorem C.1. ■

The proof of Theorem 7.3 is complete. ■

Tool

Lemma 7.4. *For $k \in \mathbb{N}$ real numbers $a_1, \dots, a_k > 0$, there is the inequality*

$$\left(\sum_{j=1}^{k} a_j\right)^2 \le k \sum_{j=1}^{k} a_j^2.$$

Proof. Observe that

$$\left(\sum_{j=1}^{k} a_j\right)^2 = \sum_{j=1}^{k} a_j^2 + \sum_{1\le j<i\le k} \underbrace{2a_j a_i}_{\le a_j^2+a_i^2} \le (1+(k-1))a_1^2 + \dots + (1+(k-1))a_k^2.$$ ∎

7.2 Perturbed case

Lemma 7.5. *The perturbed para-Darboux Hessians (5.3) defined by*

$$A^u\colon\; H_1 \to H_0, \qquad \eta \mapsto B_u^{-1}\dot\eta + \bar L_u(\eta,\dot u) - a^u\eta$$

one for each $u \in U_1$, determine a weak Hessian field A on U_1.

Theorem 7.6. *The perturbed para-Darboux weak Hessian field A is almost extendable. Moreover, the pair (F, C) defined for $u \in U_1$ by*

$$F^u\colon\; \eta \mapsto B_u^{-1}\dot\eta - a^u\eta, \qquad C^u\colon\; \eta \mapsto \bar L_u(\eta,\dot u)$$

is a decomposition.

To prove the lemma and the theorem, we need the following proposition.

Proposition 7.7. *The map $u \mapsto a^u$ is element of $C^0(U_1, \mathcal{L}(H_1))$.*

To prove the proposition, we need the following lemma.

Lemma 7.8. *Let $f \in C^1(\mathbb{R},\mathbb{R})$. Then the map*

$$F(f)\colon\; W^{1,2}\big(\mathbb{S}^1,\mathbb{R}\big) \to W^{1,2}\big(\mathbb{S}^1,\mathbb{R}\big), \qquad u \mapsto f\circ u$$

is well defined and continuous.

Proof. The composition $f\circ u$ is L^2, because since u is C^0 the image $u\big(\mathbb{S}^1\big)$ is compact, hence the continuous map f is L^2-integrable along the image. The derivative $f'|_u \cdot \dot u$ is a product of a C^0-map $f'|_u$ and an L^2-map $\dot u$. Such product is L^2. This shows well defined.

We prove continuity. Pick $\varepsilon > 0$. Since $u\colon \mathbb{S}^1 \to \mathbb{R}$ is continuous, there exists a compact interval $[a,b]$ containing the image of u. Since f and f' are continuous, there exists

$$0 < \delta < \min\left\{1, \frac{\varepsilon}{2|f'(u)|_{L^\infty}}\right\}$$

such that for all $x, y \in [a-1, b+1]$ with $|x-y| < \delta$ it holds

$$|f(x)-f(y)| \le \frac{\varepsilon}{\kappa}, \qquad \big|f'(x)-f'(y)\big| \le \frac{\varepsilon}{\kappa}, \qquad \kappa := \sqrt{10+8|u|^2_{W^{1,2}}}.$$

Now assume that $|u-v|_{W^{1,2}} \le \delta$. Since $|u-v|_{C^0} \le |u-v|_{W^{1,2}} \le \delta$ we have $|u_t - v_t| \le \delta \le 1$ whenever $t \in \mathbb{S}^1$. Hence

$$|f(u_t)-f(v_t)| \le \frac{\varepsilon}{\kappa}, \qquad \big|f'(u_t)-f'(v_t)\big| \le \frac{\varepsilon}{\kappa},$$

for every $t \in \mathbb{S}^1$. We estimate

$$\begin{aligned}
|f\circ u - f\circ v|^2_{W^{1,2}} &= |f\circ u - f\circ v|^2_{L^2} + \left|f'(u)\cdot\dot u - f'(u)\cdot\dot v + f'(u)\cdot\dot v - f'(v)\cdot\dot v\right|^2_{L^2}\\
&\le |f\circ u - f\circ v|^2_{L^2} + 2\left|f'(u)\cdot\dot u - f'(u)\cdot\dot v\right|^2_{L^2} + 2\left|f'(u)\cdot\dot v - f'(v)\cdot\dot v\right|^2_{L^2}\\
&\le \frac{\varepsilon^2}{\kappa^2} + 2\left|f'(u)\right|^2_{L^\infty}|\dot u - \dot v|^2_{L^2} + 2\frac{\varepsilon^2}{\kappa^2}|\dot v|^2_{L^2}\\
&\le \frac{\varepsilon^2}{\kappa^2} + 2\left|f'(u)\right|^2_{L^\infty}\underbrace{|u-v|^2_{W^{1,2}}}_{\le\delta^2\le\frac{\varepsilon^2}{4|f'(u)|^2_{L^\infty}}} + 4\frac{\varepsilon^2}{\kappa^2}\left(|u|^2_{W^{1,2}} + \underbrace{|u-v|^2_{W^{1,2}}}_{\le\delta\le 1}\right)\\
&\le \frac{\varepsilon^2}{\kappa^2}\left(5 + 4|u|^2_{W^{1,2}}\right) + \frac{\varepsilon^2}{2} = \varepsilon^2.
\end{aligned}$$

This proves continuity and concludes the proof of Lemma 7.8. ■

Proof of Proposition 7.7. Since the function $h\colon \mathfrak{U}\to\mathbb{R}$ is of class C^3, given $i,j=1,\dots,2n$, the matrix coefficients $\partial_i\partial_j h\colon \mathfrak{U}\to\mathbb{R}$ of the Hessian are C^1 functions. Hence, by Lemma 7.8 and the fact that multiplication $W^{1,2}\times W^{1,2}\to W^{1,2}$ is continuous, the proposition follows. ■

Proof of Lemma 7.5. By Proposition 7.7, the map $u\mapsto a^u$ lies in $C^0(U_1,\mathcal{L}(H_1))$, hence in particular in $C^0(U_1,\mathcal{L}(H_1,H_0))\cap C^0(U_2,\mathcal{L}(H_2,H_1))$. Since $A^u = A^u_0 - a^u$ and A^u_0 is a weak Hessian field according to Lemma 7.2, it follows that A is also element of $C^0(U_1,\mathcal{L}(H_1,H_0))\cap C^0(U_2,\mathcal{L}(H_2,H_1))$.

The (`Symmetry`) axiom (6.1) holds true by Lemmas 4.7 and 5.4. We show the (`Fredholm`) axiom. Since the inclusions $H_1\hookrightarrow H_0$ and $H_2\hookrightarrow H_1$ are compact, it follows that for every $u\in U_1$ the operator $a^u|_{H_2}\colon H_2\to H_1$ is compact and so is $a^u\colon H_1\to H_0$. By Lemma 7.2, the operators A^u_0 and $A^u_0|_{H_2}$ are in particular Fredholm of index zero. Since Fredholm property and index are stable under compact perturbation, it follows that the same is true for A^u and for A^u_2. This proves Lemma 7.5. ■

Proof of Theorem 7.6. Since $\mathcal{L}(H_1)\subset\mathcal{L}(H_2,H_1)\cap\mathcal{L}(H_1,H_0)$, it follows from Proposition 7.7 that the map $u\mapsto A^u$ lies in $\in C^0(U_1,\mathcal{L}(H_2,H_1)\cap\mathcal{L}(H_1,H_0))$. Since the inclusions $H_1\hookrightarrow H_0$ and $H_2\hookrightarrow H_1$ are compact, it follows that for every $u\in U_1$ the operator $a^u|_{H_2}\colon H_2\to H_1$ is compact and so is $a^u\colon H_1\to H_0$.

By the unperturbed case, Theorem 7.3, we know that $B_u^{-1}\partial_t$ as a map $H_1\to H_0$ and as a map $H_2\to H_0$ is Fredholm of index zero; this is a consequence of Theorem C.1. Since Fredholm property and index are stable under compact perturbation, the same is true for F^u and its restriction F^u_2. In particular, F satisfies axiom (`F`) in Definition 6.6.

That C satisfies axiom (`C`) was already shown in the proof of Theorem 7.3. This shows that A is almost extendable and (F,C) is a decomposition of A. ■

A Theorem of Rabier

While the result of Rabier applies in greater generality, we discuss it for Hilbert space pairs. Let (H,W) be a *Hilbert space pair*, that is H and W are Hilbert spaces such that $W\subset H$, as sets, and the inclusion map $\iota\colon W\to H$ is a compact linear operator. Let $\mathrm{i}\in\mathbb{C}$ be the imaginary unit element. Consider a family $(A(s))_{s\in\mathbb{R}}$ of bounded linear operators $A(s)\colon W\to H$. Assume that the following five conditions hold:

(H1) (H,W) is a Hilbert space pair;

(H2) the operator family $\mathbb{R}\to\mathcal{L}(W,H)$, $s\mapsto A(s)$, depends continuously on s;

(H3) there are invertible asymptotic limits $A_\mp \in \mathcal{L}(W,H)$ in the sense that

$$\lim_{s\to\mp\infty} \|A(s) - A_\mp\|_{\mathcal{L}(W,H)} = 0;$$

(H4) $\forall s \in \mathbb{R}\cup\{\mp\infty\}$ $\exists C_0(s), r_0(s) > 0$ $\forall \alpha \in \mathbb{R}$: Whenever $|\alpha| \ge r_0(s)$ the operator $A(s) - \mathrm{i}\alpha\colon W \to H$ is an isomorphism and there is the estimate

$$\left\|(A(s) - \mathrm{i}\alpha)^{-1}\right\|_{\mathcal{L}(H)} \le \frac{C_0(s)}{|\alpha|};$$

(H5) $\mathrm{i}\mathbb{R} \cap \operatorname{spec} A_\mp = \varnothing$.

Theorem A.1 (Rabier [14])**.** *Under the assumptions* (H1)–(H5) *the operator*

$$D_A := \partial_s + A\colon\ W^{1,p}_H \cap L^p_W \to L^p_H$$

is Fredholm for every $p \in (1,\infty)$.

A.1 The unparametrized Rabier condition (H4)

In this appendix, we do not deal with operator families parametrized by $s \in \mathbb{R}$, but with individual operators $A \in \mathcal{L}(W,H)$ on a Hilbert space pair (H,W). An exemption is Corollary A.4.

Definition A.2. A bounded linear operator $A\colon W \to H$ *satisfies the Rabier condition* (H4), more precisely [14, equation (1.4)] in the case of Hilbert spaces, if there are constants $C_0, r_0 > 0$ such that $\forall \alpha \in \mathbb{R}$ with $|\alpha| \ge r_0$ it holds that

$$\mathrm{i}\alpha \notin \operatorname{spec} A := \{\lambda \in \mathbb{C} \mid A - \lambda\colon W \to H \text{ is not bijective}\}$$

and

$$\left\|\alpha(A - \mathrm{i}\alpha)^{-1}\right|\_\mathcal{L}(H) \le C_0. \tag{A.1}$$

Here and throughout, we write $\mathrm{i}\alpha$ to abbreviate $\mathrm{i}\alpha\iota$.

Symmetric Fredholm operators of index zero

Lemma A.3. *Suppose $A \in \mathcal{L}(W,H)$ is (a) H-symmetric and (b) Fredholm of index zero. Then for every real number $\alpha \ne 0$, there is the estimate*

$$\left\|\alpha(A - \mathrm{i}\alpha)^{-1}\right\|_{\mathcal{L}(H)} \le 1.$$

In particular, A satisfies the Rabier condition (H4) *for $C_0 = 1$ and every $r_0 > 0$.*

Proof. By (a) and (b), the spectrum of A is real and there exists an ONB $\{e_n\}_{n\in\mathbb{N}}$ of H and $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $Ae_n = a_n e_n$ for every $n \in \mathbb{N}$; see, e.g., [4, Appendix D]. Let $\alpha \in \mathbb{R}\setminus\{0\}$, then

$$\alpha(A - \mathrm{i}\alpha)^{-1} e_n = \frac{\alpha}{a_n - \mathrm{i}\alpha} e_n.$$

Let $\xi \in H$ and write it in the form $\xi = \sum_n \xi_n e_n$ for unique real numbers ξ_n. Suppose that ξ is of unit norm $|\xi|_H^2 = \sum_n \xi_n^2 = 1$. Then

$$\alpha(A - \mathrm{i}\alpha)^{-1}\xi = \sum_n \frac{\alpha\xi_n}{a_n - \mathrm{i}\alpha} e_n$$

and we estimate

$$\left|\alpha(A-\mathrm{i}\alpha)^{-1}\xi\right|_H^2 = \sum_n \frac{\alpha^2\xi_n^2}{|a_n-\mathrm{i}\alpha|_{\mathbb{C}}^2} = \sum_n \frac{\alpha^2}{a_n^2+\alpha^2}\xi_n^2 \le \sum_n \xi_n^2 = 1.$$

This proves Lemma A.3. ∎

Corollary A.4. *Let $(A(s)\colon W\to H)_{s\in\mathbb{R}}$ be a family of bounded linear operators such that each family member $A(s)$ is H-symmetric and Fredholm of index zero and* (H1)–(H3) *are satisfied. Then the operator*

$$D_A := \partial_s + A\colon\ W^{1,p}_H \cap L^p_W \to L^p_H$$

is Fredholm for every $p\in(1,\infty)$.

Proof. By Rabier's Theorem A.1, it remains to verify (H4) and (H5). (H4) holds by Lemma A.3. We prove (H5). Since $A(s)$ is H-symmetric for every $s\in\mathbb{R}$, the same holds for the asymptotics $A_\mp$. Therefore, the spectrum of $A_\mp$ is real. Since $A_\mp$ are invertible by (H3) zero does not belong to the spectrum of $A_\mp$ and therefore $\mathrm{i}\mathbb{R}\cap\operatorname{spec}A_\mp=\emptyset$. ∎

Reformulation

Lemma A.5. *Let $A\colon W\to H$ be a bounded linear operator. Then A satisfies the Rabier condition* (H4) *if and only if it satisfies the following two conditions:*

(i) A is Fredholm of index zero.

(ii) There exists constants $C_0, r_0>0$ such that whenever $|\alpha|\ge r_0$ it holds

$$|(A-\mathrm{i}\alpha)\xi|_H \ge C_0\alpha|\xi|_H \tag{A.2}$$

for every $\xi\in H$.

Proof. '⇐' Since $A\colon W\to H$ is Fredholm of index zero and inclusion $W\hookrightarrow H$ is compact, it follows that $A-\mathrm{i}\alpha\colon W\to H$ is also Fredholm of index zero for every $\alpha\in\mathbb{R}$. If r_0 is as in (ii) and $\alpha\ge r_0$, then $A-\mathrm{i}\alpha$ is injective by the estimate in (ii), hence surjective (Fredholm index zero). Therefore, by the open mapping theorem, the operator $A-\mathrm{i}\alpha$ is an isomorphism. In particular, the inverse $(A-\mathrm{i}\alpha)^{-1}$ is bounded. Now (A.1) follows from (A.2).

'⇒' Suppose (H4) holds true. Hence, if $\alpha\ge r_0$, then $A-\mathrm{i}\alpha$ is bijective and therefore it is Fredholm of index zero. Since inclusion $W\hookrightarrow H$ is compact, it follows that A is Fredholm of index zero as well. This shows (i) and (A.2) follows from (A.1). This proves Lemma A.5. ∎

Remark A.6 (spectrum consists of eigenvalues)**.** $\lambda\in\operatorname{spec}A$ is called *eigenvalue* of $A\in\mathcal{L}(W,H)$ if $A-\lambda\iota\colon W\to H$ is not injective. Observe that

$$A \text{ satisfies (H4)} \quad\Rightarrow\quad \operatorname{spec}A=\{\text{eigenvalues of } A\}.$$

To see this, suppose A satisfies (H4) and pick $\lambda\in\operatorname{spec}A$. The latter means that $A-\lambda\iota$ is not injective or not surjective. Now A is Fredholm of index zero by (i) and so is $A-\lambda\iota$, as shown in the beginning of the previous proof. In particular, not injective and not surjective are equivalent for the operator $A-\lambda\iota$.

Stability under compact perturbation

The following proposition is a version of [13, Theorem 3.5].

Proposition A.7 (compact perturbation)**.** *Let $A\colon W\to H$ be a bounded linear operator satisfying the Rabier condition* (H4) *and $K\colon W\to H$ a compact linear operator. Then $A+K\colon W\to H$ satisfies the Rabier condition* (H4) *as well.*

In the proof, we utilize the following notions. Let $A\colon W\to H$ be a bounded linear operator. A bounded linear operator $T\colon W\to H$ is called ***A**-bounded* if there exist constants $a\ge 0$ and $b(a)\ge 0$ such that

$$|T\xi|_H\le a|A\xi|_H+b(a)|\xi|_H \tag{A.3}$$

for every $\xi\in W$. The ***A**-bound* of T is the infimum of all possible values $a\ge 0$, in symbols

$$\inf\{a\ge 0\mid \exists b(a) \text{ such that (A.3) holds } \forall \xi\in W\}.$$

A theorem of Hess [7, Satz 1] says that, given $A\in\mathcal{L}(W,H)$ with non-empty resolvent set, compact operators $K\colon W\to H$ have vanishing A-bound.

Proof. Let A satisfy (H4). Then A satisfies (i) and (ii) in Lemma A.5 with constants r_0, C_0. It suffices to check conditions (i) and (ii) for $A+K$ for some constants r_1, C_1. Since $K\colon W\to H$ is compact, it holds that $A+K$ is Fredholm and $\operatorname{index}(A+K)=\operatorname{index}A=0$. This proves (i). To prove (ii) we define

$$\varepsilon:=\min\left\{\frac{1}{2},\frac{C_0}{4}\right\},\qquad r_1:=\max\left\{r_0,\frac{1}{2}+4\frac{b(\varepsilon)}{C_0}\right\},\qquad C_1:=\frac{C_0}{8}$$

and for $|\alpha|\ge r_1\ge r_0$ we estimate

$$\begin{aligned}
|(A+K-\mathrm{i}\alpha)\xi|_H &\ge |(A-\mathrm{i}\alpha)\xi|_H-|K\xi|_H\\
&\ge |(A-\mathrm{i}\alpha)\xi|_H-\varepsilon|(A-\mathrm{i}\alpha+\mathrm{i}\alpha)\xi|_H-b(\varepsilon)|\xi|_H\\
&\ge (1-\varepsilon)|(A-\mathrm{i}\alpha)\xi|_H-(\varepsilon|\alpha|+b(\varepsilon))|\xi|_H\\
&\ge ((1-\varepsilon)C_0|\alpha|-\varepsilon|\alpha|-b(\varepsilon))|\xi|_H\\
&\ge \left(\frac{C_0}{4}|\alpha|-b(\varepsilon)\right)|\xi|_H\\
&\ge \frac{C_0}{8}|\xi|_H.
\end{aligned}$$

In inequality 2, we used (A.3) and then added zero $-\mathrm{i}\alpha+\mathrm{i}\alpha$. Inequality 3 is by the triangle inequality. Inequality 4 is by the hypothesis (A.2) on A which applies since $|\alpha|\ge r_0$. Inequality 5 is by choice of ε. Inequality 6 uses $|\alpha|\ge r_1$ and the choice of r_1. This proves Proposition A.7. ∎

Level operator

Lemma A.8. *Let (H_0,H_1,H_2) be a Hilbert space triple. Suppose $F\colon H_1\to H_0$ is a bounded linear operator which satisfies the Rabier estimate* (H4)*. If F restricts to a map $F_2=F|_{H_2}\colon H_2\to H_1$ and F_2 is Fredholm of index zero, then F_2 satisfies the Rabier estimate* (H4) *as well.*

Proof. Let $C_0,r_0>0$ be the constants in the Rabier condition (H4). Fix $\alpha_0\ge r_0$. Then $F-\mathrm{i}\alpha_0\colon H_1\to H_0$ is bijective, thus an isomorphism. Therefore, maybe after replacing the norm of H_1 by an equivalent norm, we can assume without loss of generality that for every $\xi \in H_1$ the H_1-norm is given by $|\xi|_1 = |(F - \mathrm{i}\alpha_0)\xi|_0$. Hence for $\alpha \ge r_0$ and $\xi \in H_1$, we write and estimate

$$\begin{aligned} |(F-\mathrm{i}\alpha)\xi|_1 &= |(F-\mathrm{i}\alpha_0)(F-\mathrm{i}\alpha)\xi|_0 \\ &= |(F-\mathrm{i}\alpha)(F-\mathrm{i}\alpha_0)\xi|_0 \\ &\ge C_0\alpha|(F-\mathrm{i}\alpha_0)\xi|_0 \\ &= C_0\alpha|\xi|_1. \end{aligned}$$

The inequality is by (A.2) for F. This shows that F_2 satisfies (A.2). But F_2 is Fredholm of index zero by hypothesis. Hence F_2 satisfies (i) and (ii) in Lemma A.5 and therefore (H4). This proves Lemma A.8. ∎

B Compact non-continuous perturbations

Lemma B.1. *For a Hilbert space H, an interval $I = [-T, T]$, and $c \in L^2(I, H)$ multiplication $m_c\colon W^{1,2}(I, \mathbb{R}) \to L^2(I, H)$, $\xi \mapsto c\xi$ is a compact operator.*

Proof. The multiplication operator is a composition

$$m_c = m_c^0 \circ \iota\colon\ W^{1,2}(I,\mathbb{R}) \xrightarrow{\iota} C^0(I,\mathbb{R}) \xrightarrow{m_c^0} L^2(I,H)$$

of the compact Sobolev inclusion $\iota\colon W^{1,2}(I,\mathbb{R}) \hookrightarrow C^0(I,\mathbb{R})$ followed by the multiplication m_c^0 which is bounded as we show next. Indeed, multiplication

$$L^2(I,H) \times C^0(I,\mathbb{R}) \to L^2(I,H), \qquad (c,\xi) \mapsto c\xi$$

is a continuous map since

$$\left\|m_c^0\xi\right\|_{L^2(I,H)} = \|c\xi\|_{L^2(I,H)} \le \|c\|_{L^2(I,H)}\|\xi\|_{C^0(I,\mathbb{R})}.$$

Hence the map m_c is the composition of a compact and a continuous linear map, and therefore it is compact. This proves Lemma B.1. ∎

Proposition B.2. *Assume H is a Hilbert space and $c \in L^2(\mathbb{R}, H)$. Then multiplication $m_c\colon W^{1,2}(\mathbb{R},\mathbb{R}) \to L^2(\mathbb{R}, H)$, $\xi \mapsto c\xi$ is a compact operator.*

Proof. Given $n \in \mathbb{N}$, let $\chi_n := \chi_{[-n,n]}$ be the characteristic function on the interval $[-n, n]$, that is χ_n is 1 on $[-n, n]$ and 0 else. Then the multiplication operator $m_c^n := m_{\chi_n c}$ is compact by Lemma B.1.

To show that m_c is compact, we show that the sequence of compact operators m_c^n converges in the operator norm topology to m_c. Indeed,

$$\begin{aligned} \|(m_c - m_c^n)\xi\|_{L^2(\mathbb{R},H)} &= \|(c-\chi_n c)\xi\|_{L^2(\mathbb{R},H)} \\ &= \left\|\chi_{\mathbb{R}\setminus[-n,n]}c\xi\right\|_{L^2(\mathbb{R},H)} \\ &\le \left\|\chi_{\mathbb{R}\setminus[-n,n]}c\right\|_{L^2(\mathbb{R},H)}\|\xi\|_{C^0(\mathbb{R},\mathbb{R})} \\ &\le \underbrace{\|c\|_{L^2(\mathbb{R}\setminus[-n,n],H)}}_{\to 0,\text{ as } n\to\infty}\|\xi\|_{W^{1,2}(\mathbb{R},\mathbb{R})}. \end{aligned}$$

This proves Proposition B.2. ∎

Let (H_1, H_2) be a *Hilbert space pair*, see, e.g., [3], that is H_1 and H_2 are both infinite-dimensional Hilbert spaces such that $H_2 \subset H_1$ and inclusion $\iota\colon H_2 \hookrightarrow H_1$ is compact and dense. We define $W^{1,2}_{H_1}$ and $L^2_{H_i}$ by (5.4).

Theorem B.3. *Let $C \in L^2(\mathbb{R}, \mathcal{L}(H_1))$. Then pointwise multiplication*

$$M_C\colon\ W^{1,2}_{H_1} \cap L^2_{H_2} \to L^2_{H_1}, \qquad \xi \mapsto [s \mapsto C(s)\xi(s)]$$

is a compact operator.

Remark B.4. In the special case where C is continuous, Theorem B.3 was proved by Robbin and Salamon [15, Lemma 3.18]. As in our proof, they use a sequence of compact operators which converges to the operator M_C in the operator topology. In contrast to our sequence, they first multiply with C and then project, while we first project and then multiply by C.

Definition B.5 (intersection of Hilbert spaces)**.** The intersection $H \cap W$ of two Hilbert spaces H and W is itself a Hilbert space with inner product and norm

$$\langle\cdot,\cdot\rangle_{H\cap W} := \langle\cdot,\cdot\rangle_H + \langle\cdot,\cdot\rangle_W\,, \qquad \|\cdot\|_{H\cap W} := \sqrt{\|\cdot\|_H^2 + \|\cdot\|_W^2}.$$

Proof of Theorem B.3. The Hilbert space pair (H_1, H_2) is isometric to the pair $\left(\ell^2, \ell^2_h\right)$ where $h\colon \mathbb{N} \to (0,\infty)$ is a monotone unbounded function; see [3, Theorem A.4]. In the following, we identify the Hilbert space pairs

$$(H_1, H_2) \simeq \left(\ell^2, \ell^2_h\right).$$

We denote by $\mathcal{E} = \{e_k\}_{k\in\mathbb{N}}$ the canonical basis of ℓ^2 and by

$$\pi_n\colon\ H_1 \to \mathbb{R}^n, \qquad \xi = \sum_{k=1}^\infty \xi_k e_k \mapsto (\xi_1, \dots, \xi_n)$$

the orthogonal projection of H_1 to the n-dimensional subspace of H_1 identified with $\mathbb{R}^n$ under the isometry mentioned above. As the basis $\mathcal{E}$ is still orthogonal in ℓ^2_h, the restriction $\pi_n|_{H_2}\colon H_2 \to \mathbb{R}^n$ is again the orthogonal projection.

For $n \in \mathbb{N}$, we define the operator

$$M_C^n\colon\ W^{1,2}_{H_1} \cap L^2_{H_2} \to L^2_{H_1}, \qquad \xi \mapsto [s \mapsto (M_C^n\xi)(s) := C(s)\pi_n\xi(s)]$$

where $\iota_n = \pi_n^*\colon \mathbb{R}^n \hookrightarrow H_1$ is inclusion.

Step 1. For every $n \in \mathbb{N}$, the operator M_C^n is compact.

Step 1 follows from Proposition B.2 for $m_c = M_C^n$ and where $\tilde\xi(s) = \pi_n\xi(s)$ now takes values in $\mathbb{R}^n$ instead of $\mathbb{R}$.

Step 2. As $n \to \infty$, the operators M_C^n converge to M_C in the operator norm

$$\lim_{n\to\infty} \sup_{\|\xi\|_{W^{1,2}_{H_1}\cap L^2_{H_2}}=1} \|(M_C^n - M_C)\xi\|_{L^2_{H_1}} = 0.$$

To prove Step 2, we write the difference in the form

$$M_C - M_C^n = \mathcal{M}_C \circ P_n,$$

where

$$P_n\colon\ W^{1,2}_{H_1} \cap L^2_{H_2} \to L^\infty_{H_1}, \qquad \xi \mapsto [s \mapsto (\mathbb{1} - \pi_n)\xi(s)]$$

and

$$\mathcal{M}_C\colon\ L^\infty_{H_1} \to L^2_{H_1}, \qquad \eta \mapsto [s \mapsto C(s)\eta(s)].$$

The operator $\mathcal{M}_C$ is bounded: Indeed, for $\eta \in L^\infty_{H_1}$ of unit norm, we estimate

$$\begin{aligned}\|\mathcal{M}_C\eta\|^2_{L^2_{H_1}} &= \int_{-\infty}^{\infty} |C(s)\eta(s)|^2_{H_1}\,\mathrm{d}s\\ &\le \int_{-\infty}^{\infty} \|C(s)\|^2_{\mathcal{L}(H_1)}\Big(\sup_{s\in\mathbb{R}}|\eta(s)|_{H_1}\Big)^2\,\mathrm{d}s\\ &= \|C\|^2_{L^2(\mathbb{R},\mathcal{L}(H_1))}.\end{aligned}$$

Since

$$\|M_C - M_C^n\|_{\mathcal{L}(W^{1,2}_{H_1}\cap L^2_{H_2},L^2_{H_1})} \le \|\mathcal{M}_C\|_{\mathcal{L}(L^\infty_{H_1},L^2_{H_1})}\cdot\|P_n\|_{\mathcal{L}(W^{1,2}_{H_1}\cap L^2_{H_2},L^\infty_{H_1})},$$

it suffices to show that the norm of P_n converges to zero, as $n\to\infty$.

Claim. *If $\eta \in W^{1,2}_{H_1}\cap L^2_{H_2}$ satisfies the conditions*

$$\text{(i) } \|\dot\eta\|_{L^2_{H_1}} \le 1, \qquad \text{(ii) } \|\eta\|_{L^2_{H_2}} \le 1, \qquad \text{(iii) } \forall s\in\mathbb{R}:\ \pi_n\eta(s) = 0,$$

then $\|\eta\|_{L^\infty_{H_1}} \le \big(\frac{3}{h_{n+1}}\big)^{\frac14}$.

Proof. Condition (iii) means that η is for any $s\in\mathbb{R}$ of the form

$$\eta(s) \stackrel{\text{(iii)}}{=} (0,\dots,0,\eta_{n+1}(s),\eta_{n+2}(s),\dots)$$

which, as in (B.3), provides the estimate

$$|\eta(s)|^2_{\ell^2_h} \ge h_{n+1}|\eta(s)|^2_{\ell^2}. \tag{B.1}$$

Now we prove the estimate

$$|\eta(s)|_{\ell^2} \ge |\eta(0)|_{\ell^2} - \sqrt{|s|} \tag{B.2}$$

for every $s\in\mathbb{R}$. We show this estimate for $s\ge 0$, in case $s\le 0$ the argument is similar. To this end pick $s\ge 0$. Use Cauchy–Schwarz inequality to estimate

$$\int_0^s 1\cdot|\dot\eta(t)|_{\ell^2}\,\mathrm{d}t \le \sqrt{\int_0^s 1\,\mathrm{d}t}\cdot\sqrt{\int_0^s |\dot\eta(t)|_{\ell^2}\,\mathrm{d}t} \le \sqrt{s}\cdot\|\dot\eta\|_{L^2_{H_1}} \stackrel{\text{(i)}}{\le} \sqrt{s}.$$

This estimate and the fundamental theorem of calculus yield the claim, namely

$$|\eta(s)|_{\ell^2} = \left|\eta(0) + \int_0^s \dot\eta(t)\,\mathrm{d}t\right|_{\ell^2} \ge |\eta(0)|_{\ell^2} - \int_0^s 1\cdot|\dot\eta(t)|_{\ell^2}\,\mathrm{d}t \ge |\eta(0)|_{\ell^2} - \sqrt{s}.$$

This proves the estimate (B.2). Next we estimate

$$\begin{aligned}1 &\stackrel{\text{(ii)}}{\ge} \|\eta\|^2_{L^2_{H_2}}\\ &= \int_\infty^\infty |\eta(s)|^2_{\ell^2_h}\,\mathrm{d}s\\ &\stackrel{3}{\ge} h_{n+1}\int_\infty^\infty |\eta(s)|^2_{\ell^2}\,\mathrm{d}s\\ &\ge h_{n+1}\int_{-|\eta(0)|^2_{\ell^2}}^{|\eta(0)|^2_{\ell^2}} |\eta(s)|^2_{\ell^2}\,\mathrm{d}s\\ &\stackrel{5}{\ge} 2h_{n+1}\int_0^{|\eta(0)|^2_{\ell^2}} \big(|\eta(0)|_{\ell^2} - \sqrt{s}\big)^2\mathrm{d}s\\ &= \frac{1}{3}h_{n+1}|\eta(0)|^4_{\ell^2}.\end{aligned}$$

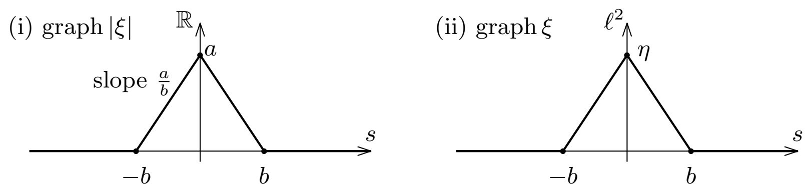

Figure 2. Graphs of $|\xi|$ and ξ in Example B.6.

In Step 3, we used (B.1). In Step 5, we used the claim (B.2). To obtain the final identity, we calculated the integral.

We rewrite the obtained estimate in the form $|\eta(0)|_{\ell^2} \le (3/h_{n+1})^{\frac{1}{4}}$. For $r \in \mathbb{R}$, we define $\eta_r(s) := \eta(s+r)$. Then conditions (i)–(iii) hold as well for η_r. Therefore,

$$|\eta(r)|_{\ell^2} = |\eta_r(0)|_{\ell^2} \le (3/h_{n+1})^{\frac{1}{4}}$$

for every $r \in \mathbb{R}$. Consequently, $\|\eta\|_{L^\infty_{H_1}} \le (3/h_{n+1})^{\frac{1}{4}}$. This proves the claim. ∎

Operator norm of $\boldsymbol{P_n}$. Pick $\xi \in W^{1,2}_{H_1} \cap L^2_{H_2}$ such that $\|\xi\|_{W^{1,2}_{H_1} \cap L^2_{H_2}} \le 1$. Now we verify conditions (i)–(iii) for $\eta = P_n\xi$. Condition (i) is satisfied, indeed

$$\left\|P_n\dot{\xi}\right\|_{L^2_{H_1}} = \left\|(\mathbb{1} - \pi_n)\dot{\xi}\right\|_{L^2_{H_1}} \le \left\|\dot{\xi}\right\|_{L^2_{H_1}} \le \|\xi\|_{W^{1,2}_{H_1}} \le \|\xi\|_{W^{1,2}_{H_1} \cap L^2_{H_2}} \le 1.$$

Inequality one uses that the projection is H_1-orthogonal. In inequality three, we used Definition B.5 of the norm in an intersection space. Condition (ii) is satisfied by the same arguments

$$\|P_n\xi\|_{L^2_{H_2}} = \|(\mathbb{1} - \pi_n)\xi\|_{L^2_{H_2}} \le \|\xi\|_{L^2_{H_2}} \le 1.$$

In inequality one, we used that the projection is also H_2-orthogonal. Condition (iii) is satisfied due to the projection property $\pi_n^2 = \pi_n$, namely

$$\pi_n P_n \xi(s) = \pi_n(\mathbb{1} - \pi_n)\xi(s) = 0.$$

Thus, by the claim, we get $\|P_n\xi\|_{L^\infty_{H_1}} \le \left(\frac{3}{h_{n+1}}\right)^{\frac{1}{4}}$. Therefore, the operator norm

$$\|P_n\|_{\mathcal{L}(W^{1,2}_{H_1} \cap L^2_{H_2}, L^\infty_{H_1})} \le \left(\frac{3}{h_{n+1}}\right)^{\frac{1}{4}}$$

converges to zero, as $n \to \infty$, since the growth function h is unbounded. This concludes the proof of Step 2.

Steps 1 and 2 together imply Theorem B.3 by the standard fact that the subspace of compact operators is closed in the space of bounded linear operators; see, e.g., [6, Theorem A.1]. This proves Theorem B.3. ∎

Example B.6. Recall that we identify $(H_1, H_2) \simeq (\ell^2, \ell^2_h)$. Pick an element $\eta \in \ell^2$ which is of the form $\eta = (0, \dots, 0, \eta_{n+1}, \eta_{n+2}, \dots)$. Set $a := |\eta|_{\ell^2}$ and pick $b > 0$. Define a map $\xi \colon \mathbb{R} \to \ell^2$ by

$$\xi(s) := \begin{cases} \left(\frac{s}{b} + 1\right)\eta, & s \in [-b, 0], \\ \left(-\frac{s}{b} + 1\right)\eta, & s \in [0, b], \\ 0, & \text{else}, \end{cases}$$

as illustrated by Figure 2.

Note that $|\dot\xi(s)|_{\ell^2}=a/b$ whenever $s\in(-b,0)\cup(0,b)$. Suppose that ξ satisfies the conditions (a) $\left\|\dot\xi\right\|_{L^2_{H_1}}=1$ and (b) $\|\xi\|_{L^2_{H_2}}=1$. By (a) and calculation,

$$1=\left\|\dot\xi\right\|^2_{L^2_{H_1}}=2\int_0^b\left(\frac{a}{b}\right)^2\mathrm{d}s=\frac{2a^2}{b}\quad\Rightarrow\quad b=2a^2$$

and

$$\|\xi\|^2_{L^2_{H_1}}=2\int_0^b\left(\frac{a}{b}s\right)^2\mathrm{d}s=2a^2\frac{b}{3}=\frac{4}{3}a^4.$$

Observe that

$$|\eta|^2_{\ell^2_h}=\sum_{k=n+1}^\infty h_k\eta_k^2\ge h_{n+1}\sum_{k=n+1}^\infty\eta_k^2=h_{n+1}|\eta|^2_{\ell^2} \tag{B.3}$$

and use the unit norm condition (b) to obtain

$$1=\|\xi\|^2_{L^2_{H_2}}\ge h_{n+1}\|\xi\|^2_{L^2_{H_1}}=\frac{4}{3}h_{n+1}a^4,$$

which implies that $a\le\left(\frac{3}{4h_{n+1}}\right)^{\frac14}\longrightarrow 0$, as $n\to\infty$.

C Slice-wise index zero Fredholm operators

The space of symplectic bilinear forms on $\mathbb{R}^{2n}$ is denoted by

$$\Omega\bigl(\mathbb{R}^{2n}\bigr)=\bigl\{\omega\colon\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}\text{ bilinear, skew-symmetric, non-degenerate}\bigr\}.$$

A loop of symplectic forms

$$\omega\in W^{1,2}\bigl(\mathbb{S}^1,\Omega\bigl(\mathbb{R}^{2n}\bigr)\bigr),\qquad t\mapsto\omega_t \tag{C.1}$$

determines a loop of invertible matrices $B_\omega\in W^{1,2}\bigl(\mathbb{S}^1,\mathrm{GL}\bigl(2n,\mathbb{R}\bigr)\bigr)$ by

$$\omega_t(\cdot,B_\omega(t)\cdot)=\langle\cdot,\cdot\rangle$$

pointwise for every $t\in\mathbb{S}^1$ where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^{2n}$.

Theorem C.1. *Fix a loop of symplectic bilinear forms $\omega\in W^{1,2}\bigl(\mathbb{S}^1,\Omega\bigl(\mathbb{R}^{2n}\bigr)\bigr)$. Then the operator*

$$F_{B_\omega^{-1}}=B_\omega^{-1}\partial_t\colon\ H_k\to H_{k-1},\qquad H_k:=W^{k,2}\bigl(\mathbb{S}^1,\mathbb{R}^{2n}\bigr),$$

is Fredholm of index zero whenever $k=1,2$.

The proof of Theorem C.1 will be given at the end of Appendix C.

Semi-Fredholm estimates

Lemma C.2. *Consider two loops of invertible matrices*

$$C \in C^0(\mathbb{S}^1, \mathrm{GL}(k,\mathbb{R})), \qquad L \in W^{1,2}(\mathbb{S}^1, \mathrm{GL}(k,\mathbb{R})).$$

Then the corresponding linear operators

$$\begin{aligned} &(a)\ F_C\colon\ W^{1,2}(\mathbb{S}^1,\mathbb{R}^k) \to L^2(\mathbb{S}^1,\mathbb{R}^k), \qquad \xi \mapsto [t \mapsto C(t)\dot\xi(t)] \\ &(b)\ F_L\colon\ W^{2,2}(\mathbb{S}^1,\mathbb{R}^k) \to W^{1,2}(\mathbb{S}^1,\mathbb{R}^k), \qquad \xi \mapsto [t \mapsto L(t)\dot\xi(t)] \end{aligned} \tag{C.2}$$

are semi-Fredholm operators.[7]

Proof. (a) Since $C(t)$ is invertible and the circle is compact, there exists $\delta > 0$ such that $|C(t)v| \geq \delta|v|$ for all $t \in \mathbb{S}^1$ and $v \in \mathbb{R}^k$. Hence we estimate

$$|F_C\xi|_{L^2}^2 = \int_0^1 \left|C(t)\dot\xi(t)\right|^2 \mathrm{d}t \geq \delta^2 \left|\dot\xi\right|_{L^2}^2 = \delta^2|\xi|_{W^{1,2}}^2 - \delta^2|\xi|_{L^2}^2.$$

Therefore,

$$|\xi|_{W^{1,2}}^2 \leq \frac{1}{\delta^2}|F_C\xi|_{L^2}^2 + |\iota\xi|_{L^2}^2$$

for every $\xi \in W^{1,2}$. Since the inclusion map $\iota\colon W^{1,2} \to L^2$ is compact, this is a semi-Fredholm estimate, see, e.g., [9, Lemma A.1.1]. It implies finite dimension of the kernel and closedness of the image of the operator F_C.

(b) Since $W^{1,2}$ embeds in C^0, the map L is continuous. Since $L(t)$ is invertible and the circle is compact, there exists $\delta > 0$ such that $|L(t)v| \geq \delta|v|$ for all $t \in \mathbb{S}^1$ and $v \in \mathbb{R}^k$. Hence we estimate

$$\begin{aligned} |F_L\xi|_{W^{1,2}}^2 &= \left|L\dot\xi\right|_{L^2}^2 + \left|\dot L\dot\xi + L\ddot\xi\right|_{L^2}^2 \\ &\geq \delta^2\left|\dot\xi\right|_{L^2}^2 + \left|\dot L\dot\xi\right|_{L^2}^2 \underline{+2\left\langle \sqrt{2}\dot L\dot\xi, \frac{1}{\sqrt{2}}L\ddot\xi\right\rangle} + \left|L\ddot\xi\right|_{L^2}^2 \\ &\geq \delta^2\left|\dot\xi\right|_{L^2}^2 + \left|\dot L\dot\xi\right|_{L^2}^2 \underline{-2\left|\dot L\dot\xi\right|_{L^2}^2 - \frac{1}{2}\left|L\ddot\xi\right|_{L^2}^2} + \left|L\ddot\xi\right|_{L^2}^2 \\ &= \delta^2\left|\dot\xi\right|_{L^2}^2 + \frac{1}{2}\left|L\ddot\xi\right|_{L^2}^2 - \left|\dot L\dot\xi\right|_{L^2}^2 \\ &\geq \delta^2\left|\dot\xi\right|_{L^2}^2 + \frac{\delta^2}{2}\left|\ddot\xi\right|_{L^2}^2 - \left|\dot L\right|_{L^2}^2\left|\dot\xi\right|_{C^0}^2 \\ &\geq \frac{\delta^2}{2}|\xi|_{W^{2,2}}^2 - \frac{\delta^2}{2}|\xi|_{L^2}^2 - |\dot L|_{L^2}^2|\xi|_{C^1}^2 \\ &\geq \frac{\delta^2}{2}|\xi|_{W^{2,2}}^2 - \left(\frac{\delta^2}{2} + |\dot L|_{L^2}^2\right)|\xi|_{C^1}^2. \end{aligned}$$

Therefore,

$$|\xi|_{W^{2,2}}^2 \leq \frac{2}{\delta^2}|F_L\xi|_{L^2}^2 + \left(1 + \frac{2}{\delta^2}|\dot L|_{L^2}^2\right)|\xi|_{C^1}^2$$

for every $\xi \in W^{1,2}$. Since the inclusion map $W^{2,2} \to C^1$ is compact, this is a semi-Fredholm estimate, see, e.g., [9, Lemma A.1.1]. It implies finite dimension of the kernel and closedness of the image of the operator F_L. This proves Lemma C.2. ∎

[7] Semi-Fredholm means finite-dimensional kernel and closed range.

Index is conjugation invariant

Let $\mathrm{GL}^+(k,\mathbb{R})$ be the open subset of $\mathbb{R}^{k\times k}$ that consists of all real $k\times k$ matrices which are of positive determinant, thus invertible. Clearly $\mathbb{1}\in\mathrm{GL}^+(k,\mathbb{R})$,

Lemma C.3. *Consider loops of invertible matrices of positive determinant*

$$(a)\ C,G\in C^0\big(\mathbb{S}^1,\mathrm{GL}^+(k,\mathbb{R})\big),\qquad (b)\ L,H\in W^{1,2}\big(\mathbb{S}^1,\mathrm{GL}^+(k,\mathbb{R})\big).$$

Then the following loops are homotopic:

$$C\sim GCG^{-1},\qquad L\sim HLH^{-1},$$

and for the operators (C.2) *the semi-Fredholm index is conjugation invariant*

$$\operatorname{index}F_C=\operatorname{index}F_{GCG^{-1}},\qquad \operatorname{index}F_L=\operatorname{index}F_{HLH^{-1}}.$$

Proof. We prove part (a). Part (b) then follows by the same arguments by noting that multiplication $W^{1,2}\times W^{1,2}\to W^{1,2}$ is continuous.

Since C and G take values in GL^+, we assume without loss of generality that the loops C and G are based at $\mathbb{1}$, in symbols $C(0)=\mathbb{1}=G(0)$.[8]

By Lemma C.4, we get the based homotopies $\approx$ in Steps 1 and 3 in the calculation

$$G^{-1}CG\approx G^{-1}\#(CG)\sim(CG)\#G^{-1}\approx CGG^{-1}=C.$$

The (free) homotopy $\sim$ in Step 2 is the map defined for $r,t\in[0,1]$ by $h^r(t):=(w\#v)\big(\tfrac{r}{2}+t\big)$. It deforms $h^0=w\#v$ to $h^1=v\#w$ and moves the time 0 point from $v(0)$ along v to $v(1)$. But the semi-Fredholm index is a homotopy invariant; e.g., [11, Section 18, Corollary 3]. This proves Lemma C.3. ∎

Lemma C.4. *Let $\mathbb{R}^{k\times k}$ be the space of real $k\times k$ matrices. Let G and H be two loops in $\mathbb{R}^{k\times k}$ based at $\mathbb{1}=G(0)=H(0)$. Then the pointwise matrix product is based homotopic to the concatenation, in symbols $HG\approx H\#G$.*

Proof. Let $r\in[0,1]$ and $t\in[0,1]$. Defining

$$G^r(t):=\begin{cases}G((1+r)t), & t\in\big[0,\tfrac{1}{1+r}\big],\\ \mathbb{1}, & t\in\big[\tfrac{1}{1+r},1\big],\end{cases}$$

then $G^1(t)$ is $G(2t)$ on $\big[0,\tfrac12\big]$ and $\mathbb{1}$ on $\big[\tfrac12,1\big]$. Defining

$$H^r(t):=\begin{cases}\mathbb{1}, & t\in\big[0,\tfrac{r}{1+r}\big],\\ H((1+r)t-r), & t\in\big[\tfrac{r}{1+r},1\big],\end{cases}$$

then $H^1(t)$ is $\mathbb{1}$ on $\big[0,\tfrac12\big]$ and $H\big(2(t-\tfrac12)\big)$ on $\big[\tfrac12,1\big]$. Note that the homotopy defined by $h^r(t):=H^r(t)G^r(t)$ deforms the map

$$t\mapsto h^0(t)=H^0(t)G^0(t)=H(t)G(t)$$

to the map

$$t\mapsto h^1(t)=H^1(t)G^1(t)=(H\#G)(t).$$

Moreover, the homotopy moves through based loops

$$h^r(0)=H^r(0)G^r(0)=\mathbb{1}G(0)=\mathbb{1},\qquad h^r(1)=H^r(1)G^r(1)=H(1)\mathbb{1}=\mathbb{1}.$$

This proves Lemma C.4. ∎

[8] If not, fix $\gamma\in C^0\big([0,1],GL^+(k,\mathbb{R})\big)$ from $\gamma(0)=\mathbb{1}$ to $\gamma(1)=C(0)$. Define $\overline{C}$ by following γ, then C, then γ backwards (notation γ^-), all at 3-fold speed. Retract γ and γ^- to C_0.

Fredholm index zero

Pick a $W^{1,2}$-loop ω as in (C.1). The loop ω determines a $W^{1,2}$-loop $B = B_\omega$ which, by (3.2), takes values in $\mathrm{GL}^+(2n,\mathbb{R})$ and which itself, by Lemma 3.3, determines a $W^{1,2}$-loop J_B of almost complex structures on $\mathbb{R}^{2n}$ compatible with ω in the sense that $g_{J_B} := \omega(\cdot, J_B \cdot)$ is a loop of Riemannian metrics. Convex combination deforms the Euclidean metric to g_{J_B}, namely

$$(1-r)\langle\cdot,\cdot\rangle + r g_{J_B} =: g^{(r)} = \omega\big(\cdot, B^{(r)}\cdot\big), \qquad r \in [0,1]. \tag{C.3}$$

The identity determines a homotopy $r \mapsto B^{(r)}$ from the loop $B^{(0)} = B$ to $B^{(1)} = J_B$. Abbreviate

$$J(t) := J_{B(t)}, \qquad t \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}.$$

Lemma C.5. *There exists a loop* $\Psi \in W^{1,2}\big(\mathbb{S}^1, \mathrm{GL}^+(2n,\mathbb{R})\big)$ *such that*

$$\Psi(t) J(0) \Psi(t)^{-1} = J(t), \qquad \forall t \in \mathbb{S}^1. \tag{C.4}$$

Proof. The proof has three steps.

Step 1. There exists a unique map $\psi\colon \mathbb{S}^1 \to \mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$ such that for any $t \in \mathbb{S}^1$ and any representative $\Psi(t) \in \mathrm{GL}(2n,\mathbb{R})$ of $\psi(t)$, we have

$$\Psi(t) J(0) \Psi(t)^{-1} = J(t). \tag{C.5}$$

Proof. Since two complex structures are conjugated, see, e.g., [9, Proposition 2.5.1], for every t there exists $\Psi(t) \in \mathrm{GL}(2n,\mathbb{R})$ such that $\Psi(t) J(0) \Psi(t)^{-1} = J(t)$.

If $\tilde\Psi(t)$ is another element of $\mathrm{GL}(2n,\mathbb{R})$ such that $\tilde\Psi(t) J(0) \tilde\Psi(t)^{-1} = J(t)$, then

$$\tilde\Psi(t)^{-1} \Psi(t) J(0) \big(\tilde\Psi(t)^{-1}\Psi(t)\big)^{-1} = J(0).$$

This shows that $\tilde\Psi(t)^{-1}\Psi(t)$ preserves $J(0)$ and hence this is an element of $\mathrm{GL}(n,\mathbb{C})$. Therefore, the map denoted and defined by

$$\psi\colon\ \mathbb{S}^1 \to \frac{\mathrm{GL}(2n,\mathbb{R})}{\mathrm{GL}(n,\mathbb{C})}, \qquad t \mapsto [\Psi(t)] \tag{C.6}$$

is well defined, namely independent of the choice of $\Psi(t)$. In view of (C.5), the subgroup $\mathrm{GL}(n,\mathbb{C})$ acts from the right on $\mathrm{GL}(2n,\mathbb{R})$. This proves Step 1. ■

Step 2. The canonical map ψ defined by (C.6) is of class $W^{1,2}$.

Proof. Let $t \in \mathbb{S}^1$ and choose vectors $v_1,\dots,v_n \in \mathbb{R}^{2n}$ such that $v_1, J(t)v_1, \dots, v_n, J(t)v_n$ is a basis of $\mathbb{R}^{2n}$. Since J is of class $W^{1,2}$, it is continuous. Therefore, since linear independence is an open property, there exists $\varepsilon > 0$ such that for any $t' \in (t-\varepsilon, t+\varepsilon)$ the vectors $v_1, J(t')v_1, \dots, v_n, J(t')v_n$ still form a basis of $\mathbb{R}^{2n}$. Choose further vectors $w_1,\dots,w_n \in \mathbb{R}^{2n}$ such that $w_1, J(0)w_1, \dots, w_n, J(0)w_n$ is a basis of $\mathbb{R}^{2n}$. For $t' \in (t-\varepsilon, t+\varepsilon)$, define $\Psi(t') \in \mathrm{GL}(2n,\mathbb{R})$ by the requirement

$$\Psi(t') w_j = v_j, \qquad \Psi(t') J(0) w_j = J(t') v_j, \qquad j = 1,\dots,n.$$

These conditions are equivalent to

$$\Psi(t') J(0) = J(t') \Psi(t').$$

Moreover, since $t \mapsto J(t)$ is of class $W^{1,2}$, it follows that the map

$$(t-\varepsilon, t+\varepsilon) \to \mathrm{GL}(2n,\mathbb{R}), \qquad t' \mapsto \Psi(t'),$$

which locally represents ψ, is of class $W^{1,2}$, too. Therefore, the map ψ is locally $W^{1,2}$ and since $\mathbb{S}^1$ is compact, it is globally $W^{1,2}$. This proves Step 2. ■

Step 3. The loop ψ in (C.6) lifts to a $W^{1,2}$-loop Ψ in $\mathrm{GL}^+(2n,\mathbb{R})$ based at $\mathbb{1}$.

Proof. As the Lie group $\mathrm{GL}(n,\mathbb{C})$ is closed in $\mathrm{GL}(2n,\mathbb{R})$ we have a fiber bundle

$$\mathrm{GL}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}(2n,\mathbb{R}) \to \frac{\mathrm{GL}(2n,\mathbb{R})}{\mathrm{GL}(n,\mathbb{C})}$$

see, e.g., [2, Chapter II, end of Section 13]. By choosing a connection on the fiber bundle, we can lift the path ψ, see (C.6), to a path $\tilde\Psi \in W^{1,2}([0,1],\mathrm{GL}(2n,\mathbb{R}))$ such that $\tilde\Psi(0)=\mathbb{1}$ and $\big[\tilde\Psi(t)\big]=\psi(t)$ for every $t\in[0,1]$. The path $\tilde\Psi$ is not necessarily a loop. But since $\big[\tilde\Psi(1)\big]=\psi(1)=\psi(0)=\big[\tilde\Psi(0)\big]$, the initial and the end point of $\tilde\Psi$ differ by an element in $\mathrm{GL}(n,\mathbb{C})$. Hence there exists $\Phi_1\in\mathrm{GL}(n,\mathbb{C})$ such that $\tilde\Psi(1)=\tilde\Psi(0)\Phi_1=\Phi_1$. Since $\mathrm{GL}(n,\mathbb{C})$ is connected, there exists a smooth path $\Phi\colon[0,1]\to\mathrm{GL}(n,\mathbb{C})$ from $\Phi(0)=\mathbb{1}$ to $\Phi(1)=\Phi_1$. The path defined for $t\in[0,1]$ by

$$\Psi(t):=\tilde\Psi(t)\Phi(t)^{-1}$$

has the following properties. Firstly, it is a loop in $\mathrm{GL}^+(2n,\mathbb{R})$ since $\Psi(0):=\tilde\Psi(0)\Phi(0)^{-1}=\mathbb{1}$ and $\Psi(1):=\tilde\Psi(1)\Phi(1)^{-1}=\Phi(1)\Phi(1)^{-1}=\mathbb{1}$. Secondly, it is of class $W^{1,2}$ since $\tilde\Psi$ is of class $W^{1,2}$ and Φ is smooth. Thirdly, it is a lift of ψ since $[\Psi(t)]=\big[\tilde\Psi(t)\Phi(t)^{-1}\big]=\big[\tilde\Psi(t)\big]=\psi(t)$. Therefore, (C.4) holds by (C.5). This proves Step 3. ∎

This proves Lemma C.5. ∎

Proof of Theorem C.1

We show Theorem C.1 for $k=2$, the case $k=1$ is analogous. To this end, given a loop $\omega\in W^{1,2}\big(\mathbb{S}^1,\Omega(\mathbb{R}^{2n})\big)$, we get a $W^{1,2}$-loop $B=B_\omega$ in $\mathrm{GL}^+(2n,\mathbb{R})$. Consider the operator

$$F_{B^{-1}}=B^{-1}\partial_t\colon\ W^{2,2}\big(\mathbb{S}^1,\mathbb{R}^{2n}\big)\to W^{1,2}\big(\mathbb{S}^1,\mathbb{R}^{2n}\big).$$

By Lemma C.2, this is a semi-Fredholm operator. By (C.3), the loop $t\mapsto B(t)$ is homotopic in $\mathrm{GL}^+(2n,\mathbb{R})$ to the loop $t\mapsto J_{B(t)}$ of almost complex structures. Therefore, the loop B^{-1} is homotopic to the loop $(J_B)^{-1}=-J_B$.

Since the semi-Fredholm index is a homotopy invariant, see, e.g., [11, Section 18, Corollary 3], we have $\operatorname{index}F_{B^{-1}}=\operatorname{index}F_{-J_B}$. By Lemma C.5, the loop J_B is conjugated in $W^{1,2}\big(\mathbb{S}^1,\mathrm{GL}^+(2n,\mathbb{R})\big)$ to the constant loop $J(0)=J_{B(0)}$. And therefore by Lemma C.3,

$$\operatorname{index}F_{-J_B}=\operatorname{index}F_{-J(0)}.$$

Since $F_{-J(0)}=-J(0)\partial_t$ is symmetric for the $W^{1,2}$ inner product, it holds $\operatorname{index}F_{-J(0)}=0$. Summarizing

$$\operatorname{index}F_{B^{-1}}=\operatorname{index}F_{-J_B}=\operatorname{index}F_{-J(0)}=0.$$

Since the index of the semi-Fredholm operator $F_{B^{-1}}$ is zero, the operator $F_{B^{-1}}$ is actually Fredholm. This proves Theorem C.1.

D Heron square root iteration

Consider the cone $\mathcal{P}$ of positive symmetric $k\times k$ matrices. The square root map $\mathcal{P}\to\mathcal{P}$, $Q\mapsto\sqrt{Q}$, is a diffeomorphism. There are two ways to construct the square root map, either using spectral calculus or the Heron iteration method. The disadvantage of the spectral calculus approach is that smoothness of the square root map is rather unexpected since eigenvalues in general depend only continuously on the matrix and the projections to the eigenspaces can even be discontinuous, if different branches of eigenvalues meet. Therefore, we define the square root map in this appendix with the help of the Heron iteration.

D.1 Real values

D.1.1 Heron iteration

Lemma D.1 (Heron method)**.** *Let $q>0$ be a positive real number. Pick a positive real number $r_1>0$. Define a sequence r_n recursively by the requirement*

$$r_1>0,\qquad r_{n+1}:=\frac{1}{2}\big(r_n+r_n^{-1}q\big),\qquad n\in\mathbb{N}.\tag{D.1}$$

Then the following is true:

(i) *The sequence $(r_n)_{n\ge 2}$ is monotone decreasing and bounded below by $\sqrt{q}$.*

(ii) *The limit $r:=\lim_{n\to\infty} r_n$ exists and $r^2=q$.*

Proof of Lemma D.1. (i) We first show that

$$\text{(a) } r_n>0,\qquad \text{(b) } r_n\ge\sqrt{q},$$

whenever $n\ge 2$. While (a) is obvious from (D.1), to see (b) note that

$$a:=\sqrt{\frac{r_{n-1}}{2}},\qquad b\overset{\text{(a)}}{:=}\sqrt{\frac{q}{2r_{n-1}}},\qquad\Rightarrow\qquad \frac{1}{2}\sqrt{q}=ab\le\frac{a^2+b^2}{2}=\frac{1}{2}\left(\frac{r_{n-1}}{2}+\frac{q}{2r_{n-1}}\right)=\frac{r_n}{2}.$$

We show that the sequence is monotone decreasing. Indeed, by (a) and (b), we get

$$r_n-r_{n+1}=r_n-\frac{1}{2}\big(r_n+r_n^{-1}q\big)=\frac{1}{2}\frac{r_n^2-q}{r_n}\ge 0$$

whenever $n\ge 2$.

(ii) follows from (i). By the monotone convergence theorem, the sequence r_n has a limit $r\ge\sqrt{q}$. On the other hand, by the recursion formula (D.1) the limit r satisfies $r=\frac{1}{2}\big(r+r^{-1}q\big)$, equivalently $r^2=q$. ∎

D.1.2 Newton–Picard iteration

To determine the square root of a positive real number $q>0$ is equivalent, to show that the function $f(r)=r^2-q$ has a unique positive zero $r_*>0$. For the latter, Newton–Picard iteration serves. Choose a point $r_n>0$ and consider the tangent line to the graph of f at the point $(r_n,f(r_n))$. If the slope $f'(r_n)\ne 0$ is non-zero, the tangent line intersects the x-axis at a point denoted and given by

$$r_{n+1}=r_n-\frac{f(r_n)}{f'(r_n)}$$

as illustrated by Figure 3. Since $f(r_n)=r_n^2-q$ and $f'(r_n)=2r_n$, Newton–Picard iteration for the function $f(r)=r^2-q$ reproduces the Heron method, indeed

$$r_1>0,\qquad r_{n+1}=r_n-\frac{r_n^2-q}{2r_n}=\frac{1}{2}\left(r_n+\frac{q}{r_n}\right),\qquad n\in\mathbb{N}.$$

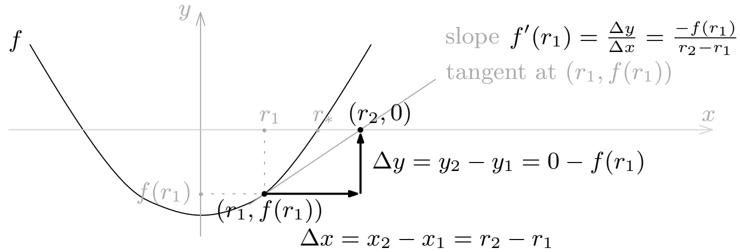

Figure 3. Newton–Picard iteration for $f(r) = r^2 - q$.

D.2 Matrix values

Lemma D.2 (Heron method)**.** *Let $Q \in \mathbb{R}^{k\times k}$ be a symmetric positive definite matrix. Define a sequence R_n of matrices recursively by the requirement*

$$R_1 := \mathbb{1}, \qquad R_{n+1} := \frac{1}{2}\big(R_n + R_n^{-1}Q\big), \qquad n \in \mathbb{N}. \tag{D.2}$$

Then the following is true:

(i) Each matrix R_n commutes with Q and is symmetric positive definite.

(ii) The limit $R := \lim_{n\to\infty} R_n$ exists and is symmetric.

(iii) $R^2 = Q$ and R is positive definite.

Remark D.3. The symmetric positive definite matrix R is called the (positive) *square root* of Q and denoted by $\sqrt{Q} := R$.

Corollary D.4. *Let $Q \in \mathbb{R}^{k\times k}$ be a symmetric positive definite matrix. Then any matrix B that commutes with Q also commutes with $\sqrt{Q}$.*

Proof of Corollary D.4. Let $(R_n)_{n\in\mathbb{N}}$ be the sequence of positive definite symmetric matrices defined by (D.2). In particular, $\lim_{n\to\infty} R_n = R =: \sqrt{Q}$. Clearly, B commutes with $R_1 = \mathbb{1}$. We show inductively that B commutes with R_{n+1} whenever $n \in \mathbb{N}$. To this end, assume that $[B, R_n] = 0$. Therefore, $\big[B, R_n^{-1}\big] = 0$. Together with the corollary hypothesis $[B, Q] = 0$, we obtain that $\big[B, R_n^{-1}Q\big] = 0$. Hence by the recursion formula (D.2), we get that

$$[B, R_{n+1}] = \frac{1}{2}\big([B, R_n] + \big[B, R_n^{-1}Q\big]\big) = 0.$$

This finishes the induction. Hence $[B, R] = \lim_{n\to\infty}[B, R_n] = 0$. This proves Corollary D.4. ∎

Proof of Lemma D.2. (i) The proof is by induction on n. For $n = 1$, this is true. Suppose (i) holds for R_n. In particular, R_n^{-1} commutes with Q. Then

$$R_{n+1}Q = \frac{1}{2}\big(R_nQ + R_n^{-1}QQ\big) = \frac{1}{2}\big(QR_n + QR_n^{-1}Q\big) = QR_{n+1}$$

and the transpose satisfies

$${R_{n+1}}^T \stackrel{1}{=} \frac{1}{2}\big({R_n}^T + \big(QR_n^{-1}\big)^T\big) = \frac{1}{2}\big(R_n + \big(R_n^{-1}\big)^T Q^T\big) \stackrel{3}{=} R_{n+1}.$$

In Step 1, we used $R_n^{-1}Q = QR_n^{-1}$, and in Step 3 that transpose of inverse is inverse of transpose. That R_{n+1} defined by (D.2) is positive definite is true since positive definiteness is preserved under composition of commuting symmetric matrices and under sum of symmetric matrices.

(ii) We follow [8]. Since Q is symmetric positive definite it is diagonalizable, that is, there exists an orthogonal matrix P such that

$$PQP^{-1} = \operatorname{diag}\bigl(q^{(1)}, \dots, q^{(k)}\bigr) =: \Lambda,$$

where $q^{(1)}, \dots, q^{(k)} \in \mathbb{R} \setminus \{0\}$ are the eigenvalues of Q. Now the iterations

$$PR_nP^{-1} = \operatorname{diag}\bigl(r_n^{(1)}, \dots, r_n^{(k)}\bigr) =: D_n$$

are diagonal as well, as follows by induction on n. For $n = 1$, this is true since

$$D_1 := PR_1P^{-1} = \mathbb{1}.$$

Suppose it is true for n, then using (D.2) in Step 2

$$D_{n+1} := PR_{n+1}P^{-1} = \frac{1}{2}\bigl(PR_nP^{-1} + PR_n^{-1}P^{-1}PQP^{-1}\bigr) = \frac{1}{2}\bigl(D_n + D_n^{-1}\Lambda\bigr)$$

is indeed a diagonal matrix. But now each diagonal position corresponds to a real-valued Heron iteration (D.1). This proves that the limit R exists.

Symmetry: Given $\xi, \eta \in \mathbb{R}^k$, then $\langle R_n\xi, \eta\rangle = \langle \xi, R_n\eta\rangle$ by symmetry of R_n. In the limit, as $n \to \infty$, we get $\langle R\xi, \eta\rangle = \langle \xi, R\eta\rangle$.

(iii) By (ii), the limit R exists and by the recursion formula (D.2) it satisfies $R = \frac{1}{2}\bigl(R + R^{-1}Q\bigr)$, equivalently $R^2 = Q$.

Positive definite: Since the R_n are symmetric positive definite, their eigenvalues are real and ≥ 0. Since $R = \lim_{n\to\infty} R_n$ is symmetric, its eigenvalues are real. Since eigenvalues depend continuously on the matrix the eigenvalues of R are ≥ 0. Let ρ be the smallest eigenvalue of R. Then $\rho \geq 0$ and ρ^2 is eigenvalue of $R^2 = Q$. Since Q is positive definite, all eigenvalues are strictly positive, in particular $\rho^2 > 0$, hence $\rho > 0$. Thus R is positive definite.

This concludes the proof of Lemma D.2. ∎

If in Corollary D.4 one assumes in addition symmetric and positive definite for the matrix B, then the proof reduces essentially to the existence of a basis composed of common eigenvectors of B and Q.

Lemma D.5. *Let $Q \in \mathbb{R}^{k\times k}$ be a symmetric positive definite matrix. Then a symmetric positive definite matrix B that commutes with Q commutes with $\sqrt{Q}$.*

Proof. Linear algebra tells the following: Two symmetric positive definite $k \times k$ matrices, in the case at hand Q and B, commute if and only if there exists an orthonormal basis $\mathcal{X} = \{\xi_1, \dots, \xi_k\}$ whose elements are eigenvectors of both matrices, say $Q\xi_i = \rho_i\xi_i$ and $B\xi_i = \lambda_i\xi_i$ for some $\rho_i, \lambda_i > 0$ and $i = 1, \dots, k$. The positive square root of Q, notation $\sqrt{Q} =: R$, is defined by $R\xi_i := \sqrt{\rho_i}\xi_i$ for $i = 1, \dots, k$. It is an exercise to check that R is symmetric positive definite; here pairwise orthogonality of the ξ_i enters. But the ON basis $\mathcal{X}$ is composed of eigenvectors of both R and B, hence both matrices commute by the linear algebra cited in the beginning of this proof.

This proves Lemma D.5. ∎

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